Lecture 2:
NNLO Computations and Two-loop Amplitudes

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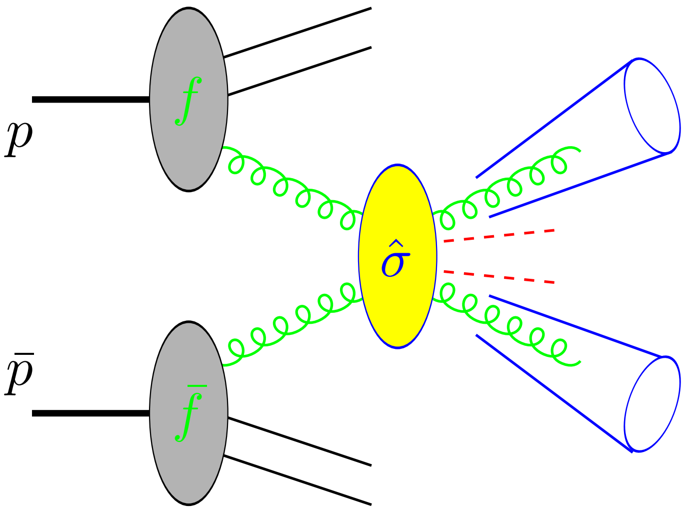
Anatomy of NNLO computations

- A hard process, such as $p\bar{p} \to$ 2 jets + anything, factorizes into parton distribution functions f and a hard cross section $\hat{\sigma}$:

$$\sigma_{p\bar{p}\to jjX} = \sum_{a,b} \int dx_1 dx_2 \; f_a(x_1;\mu_F)\, \bar{f}_b(x_2;\mu_F) \times \hat{\sigma}_{ab\to jjX}(s x_1 x_2; \mu_F, \mu_R)$$



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- Here, focus on corrections to hard processes:

$$\hat{\sigma}_{ab\to jjX} = \alpha_s^2\,\hat{\sigma}^{\rm LO}_{ab\to jj} + \alpha_s^3\,\hat{\sigma}^{\rm NLO}_{ab\to jjX} + \alpha_s^4\,\hat{\sigma}^{\rm NNLO}_{ab\to jjX} + \ldots$$

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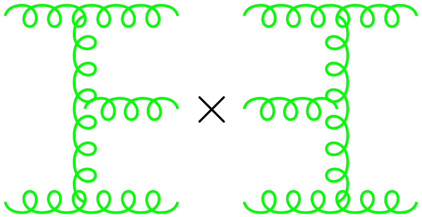
- real $\times$ real: 

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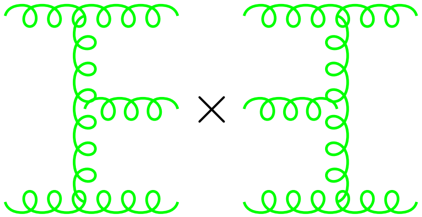

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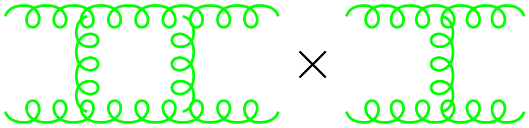

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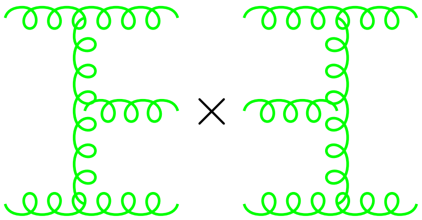

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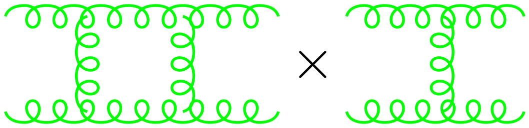

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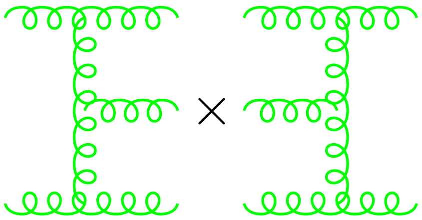

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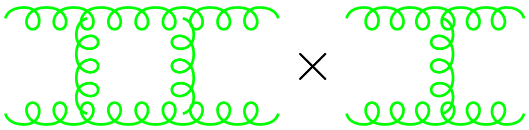

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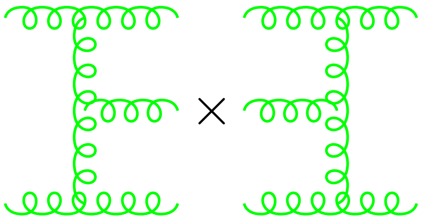

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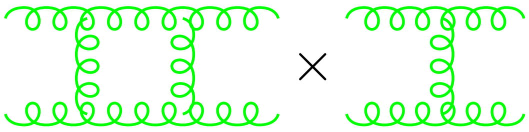

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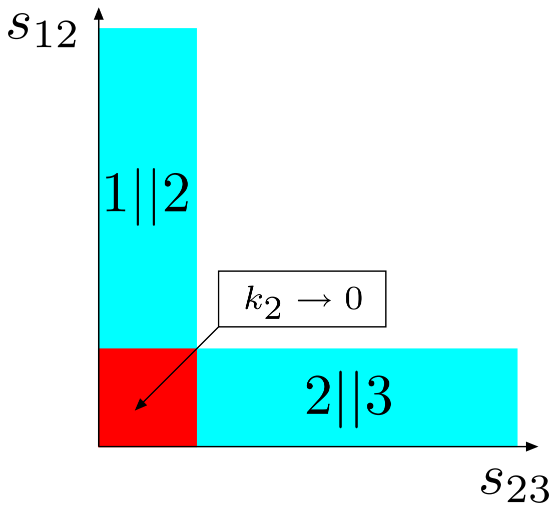

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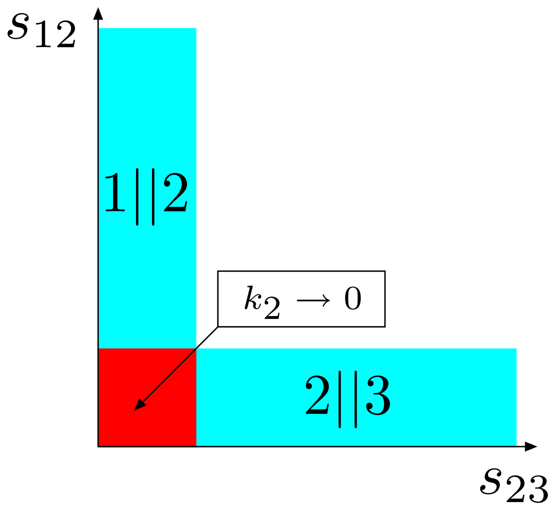

- Behavior of cross section in this singular region is universal.

NLO real $\times$ real integration (cont.)

- Often cannot do full $D = 4 - 2\epsilon$ integral analytically. Instead, exploit the universal amplitude behavior, e.g.

$$A_{n+1}^{\text{tree}}(k_1, k_2, \ldots, k_{n+1}) \to \text{Split}(z, s_{12}) \times A_n^{\text{tree}}(k_P, \ldots, k_{n+1}),$$

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- Slicing methods — Giele, Glover; Giele, Glover, Kosower

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- Some numerical issues: Have to choose $s_{\min}$ small enough that soft/collinear amplitude approximations are OK, yet large enough that $\ln(s_{\min})$ factors do not overwhelm remaining terms.

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- Dipole formalism — Catani, Seymour
  is one specific implementation, which can be applied straightforwardly to generic NLO QCD processes.

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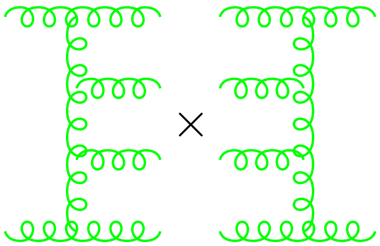

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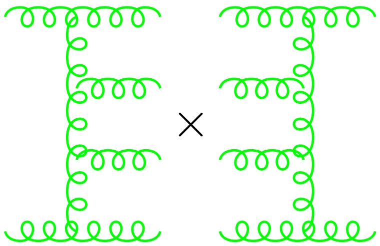

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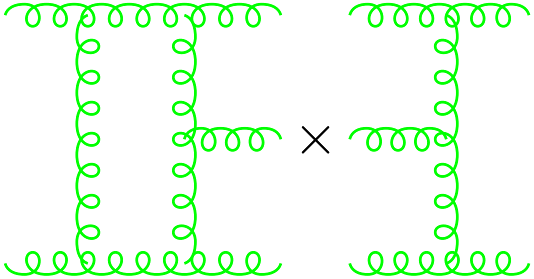

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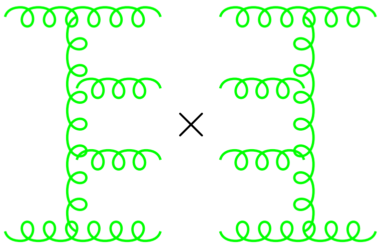

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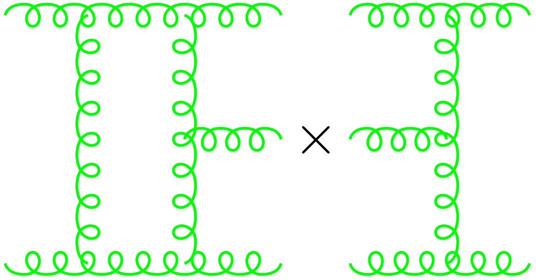

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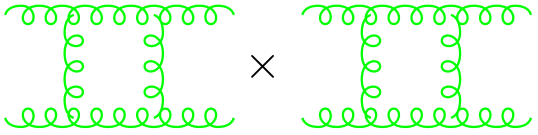

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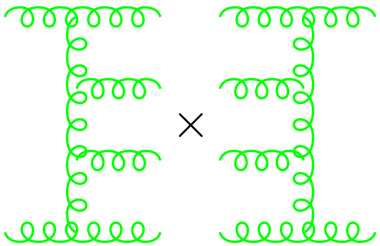

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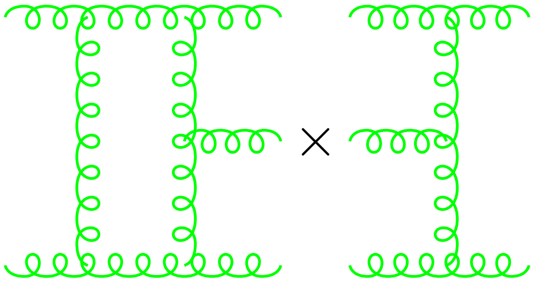
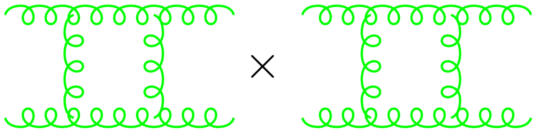
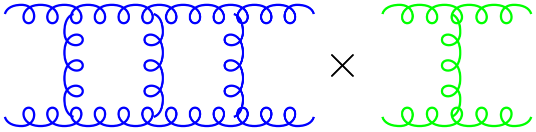


- virtual $\times$ virtual:



- doubly-virtual $\times$ real:



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  - triple collinear — Catani, Grazzini; Catani, Grazzini
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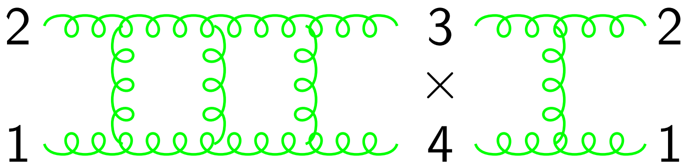
- Today, focus on two-loop QCD scattering amplitudes.

Interferences or amplitudes?

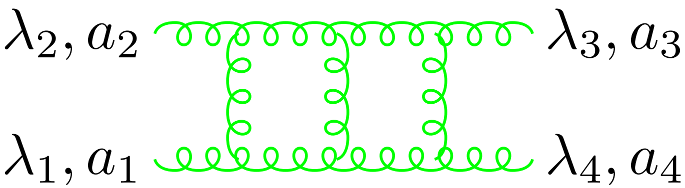
- Two basic approaches to (two)-loop QCD calculations:

1. Calculate unpolarized (doubly) virtual cross section, i.e., the interference, summed over all external helicities λ_i and colors a_i:

$$\sum_{\lambda_i, a_i}$$



2. Calculate the amplitudes, decomposed under helicities and colors:



Interference Method

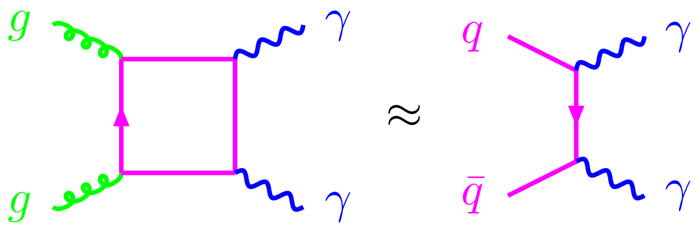
- Has been applied very successfully to all $2 \to 2$ parton scattering amplitudes in QCD:
  - $qq \to qq$ — Anastasiou, Glover, Oleari, Tejeda-Yeomans; AGOT-Y
  - $q\bar{q} \to gg$ — AGOT-Y
  - $gg \to gg$ — Glover, Oleari, Tejeda-Yeomans
- More recently, to processes with a massive vector V:
  - $V \to q\bar{q}g$ — Garland, Gehrmann, Glover, Koukoutsakis, Remiddi
  - $q\bar{q} \to Vg$, $Vq \to gq$, etc. — Gehrmann, Remiddi
- Perfectly fine for the main application: unpolarized NNLO jet cross sections (and integrating over leptonic angles, for $V \to \ell^+\ell^-$).
- For other applications, can use helicity amplitudes.

Uses of helicity amplitudes

- Polarized processes. NNLO jet cross sections at RHIC?
- Including event orientation in $\mathrm{e}^+\mathrm{e}^- \to q\bar{q}g$,
  lepton decay angle in $qg \to Wq' \to \ell\nu_\ell q'$
- Understanding formal properties of two-loop amplitudes:
  - supersymmetry
  - high energy (BFKL) limit
  - collinear limits
  - infrared singularities.
- Regularization scheme dependence at two loops:
  - conventional dim. reg. ($D = 4 - 2\epsilon$ everywhere)
  - 't Hooft-Veltman scheme ($4 - 2 = 2$ external g states)
  - FDH scheme (2 g states everywhere, SUSY scheme)

Uses of helicity amplitudes (cont.)

- Two-loop amplitudes where the interference of phenomenological interest is with a one-loop amplitude, because the tree amplitude vanishes.
  Prominent examples:
  - $\gamma\gamma \to \gamma\gamma$ fundamental QED process
  - $\gamma g \to \gamma g$ significant contrib. to $\gamma \mathrm{p} \to \gamma X$ at HERA
  - $gg \to \gamma\gamma$ background to LHC Higgs search



  after pdf convolution, due to large $g(x)$ in proton at $x \approx 0.01$. (See lecture 5.)

Known two-loop helicity amplitudes

- $q\bar{q} \to gg$ — Bern, De Freitas, LD; Glover, Tejeda-Yeomans
- $gg \to gg$ — Bern, De Freitas, LD
- $gg \to \gamma\gamma$ and $\gamma\gamma \to \gamma\gamma$ — Bern et al.; Bern et al.; Binoth et al.
- $\mathrm{e}^+\mathrm{e}^- \to q\bar{q}g$ — Garland, Gehrmann, Glover, Koukoutsakis, Remiddi

  Moch, Uwer, Weinzierl

Two-loop techniques

- Two basic, but separate, issues:
  1. integrands
  2. integrals
- Discuss these in turn. Integrals next lecture.

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- In practice, this is often the most straightforward and convenient way to proceed for two-loop QCD amplitudes of phenomenological interest, particularly in the interference method.
- Diagrams can be generated automatically. See Harlander & Steinhauser (1998) for a review.

Integrands (cont.)

- E.g., QGRAF (FORTRAN) takes as input:
  - propagators & vertices (topological)
  - initial & final states, and number of loops

  It then generates all Feynman graph topologies with different possible output formats, e.g. FORM.

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- In the interference method, the Feynman integrand has no free spinor or vector indices.
- Can use FORM to evaluate all Dirac traces and perform all index contraction in D dimensions (the conventional dimensional regularization (CDR) scheme).

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- Spinor spin sum $\sum_s u_s(a)\bar{u}_s(a) = (\not{k}_a + m)$
- Massless vector spin sum (transverse projector)

$$\sum_\lambda \varepsilon_\lambda^\mu(a)\varepsilon_\lambda^{*\,\nu}(a) = -\eta^{\mu\nu} + (k_a^\mu q_a^\nu + q_a^\mu k_a^\nu)/(k_a \cdot q_a)$$

for some "light-like axial gauge choice", or
reference momentum vector q_a, $q_a^2 = 0$. (Can take $q_a = k_b$.)

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use $\sum_\lambda \varepsilon_\lambda^\mu(a)\varepsilon_\lambda^{*\,\nu}(a) = -\eta^{\mu\nu}$ instead.
- D-dimensional Dirac algebra, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$,
with $\eta^\mu_{\ \mu} = D = 4 - 2\epsilon$, $\mathrm{Tr}\, 1 = 4$

Interference method (cont.)

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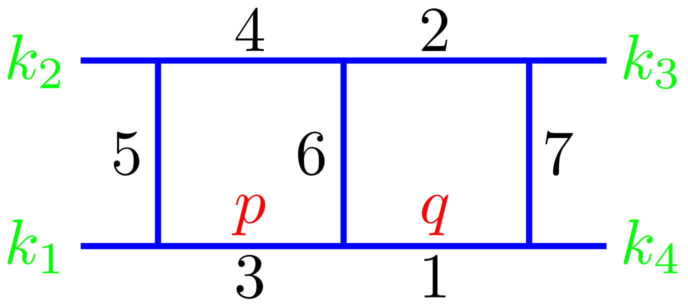
- For example, the planar double box diagram



generates $$\mathcal{I}[P] \equiv \int d^Dp\, d^Dq\, \frac{P(p\cdot q, p\cdot k_i; s,t)}{p_1^2\, p_2^2 \cdots p_7^2}$$

Helicity amplitude method I

Garland, Gehrmann, Glover, Koukoutsakis, Remiddi; Glover, Tejeda-Yeomans

1. Perform a general kinematic (and color) decomposition of the amplitude, using on-shell properties. For example, for $0 \to q(p_1, \lambda_1) + \bar{q}(p_2 \lambda_2) + g_3(p_3, \lambda_3) + g_4(p_4, \lambda_4)$,

$$
\begin{aligned}
|\mathcal{A}\rangle \;=\; & \bar{u}(p_1)\, \not{p}_3 u(p_2)(A_{11}\, \varepsilon_3 \cdot p_1\, \varepsilon_4 \cdot p_1 + A_{12}\, \varepsilon_3 \cdot p_1\, \varepsilon_4 \cdot p_2) \\
& + \bar{u}(p_1)\, \not{p}_3 u(p_2)(A_{21}\, \varepsilon_3 \cdot p_2\, \varepsilon_4 \cdot p_1 + A_{22}\, \varepsilon_3 \cdot p_2\, \varepsilon_4 \cdot p_2) \\
& + \bar{u}(p_1)\, \not{\varepsilon}_4 u(p_2)(B_1\, \varepsilon_3 \cdot p_1 + B_2\, \varepsilon_3 \cdot p_2) \\
& + \bar{u}(p_1)\, \not{\varepsilon}_3 u(p_2)(C_1\, \varepsilon_4 \cdot p_1 + C_2\, \varepsilon_4 \cdot p_2) \\
& + D_1\, \bar{u}(p_1)\, \not{\varepsilon}_3\, \not{p}_3\, \not{\varepsilon}_4 u(p_2) + D_2\, \bar{u}(p_1)\, \not{\varepsilon}_4\, \not{p}_3\, \not{\varepsilon}_3 u(p_2)
\end{aligned}
$$

using $p_1 + p_2 + p_3 + p_4 = 0$

and $\varepsilon_3 \cdot p_3 = \varepsilon_4 \cdot p_4 = \not{p}_1 u(p_1) = \not{p}_2 u(p_2) = 0$

Helicity amplitude method I (cont.)

Ward identities $|\mathcal{A}\rangle(\varepsilon_3 \to p_3) = |\mathcal{A}\rangle(\varepsilon_4 \to p_4) = 0$, relate the 10 kinematic structures, so only 5 are independent,

$$
\begin{aligned}
|\mathcal{A}\rangle &= \sum_{I,J=1,2} A_{IJ}(s_{13}, s_{23})\, T_{IJ} + B(s_{13}, s_{23})\, T \\
T_{IJ} &= \bar{u}(p_1)\ \not{p}_3 u(p_2)\, \varepsilon_3 \cdot p_I\, \varepsilon_4 \cdot p_J - \frac{s_{I3}}{2} \bar{u}(p_1)\ \not{\varepsilon}_3 u(p_2)\, \varepsilon_4 \cdot p_J \\
&\quad - \frac{s_{J4}}{2} \bar{u}(p_1)\ \not{\varepsilon}_3\ \not{p}_3\ \not{\varepsilon}_4 u(p_2) \\
T &= s_{23}\left(\bar{u}(p_1)\ \not{\varepsilon}_4 u(p_2) \varepsilon_3 \cdot p_1 + \frac{1}{2}\bar{u}(p_1)\ \not{\varepsilon}_3\ \not{p}_3\ \not{\varepsilon}_4 u(p_2)\right) \\
&\quad - s_{13}\left(\bar{u}(p_1)\ \not{\varepsilon}_4 u(p_2) \varepsilon_3 \cdot p_2 + \frac{1}{2}\bar{u}(p_1)\ \not{\varepsilon}_4\ \not{p}_3\ \not{\varepsilon}_3 u(p_2)\right)
\end{aligned}
$$

Bose symmetry relates $A_{21}(s_{13}, s_{23}) = -A_{12}(s_{23}, s_{13})$, $A_{22}(s_{13}, s_{23}) = -A_{11}(s_{23}, s_{13})$, $\qquad B(s_{13}, s_{23}) = B(s_{23}, s_{13})$

Helicity amplitude method I (cont.)

2. Find projectors $\mathcal{P}$ onto the independent tensor structures in D dimensions, by inverting a matrix $\sum_{s,\lambda}\langle T'|T\rangle$.
   For example,

$$
\begin{aligned}
\mathcal{P}(B) &= \frac{1}{2s_{12}^2 s_{13}s_{23}(D-3)}T_{11}^\dagger - \frac{1}{2s_{12}^2 s_{13}s_{23}(D-3)}T_{22}^\dagger \\
&+\frac{1}{2s_{12}^2 s_{13}s_{23}(D-4)}T^\dagger - \frac{2s_{13}-(D-4)s_{23}}{2s_{12}^2 s_{13}^2 s_{23}(D-3)(D-4)}T_{12}^\dagger \\
&+\frac{2s_{23}-(D-4)s_{13}}{2s_{12}^2 s_{13}s_{23}^2(D-3)(D-4)}T_{21}^\dagger
\end{aligned}
$$

3. Compute B, etc. as $\sum_{s,\lambda}\mathcal{P}(B)|\mathcal{A}\rangle = B$.
   Requires only the same type of $P(p\cdot q, p\cdot k_i; s, t)$ as in interference method.

Helicity amplitude method II

Bern et al.; Bern et al.

1. Use explicit four-dimensional external helicity states:
   - Spinors $u_\pm(k_a) = \frac{1}{2}(1 \pm \gamma_5) u(k_a)$
   - Massless vectors $\varepsilon^\pm_\mu(k_a, q_a) = \pm \frac{\langle q_a^\mp | \gamma_\mu | k_a^\mp \rangle}{\sqrt{2} \langle q_a^\mp | k_a^\pm \rangle}$ LD review
2. $\varepsilon^\pm_\mu$ are 4-dimensional, pick out a subspace of $D = 4 - 2\epsilon$ (best to think of $\epsilon < 0$).
3. Can divide and multiply amplitude by short spinor strings to convert to Dirac traces.
4. Polynomials P contain additional kinematic structures:
   - $\varepsilon_a \cdot p$, $\varepsilon_a \cdot q$, or
   - λ_p^2, λ_q^2, $\lambda_p \cdot \lambda_q$, where $p = (p_{[4]}, \lambda_p)$ is $D = (4, (-2\epsilon))$ decomposition of loop momentum.

Helicity amplitude method II (cont.)

- Here one has the option to let ${\eta^\mu}_\mu = D_s = 4 - 2\epsilon\delta_R$.

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- $\delta_R = 1$ corresponds to ‘t Hooft-Veltman (HV) scheme (external states 4-dimensional; internal D-dimensional). Amplitudes computed this way agree with method I (at least for $\bar{q}q \to gg$!).

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- Here one has the option to let $\eta^{\mu}{}_{\mu} = D_s = 4 - 2\epsilon\delta_R$.
- $\delta_R = 1$ corresponds to 't Hooft-Veltman (HV) scheme (external states 4-dimensional; internal D-dimensional). Amplitudes computed this way agree with method I (at least for $\bar{q}q \to gg$!).
- $\delta_R = 0$ defines four-dimensional helicity (FDH) scheme $\sim$ dimensional reduction — but $\epsilon < 0 \Rightarrow$ "room" for helicities. Equal number of fermionic and bosonic degrees of freedom. Amplitudes computed this way, in supersymmetric theories, obey supersymmetry Ward identities

  Grisaru, Pendleton, van Nieuwenhuizen (1977); Parke, Taylor (1985)

  Powerful check on amplitudes, even in QCD.

Helicity amplitude method II (cont.)

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Helicity amplitude method II (cont.)

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- It is generally necessary to do so for the complete set of QCD amplitudes required phenomenologically.
- However, for special helicity configurations, and/or theories with high degrees of supersymmetry, remarkably much can be done by hand, if one also exploits unitarity techniques
  $\Rightarrow$ next lecture.