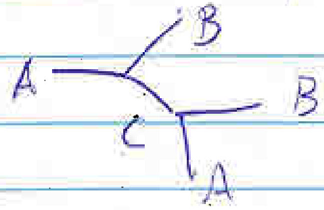
Solutions to Problem Set 3

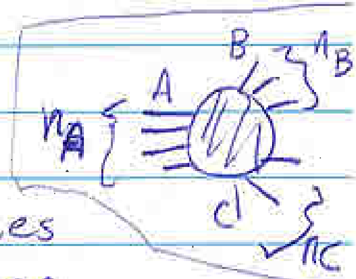
1 (a) $A \to B + B$ is not a possible process in the ABC theory because

$A \to B + \cancel{C} + B + A$.



That is, to remove C by adding another ABC vertex makes both B and A

(b) More generally, the ABC vertex changes the numbers n_A, n_B, n_C all by 1, and connecting vertices with propagators changes one number n_A, n_B, or n_C by 2. ← (this is no change, mod 2)



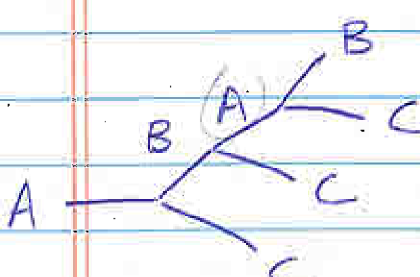
So every allowed reaction must have

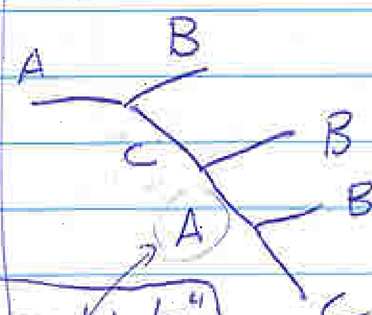
$$\boxed{n_A \pmod 2 = n_B \pmod 2 = n_C \pmod 2}$$

Or, n_A, n_B, n_C all have same parity: all odd, or all even.

(c) After $A \to B + C$ , the next most likely decay modes are

$A \to B + C + C + C$	$A \to B + B + B + C$





"virtual particle" not enough energy to make a real "A" here.

(§3.2)

conjugate, multiply on left by

(2) (a) C takes $\psi \to i\gamma^2\psi^*$ ← $i\gamma^2 = \begin{pmatrix} 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \end{pmatrix}$

So $u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \to N \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v^{(1)}$

Eq. (7.46) of Griffiths 2nd. ed. Eq. (7.47)

In the same way, it's easy to see that

$u^{(2)} \longrightarrow v^{(2)}$

So electron and positron states are exchanged by C, as desired.

(b) C takes ~~[illegible]~~

$\psi^* \to i\underbrace{\gamma^2}_{\text{real, so stays the same under complex conjugation}}\psi$

So $\bar{\psi} = \psi^{*T}\gamma^0 \to \psi^T i\gamma^{2T}\gamma^0$

Hence $\bar{\psi}\psi \to i \cdot i\, \psi^T\gamma^{2T}\gamma^0\gamma^2\psi^*$

Now the right-hand side is not a matrix, just a number. However, what I didn't tell you (so no points off for missing it!) is that the ψ's are made out of "Grassmann numbers", which means there is a minus sign when we replace the "number" by its transpose, because 2 Grassman numbers obey $\psi_1\psi_2 = -\psi_2\psi_1$

So $\bar{\psi}\psi \to i \cdot i\,(\psi^T\gamma^{2T}\gamma^0\gamma^2\psi^*)^T(-1)$

$(\gamma^2)^T = +\gamma^2$
$\gamma^{0T} = +\gamma^2$

$= +\psi^{*T}(\gamma^2)^T(\gamma^0)^T\gamma^2\psi$

$= \psi^\dagger\gamma^2\gamma^0\gamma^2\psi = -\underbrace{\psi^\dagger\gamma^0}_{\bar{\psi}}\underbrace{(\gamma^2)^2}_{-1}\psi$

$\boxed{\bar{\psi}\psi \to +\bar{\psi}\psi}$ ← this means ~~that~~ the electron mass, $m_e\bar{\psi}\psi$, preserves C

(2)(b) (CONT.) $\bar{\psi}\gamma^\mu\psi$ works similarly:

$$\bar{\psi}\gamma^\mu\psi \to i\cdot i\,(\psi^T\gamma^{2T}\gamma^0\gamma^\mu\gamma^2\psi^*)^T(-1)$$
$$= +\,\psi^{*T}(\gamma^2)^T(\gamma^\mu)^T(\gamma^0)^T\gamma^2\psi$$
$$= \bar{\psi}\,\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\psi$$

Note that $(\gamma^\mu)^T = \begin{cases} +(\gamma^\mu) & \mu=0,2 \\ -(\gamma^\mu) & \mu=1,3 \end{cases}$

But when we anticommute $\gamma^0\gamma^2$ past γ^μ we get $\begin{cases} -1 & \mu=0,2 \\ +1 & \mu=1,3 \end{cases}$

$$\Rightarrow \bar{\psi}\gamma^\mu\psi \to -\bar{\psi}\gamma^\mu\gamma^0\overset{(-)}{\gamma^2\gamma^0}\gamma^2\psi$$
$$= +\,\bar{\psi}\gamma^\mu\underbrace{(\gamma^0)^2}_{+1}\underbrace{(\gamma^2)^2}_{-1}\psi$$

$$\boxed{\bar{\psi}\gamma^\mu\psi \to -\bar{\psi}\gamma^\mu\psi}$$

So the electromagnetic current is <u>odd</u> under charge conjugation C (not surprisingly, since charges flip sign).

(c) But also $A^\mu \to -A^\mu$, because $\vec{E}\to-\vec{E}$, $\vec{B}\to-\vec{B}$

So the QED interaction is <u>invariant</u>,

$$A_\mu\bar{\psi}\gamma^\mu\psi \to +A_\mu\bar{\psi}\gamma^\mu\psi$$

$\Rightarrow$ C is a good symmetry of QED

(d) Following the same steps as above,

$$\bar{\psi}\gamma^\mu\gamma^5\psi \to \bar{\psi}\gamma^0\gamma^2\underbrace{(\gamma^5)^T}_{\gamma^5}(\gamma^\mu)^T\gamma^0\gamma^2\psi$$

(as above) $\dots \to -\bar{\psi}\gamma^5\gamma^\mu\psi$

$$= +\,\bar{\psi}\gamma^\mu\gamma^5\psi$$

(2)(d) (CONT.)

Because $\bar{\psi}\gamma^\mu\gamma^5\psi \to +\bar{\psi}\gamma^\mu\gamma^5\psi$

while $\bar{\psi}\gamma^\mu\psi \to -\bar{\psi}\gamma^\mu\psi$,

the axial coupling of the electron to the Z violates C: $Z_\mu \to -Z_\mu$ (just like A_μ)

$$\Rightarrow \quad g_A \bar{\psi}\gamma^\mu\gamma^5\psi Z_\mu \xrightarrow{C} -g_A \bar{\psi}\gamma^\mu\gamma^5\psi Z_\mu$$

[There is also a vector coupling to the Z, which preserves C: $g_V \bar{\psi}\gamma^\mu\psi Z_\mu \to +g_V \bar{\psi}\gamma^\mu\psi Z_\mu$]

- But in class we saw that under P, the axial coupling is also odd: $g_A \bar{\psi}\gamma^\mu\gamma^5\psi Z_\mu \xrightarrow{P} -g_A \bar{\psi}\gamma^\mu\gamma^5\psi Z_\mu$

- So under CP, it is <u>even</u>,

$$g_A \bar{\psi}\gamma^\mu\gamma^5\psi Z_\mu \xrightarrow{CP} +g_A \bar{\psi}\gamma^\mu\gamma^5\psi Z_\mu$$

- Also the vector coupling is <u>even</u>,

$$g_V \bar{\psi}\gamma^\mu\psi Z_\mu \xrightarrow{CP} +g_V \bar{\psi}\gamma^\mu\psi Z_\mu$$

- CP violation in the Standard Model is more subtle – it requires certain complex couplings when quarks couple to the W. In fact it only happens with at least 3 generations $\underset{1}{\begin{pmatrix} u \\ d \end{pmatrix}} \underset{2}{\begin{pmatrix} s \\ c \end{pmatrix}} \underset{3}{\begin{pmatrix} t \\ b \end{pmatrix}}$

[Kobayashi + Maskawa, Nobel Prize 2008]

(3) starting with $(\not{p}-m)u=0$, take the adjoint and multiply on the right with γ^0:

$$0 = u^\dagger(\not{p}^\dagger - m)\gamma^0$$

$$= \bar{u}\gamma^0(p_\mu \gamma^{\mu\dagger} - m)\gamma^0$$

$$= \bar{u}(p_\mu \gamma^0\gamma^{\mu\dagger}\gamma^0 - m)$$

using $\gamma^{\mu\dagger} = \begin{cases} +\gamma^\mu & \mu=0 \\ -\gamma^\mu & \mu=1,2,3, \end{cases}$

and commuting γ^0 past γ^μ, we have $\gamma^0\gamma^{\mu\dagger}\gamma^0 = +\gamma^\mu$

$$\Rightarrow \boxed{0 = \bar{u}(\not{p}-m)}$$

The same manipulations, given $(\not{p}+m)v=0$, show that $\boxed{\bar{v}(\not{p}+m)=0}$

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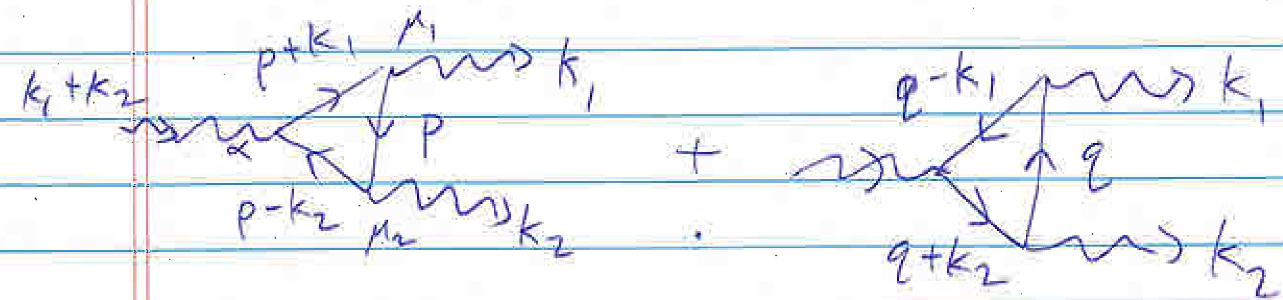
(4) The simplest argument is to use C, which is a good symmetry of QED.
Photons have $C=-1$, because $A_\mu \xrightarrow{C} -A_\mu$ and a photon is "made" from $A_\mu \sim \varepsilon_\mu e^{ik\cdot x}$
An odd number of photons is forbidden because the total $C=-1$.
Or you can say that a transition from odd # photons $\longrightarrow$ even # photons

$C = \quad -1 \longrightarrow +1$

violates C.

(4) (CONT.)

But we can also see the vanishing as a cancellation between two Feynman diagrams:



• The sum is proportional to: Let $q = -p$ here

$$\int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[(\not p - m)\gamma^{\mu_1}(\not p+\not k_1+m)\gamma^{\alpha}(\not p-\not k_2+m)\gamma^{\mu_2} + (-\not p - m)\gamma^{\mu_2}(-\not p+\not k_2+m)\gamma^{\alpha}(-\not p-\not k_1+m)\gamma^{\mu_1}\right]}{(p^2-m^2)((p-k_2)^2-m^2)((p+k_1)^2-m^2)}$$

cancel

• Notice that the two terms are the opposite of each other, except:

1) They are in reverse order (But both orders will give the same result using the Dirac trace formulas)

and

2) One should let $m \to -m$

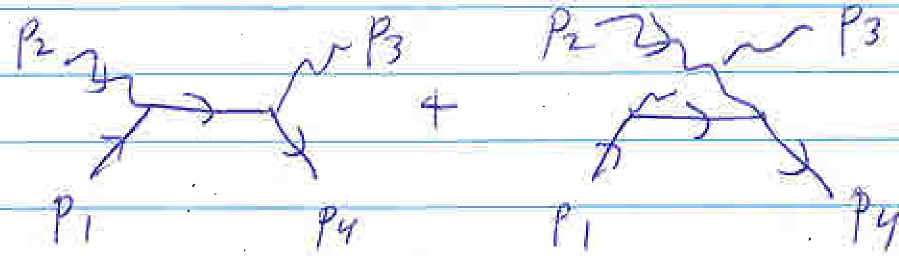
But only $(m^2)^k$ appears, because odd powers of m accompany odd numbers of γ matrices, which vanish.

$\therefore$ The two terms cancel.

So $\mathcal{M}(\gamma \to 2\gamma) = 0$

Compton scattering amplitude

(5) $\mathcal{M}(\gamma e \to \gamma e)$

$=$ 

$= \quad \mathcal{M}_1 \quad + \quad \mathcal{M}_2$

We want $\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\frac{1}{2}\sum'_{\text{spins}}\left\{|\mathcal{M}_1+\mathcal{M}_2|^2\right\}$

$$= \frac{1}{4}\sum'_{\text{spins}}\left\{|\mathcal{M}_1|^2 + \mathcal{M}_1^*\mathcal{M}_2 + \mathcal{M}_2^*\mathcal{M}_1 + |\mathcal{M}_2|^2\right\}$$

where

$$\mathcal{M}_1 = \bar{u}(p_4)\, ie\gamma^{\mu_3}\epsilon^*_{\mu_3}(p_3)\, \frac{i(\not p_1+\not p_2)}{(p_1+p_2)^2}\, ie\gamma^{\mu_2}\epsilon_{\mu_2}(p_2)\, u(p_1)$$

$$\mathcal{M}_2 = \bar{u}(p_4)\, ie\gamma^{\mu_2}\epsilon_{\mu_2}(p_2)\, \frac{i(\not p_1-\not p_3)}{(p_1-p_3)^2}\, ie\gamma^{\mu_3}\epsilon^*_{\mu_3}(p_3)\, u(p_1)$$

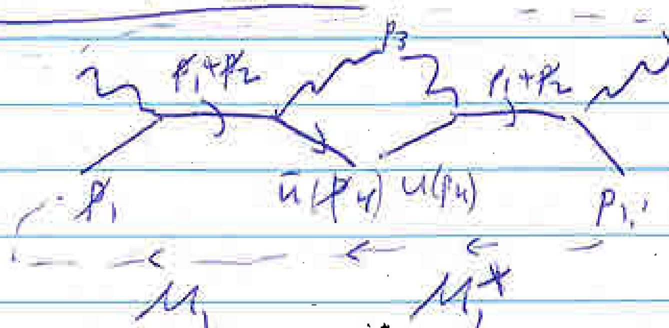
So, by Casimir's trick,

from $\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$

$$\frac{1}{4}\sum'_{\text{spins}}\left\{|\mathcal{M}_1|^2\right\} = \sum'_{s_{2,3}}\frac{e^4}{4((p_1+p_2)^2)^2}\,\mathrm{Tr}\left[\not p_4\, \not\epsilon^*(p_3)(\not p_1+\not p_2)\not\epsilon(p_2)\,\not p_1\, \not\epsilon^*(p_2)(\not p_1+\not p_2)\not\epsilon(p_3)\right]$$

from $\sum'_{s_4} u^{(s_4)}(p_4)\,\bar{u}^{(s_4)}(p_4)$ $\qquad$ from $\sum'_{s_1} u^{(s_1)}(p_1)\,\bar{u}^{(s_1)}(p_1)$

We can read this off from:



(5) (CONT.)

We do the sum over photon spins using

$$\sum_s \epsilon^{\mu}_{(s)}(q)\, \epsilon^{\nu *}_{(s)}(q) = -\eta^{\mu\nu}$$

This replaces

$$\not\epsilon(p_2) \;\ldots\; \not\epsilon^*(p_2)$$
$$\downarrow$$
$$\gamma^{\mu} \;\ldots\; \gamma_{\mu} \qquad \text{etc.}$$

$$\Rightarrow \frac{1}{4}\sum_{\text{spins}} \langle |\mathcal{M}_1|^2 \rangle = \frac{e^4}{4((p_1+p_2)^2)^2} \mathrm{Tr}\left[\not p_4 \gamma^{\mu_3} (\not p_1 + \not p_2) \gamma^{\mu_2} \not p_1 \gamma_{\mu_2} (\not p_1 + \not p_2) \gamma_{\mu_3}\right]$$

$$\gamma^{\mu_2} \not p_1 \gamma_{\mu_2} = -2\not p_1$$
$$\gamma_{\mu_3} \not p_4 \gamma^{\mu_3} = -2\not p_4$$

$$\Rightarrow \frac{1}{4}\sum_{\text{spins}} \{ |\mathcal{M}_1|^2 \} = \frac{e^4}{((p_1+p_2)^2)^2} \mathrm{Tr}\left[\not p_4 (\underbrace{\not p_1}_{0} + \not p_2) \not p_1 (\underbrace{\not p_1}_{0} + \not p_2)\right]$$

But $\not p_1 \not p_1 = p_1^2 = 0$

$$= \frac{e^4}{((p_1+p_2)^2)^2} \mathrm{Tr}\left[\not p_4 \not p_2 \not p_1 \not p_2\right]$$

$$= \frac{e^4}{((p_1+p_2)^2)^2} 4\left[2 p_4\cdot p_2\, p_1\cdot p_2 - p_4\cdot p_1\, \underbrace{p_2^2}_{0}\right]$$

$$= \frac{2e^4\, -(p_2-p_4)^2}{(p_1+p_2)^2} \Rightarrow \boxed{\frac{1}{4}\sum_{\text{spins}} \langle |\mathcal{M}_1|^2 \rangle = 2e^4\left(\frac{-u}{s}\right)}$$

$u = (p_1-p_3)^2 = (p_2-p_4)^2$, $s = (p_1+p_2)^2$

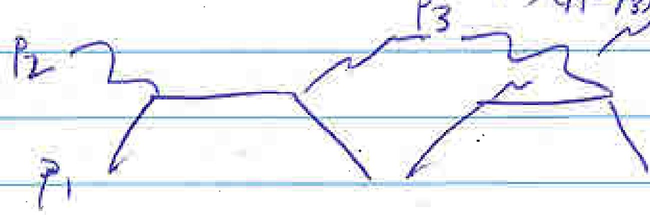
And $\frac{1}{4}\sum_{\text{spins}} \{ |\mathcal{M}_2|^2 \} = \left[\begin{array}{l}\text{same with } p_2 \leftrightarrow -p_3 \\ \text{i.e. } s \leftrightarrow u\end{array}\right]$

$$= 2e^4\left(\frac{s}{-u}\right)$$

(5) (CONT.)

Finally, the cross term $\frac{1}{4}\sum_{\text{spins}} \langle M_2^* M_1 \rangle$ has the same form, but $\epsilon_2 \leftrightarrow \epsilon_3$, etc. in only one of the factors

$$\frac{1}{4}\sum_{\text{spins}} \langle M_2^* M_1 \rangle = \frac{e^4/4}{(p_1+p_2)^2 (p_1-p_3)^2} \sum_{\epsilon_2,\epsilon_3} \mathrm{Tr}\left[\not p_4 \not\epsilon^*(p_3)(\not p_1+\not p_2)\not\epsilon(p_2)\not p_1 \not\epsilon(p_3)(\not p_1-\not p_3)\not\epsilon^*(p_2)\right]$$



$$= \frac{e^4/4}{(p_1+p_2)^2(p_1-p_3)^2} \mathrm{Tr}\left[\not p_4 \gamma^{\mu_3}(\not p_1+\not p_2)\underbrace{\gamma^{\mu_2}\not p_1}_{2p_1^{\mu_2} - \not p_1 \gamma^{\mu_2}} \gamma_{\mu_3}(\not p_1-\not p_3)\gamma_{\mu_2}\right]$$

The trace is, after anticommuting γ^{μ_2} past $\not p_1$,

$$2\,\mathrm{Tr}\left[\not p_4 \gamma^{\mu_3}(\not p_1+\not p_2)\gamma_{\mu_3}(\not p_1-\not p_3)\not p_1\right]$$

$$- \mathrm{Tr}\left[\not p_4 \gamma^{\mu_3}(\not p_1+\not p_2)\not p_1 \gamma^{\mu_2}\gamma_{\mu_3}(\not p_1-\not p_3)\gamma_{\mu_2}\right]$$

use $\gamma^a \not a \not b \gamma_a = 4a\cdot b$

$$= -4\,\mathrm{Tr}\left[\not p_4 (\underbrace{\not p_1+\not p_2}_{\not p_3+\not p_4})(\cancel{\not p_1}-\not p_3)\not p_1\right] \quad p_1^2=0$$

$$-4\,\mathrm{Tr}\left[\not p_4 (\not p_1-\not p_3)(\underbrace{\cancel{\not p_1}}_{0}+\not p_2)\not p_1\right]$$

$$= +4\,\mathrm{Tr}\left[\not p_4(\not p_3+\not p_4)\not p_3 \not p_4\right] \to 0$$

$$+4\,\mathrm{Tr}\left[\not p_4(\not p_2-\not p_4)\not p_2 \not p_1\right] \to 0$$

$\therefore \frac{1}{4}\sum_{\text{spins}} \langle M_2^* M_1 \rangle = 0$ And similarly, $\frac{1}{4}\sum_{\text{spins}} \langle M_1^* M_2 \rangle = 0$

(§3.10)

⑤ (CONT.) Adding up the nonvanishing terms,

$$\boxed{\langle|\mathcal{M}|^2\rangle_{\gamma e\to\gamma e} = 2e^4\left[\frac{-u}{s} + \frac{s}{-u}\right]}$$

$$\boxed{u = -\tfrac{1}{2}s(1-\cos\theta)}$$

$$\boxed{t = -\tfrac{1}{2}s(1+\cos\theta)}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2}\frac{\langle|\mathcal{M}|^2\rangle}{s}\frac{|p_f|}{|p_i|}$$

$\uparrow$ $\nearrow$ $\cdot$ $\downarrow 1$

in the high energy, or $m_e\to 0$, limit.

$$\frac{d\sigma}{d(\cos\theta)} = 2\pi\frac{d\sigma}{d\Omega} = \frac{2\pi}{(8\pi)^2 s}\,2e^4\left(-\frac{u}{s} + \frac{s}{-u}\right)$$

$(4\pi\alpha)^2$

$$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[\frac{1-\cos\theta}{2} + \frac{2}{1-\cos\theta}\right]}$$

The total cross section is <u>not finite</u> in this approximation, because $\int_{-1}^{1}\frac{d\cos\theta}{1-\cos\theta}$ has a log divergence at the upper endpoint.
(If one keeps $m_e \neq 0$, then the total cross section becomes finite.)

Use Crossing symmetry

$$\boxed{\begin{array}{l} p_2 \leftrightarrow -p_4 \\ s \leftrightarrow t \\ u \leftrightarrow u \end{array}}$$

must be > 0

for $e^+e^-\to\gamma\gamma$

$$\Rightarrow \langle|\mathcal{M}|^2\rangle_{\substack{e^+e^-\to\gamma\gamma \\ \gamma\gamma\to e^+e^-}} = 2e^4\left[+\frac{u}{t} + \frac{t}{u}\right]$$

$\gamma\gamma\to e^+e^-$ is <u>same</u> because $p_2\leftrightarrow p_3$ leaves $p_1\leftrightarrow p_4$ s, t, u invariant.

$$\Rightarrow \boxed{\frac{d\sigma}{d\cos\theta} = \left(\frac{1}{2}\right)\frac{\pi\alpha^2}{s}\left[\frac{1-\cos\theta}{1+\cos\theta} + \frac{1+\cos\theta}{1-\cos\theta}\right]}$$ for $e^+e^-\to\gamma\gamma$ and $\gamma\gamma\to e^+e^-$

"1/2" is only for $\gamma\gamma$ final state — due to Bose symmetry