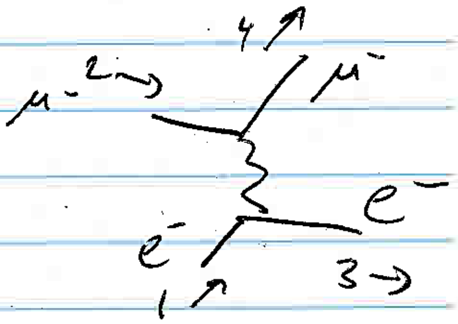
Lecture 10 ⟺ still more QED

Crossing Symmetry again:

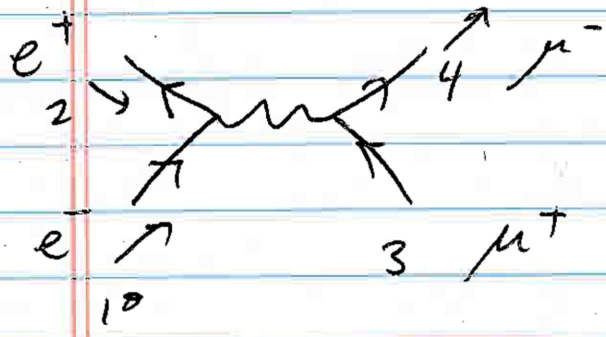
$e^- \mu^- \to e^- \mu^-$



$$\langle |\mathcal{M}|^2 \rangle_{e\mu\to e\mu} = 2e^4\left[\left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2\right]$$

$$s = (p_1+p_2)^2$$
$$t = (p_1 - p_4)^2$$
$$u = (p_1 - p_3)^2$$

$e^+e^- \to \mu^+\mu^-$



Related by crossing

$$\left\{\begin{array}{l} p_1 \to p_1 \\ p_2 \to -p_3 \\ p_3 \to -p_2 \\ p_4 \to p_4 \end{array}\right\}$$

$$s = (p_1+p_2)^2$$
$$t = (p_1-p_4)^2$$
$$u = (p_1-p_3)^2$$

$$\boxed{\begin{array}{l} \text{or} \\ t \leftrightarrow t \\ s \leftrightarrow u \end{array}}$$

$$\boxed{\begin{array}{l} t = -\frac{s}{2}(1+\cos\theta) \\ u = -\frac{s}{2}(1-\cos\theta) \end{array}}$$

$$\Rightarrow \boxed{\begin{aligned} \langle |\mathcal{M}|^2 \rangle_{e^+e^-\to\mu^+\mu^-} &= 2e^4\left[\left(\frac{u}{s}\right)^2 + \left(\frac{t}{s}\right)^2\right] \\ &= \frac{e^4}{2}\left[(1-\cos\theta)^2 + (1+\cos\theta)^2\right] \\ &= e^4\left[1+\cos^2\theta\right] \end{aligned}}$$

$$\boxed{\frac{4\pi}{3}\frac{\alpha^2}{s}}$$

$$\frac{d\sigma}{\ } = \frac{e^4(1+\cos^2\theta)}{\ } = \frac{\alpha^2}{\ }(1+\cos^2\theta) \qquad \sigma(e^+e^- \to \mu^+\mu^-) = 2\pi\,\alpha^2 \int_{\ }^{1} d(\cos\theta)\,(1+\cos^2\theta)$$

External Photons & QED Ward Identity

Electromagnetic current is <u>conserved</u>:

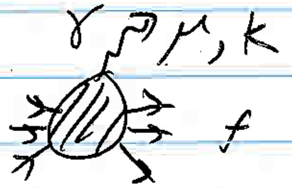
$$\partial_t \rho = -\vec{\nabla}\cdot\vec{J}$$

or $\boxed{\partial_\mu J^\mu = 0}$

Interaction term $\propto \int d^4x \; e J^\mu A_\mu$

means that the matrix element for making a photon with pol. ϵ^μ

is $\mathcal{M}^\mu(k) \sim \int d^4x \, e^{ik\cdot x} \langle j | J^\mu(x) | i \rangle$



Then

$$k_\mu \mathcal{M}^\mu(k) \sim \int d^4x \, e^{ik\cdot x} \langle j | \partial_\mu J^\mu(x) | i \rangle$$
$$= 0$$

$\boxed{k_\mu \mathcal{M}^\mu(k) = 0 \quad \text{QED Ward Identity}}$

(Not true in QCD because gluons also "charged",)

Suppose $k^\mu = (\overset{0}{k}, \overset{1}{0}, \overset{2}{0}, \overset{3}{k})$

Then $k\mathcal{M}^0 - k\mathcal{M}^3 = 0$

. Take ϵ^μ to be in 1-2 plane

$$\Rightarrow \sum_{1,2} \epsilon_\mu^{*(i)} \epsilon_\nu^{(i)} \mathcal{M}^\mu(k) \mathcal{M}^{\nu *}(k) = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2$$

$$= -g_{\mu\nu} \mathcal{M}^\mu(k) \mathcal{M}^{\nu *}(k) = -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2$$

QED Ward Identity (CONT.)

- Thus we do not need to literally restrict the sum over polarizations to some two transverse pol's (x, y) or (+, −)

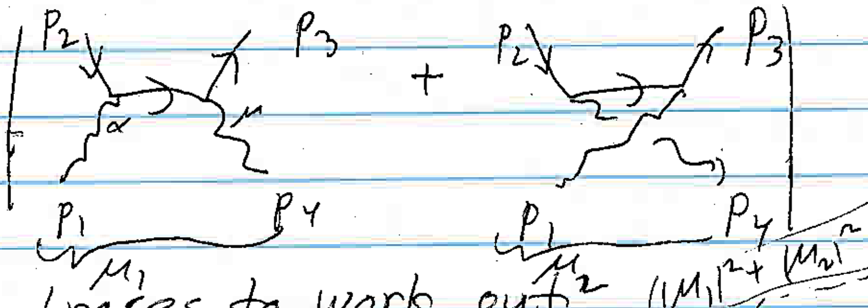
- We can include the "longitudinal polarizations" ~ t, z

$$\Rightarrow \boxed{\sum_{1,2} \epsilon_\mu^{*(i)} \epsilon_\nu^{(i)} \longrightarrow -\eta_{\mu\nu} \quad \text{is OK}}$$

- simplifies the calculation a lot!

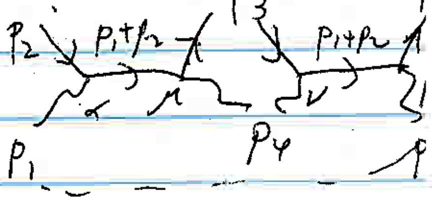
[Note problem 5 changed!]

Problem 5 has 2 Feynman graphs:

$|\mathcal{M}|^2 =$ 

$$|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2(\mathcal{M}_1^* \mathcal{M}_2 + \mathcal{M}_2^* \mathcal{M}_1)$$

⇒ 2·2 = 4 traces to work out.

The first one is $|\mathcal{M}_1|^2 =$  $=$ 

$$= \frac{\# e^4}{((p_1+p_2)^2)^2} \mathrm{Tr}\left[\not p_3 \gamma^\mu (\not p_1+\not p_2) \gamma^\alpha \not p_2 \gamma^\beta (\not p_1+\not p_2) \gamma^\nu\right]$$

$$\left(-\eta_{\alpha\beta} + \frac{p_{1\alpha} r_{1\beta} + r_{1\alpha} p_{1\beta}}{p_1 \cdot r_1}\right)\left(-\eta_{\mu\nu} + \frac{p_4^\mu r_4^\nu + r_4^\mu p_4^\nu}{p_4 \cdot r_4}\right)$$

(9.4)

from η term

$$\Rightarrow |\mathcal{M}_1|^2 = \frac{1}{4}\frac{(4\pi\alpha)^2}{s^2}\,\mathrm{Tr}\Big[\Big(2\not{p}_3 + \frac{1}{p_4\cdot r_4}\overbrace{(\not{r}_4\not{p}_3\not{r}_4 + \not{r}_4\not{p}_3\not{p}_4)}^{\mu}\Big)$$

$$\cdot(\not{p}_1+\not{p}_2)\Big(2\not{p}_2 + \frac{1}{p_1\cdot r_1}\underbrace{(\not{r}_1\not{p}_2\not{r}_1 + \not{r}_1\not{p}_2\not{p}_1)}_{\text{generate many terms}}\Big)(\not{p}_1+\not{p}_2)\Big]$$

– but all cancel against terms from
$|\mathcal{M}_2|^2$, $\mathcal{M}_1^*\mathcal{M}_2$, $\mathcal{M}_2^*\mathcal{M}_1$
when the dust settles!

- You can keep only the η terms for problem 5.

(11.3) → Ph 152A notes

9.5

$p_1 \cdot p_2 = p_3 \cdot p_4 = \frac{s}{2}$

$p_1 \cdot p_4 = p_2 \cdot p_3 = -t/2$

$(p_1 - p_3)^2 = u$

$\boxed{\begin{matrix} m, M \to 0 \\ \text{HE limit} \end{matrix}}$

$$\langle |M|^2 \rangle = 2e^4\left[\left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2 s} 2e^4\left[\left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2\right]$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\frac{s^2+t^2}{u^2}} \leftarrow \boxed{\begin{matrix} t = -s/2(1+\cos\theta) \\ u = -s/2(1-\cos\theta) \end{matrix}}$$

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(2) 2nd approach: With longitudinal polarization, and/or high energy limit, compute helicity amplitudes

$\boxed{\sigma \cdot \hat{p} = \pm 1}$ Helicity states for $m=0$

$$u^{\pm} = N \begin{pmatrix} u_A \\ \pm u_A \end{pmatrix}$$

$$\boxed{\begin{matrix} u_B = \frac{p \cdot \sigma}{E} u_A \\ = \pm u_A \end{matrix}}$$

$$N = \sqrt{\frac{1}{2|p|(|p| \pm p_z)}}$$

$$u_A = \begin{pmatrix} p_z \pm |\vec{p}| \\ p_x + i p_y \end{pmatrix}$$

$$\sigma \cdot \vec{p}\, u_A = \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix}\begin{pmatrix} p_z \pm |\vec{p}| \\ p_x + i p_y \end{pmatrix}$$

$$= \begin{pmatrix} (\overbrace{p_z^2 + p_x^2 + p_y^2}^{|p|^2}) \pm p_z|\vec{p}| \\ \pm |\vec{p}|(p_x + i p_y) \end{pmatrix} = \pm |\vec{p}|\, u_A$$

- Also chirality (γ^5) eigenstates (for $m=0$)

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \boxed{\gamma^5 u_{\pm} = \pm u_{\pm}}$$

helicity = chirality for $m=0$, $\sigma \cdot \hat{p} = +1 \Leftrightarrow \frac{1}{2}(1+\gamma_5)u = u$

QED interaction preserves chirality

$$\{\gamma^\mu, \gamma^5\} = 0$$

$$\Rightarrow \quad \bar{u}_{3(-)} \gamma^\mu u_{1(+)} = \bar{u}_{3(-)} \gamma^\mu \tfrac{1}{2}(1+\gamma_5) u_{1(+)}$$

$$= u^{+}_{3(-)} \tfrac{1}{2}(1-\gamma_5^{+}) \gamma^0 \gamma^\mu \tfrac{1}{2}(1+\gamma_5) u_{1(+)}$$

$$= \bar{u}_{3(-)} \overrightarrow{\tfrac{1}{2}(1+\gamma_5) \gamma^\mu} \tfrac{1}{2}(1+\gamma_5) u_{1(+)}$$

$$= \bar{u}_{3(-)} \gamma^\mu \tfrac{1}{2}(1-\gamma_5) \tfrac{1}{2}(1+\gamma_5) u_{1(+)} = 0$$

∴ Only $\bar{u}_{3(+)} \gamma^\mu u_{1(+)} \bar{u}_{4(+)} \gamma_\mu u_{2(+)}$

and $\bar{u}_{3(+)} \gamma^\mu u_{1(+)} \bar{u}_{4(-)} \gamma_\mu u_{2(-)}$

⊕ amp's with all spins flipped together are nonzero.

these are related by parity:

P : $\hat{p} \to -\hat{p}$

$\vec{s} \to +\vec{s}$

$\Rightarrow \vec{s}\cdot\hat{p} \to -\vec{s}\cdot\hat{p}$

- There are some nifty methods ("helicity techniques") to compute helicity amplitudes directly.
- However, we don't have time to develop this formalism completely.
- Instead we'll compute $|\mathcal{M}|^2$ for helicity states, then take the square root.

Use $\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4i\,\varepsilon^{\mu\nu\lambda\sigma}$ (9.7)

$\varepsilon^{\mu\nu\lambda\sigma} = \begin{cases} -1 & \mu\nu\lambda\sigma \text{ an } \underline{\text{even}} \text{ perm. of } 0123 \\ +1 & \quad\text{"}\quad \underline{\text{odd}} \quad\text{"}\quad\text{"}\quad\text{"} \\ 0 & \text{otherwise} \end{cases}$

- For $|\mathcal{M}(e_1^+\mu_2^\pm \to e_3^+\mu_4^\pm)|^2 \equiv |\mathcal{M}|^2_\pm$

we need

$$\mathrm{Tr}\left[\gamma^\mu \underbrace{\tfrac{1}{2}(1+\gamma_5)}_{\text{from } \frac{1}{2}(1+\gamma_5)u_1\bar{u}_1} \not{p}_1 \gamma^\nu \cancel{\tfrac{1}{2}(1+\gamma_5)} \not{p}_3\right]$$

(not needed because of helicity conservation)

$$* \ \mathrm{Tr}\left[\gamma_\mu \tfrac{1}{2}(1\pm\gamma_5)\not{p}_2\gamma_\nu \cancel{\tfrac{1}{2}(1\pm\gamma_5)}\not{p}_4\right]$$

$$= \tfrac{1}{4}\mathrm{Tr}[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3]\,\mathrm{Tr}[\gamma_\mu\not{p}_2\gamma_\nu\not{p}_4]$$

$$\pm\tfrac{1}{4}\mathrm{Tr}[\gamma^\mu\gamma_5\not{p}_1\gamma^\nu\not{p}_3]\,\mathrm{Tr}[\gamma_\mu\gamma_5\not{p}_2\gamma_\nu\not{p}_4]$$

odd terms in γ_5 $\Rightarrow$ linear in $\varepsilon_{\mu\nu\sigma\rho}$. But antisym $\Rightarrow$ $\varepsilon_{\mu\nu\sigma\rho}p_1^\mu p_2^\nu p_3^\sigma p_i^\rho = 0$ $i=1,3,3$ $p_4 = -p_3+p_1+p_2$

$$= \tfrac{1}{4}(\text{unpolarized trace})$$

$$\pm\tfrac{1}{4}(4i)^2\,\varepsilon^{\mu\nu}{}_{\alpha\beta}\,p_1^\alpha p_3^\beta\,\varepsilon_{\mu\nu\gamma\delta}\,p_2^\gamma p_4^\delta$$

Use identity $\varepsilon^{\mu\nu}{}_{\alpha\beta}\,\varepsilon_{\mu\nu\gamma\delta} = -2(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})$

$$= \tfrac{1}{4}\cdot 32\,(p_1\cdot p_2\,p_3\cdot p_4 + p_1\cdot p_4\,p_2\cdot p_3)$$

$$\pm(-4)\cdot(-2)\,(p_1\cdot p_2\,p_3\cdot p_4 - p_1\cdot p_4\,p_2\cdot p_3)$$

(no $\frac{1}{4}$ for spin avg. here)

$$\Rightarrow |\mathcal{M}|^2_+ = \frac{e^4\cdot 16(p_1\cdot p_2)^2}{((p_1-p_3)^2)^2} = 4e^4\left(\frac{s}{u}\right)^2$$

$$|\mathcal{M}|^2_- = \frac{e^4}{((p_1-p_3)^2)^2}\cdot 16(p_1\cdot p_4)^2 = 4e^4\left(\frac{t}{u}\right)^2$$

Take square root

$$\Rightarrow \mathcal{M}(e_1^+\mu_2^+ \to e_3^+\mu_4^+) = 2e^2\,\frac{s}{u}\,(\text{phase})$$

$$\mathcal{M}(e_1^+\mu_2^- \to e_3^+\mu_4^-) = 2e^2\,\frac{t}{u}\,(\text{phase})$$

doesn't usually matter, if no $\perp$ pol.

- We can understand the 2nd formula by a helicity argument:

$$t = -\frac{s}{2}(1+\cos\theta)$$
$$u = -\frac{s}{2}(1-\cos\theta)$$

- $u \to 0$ for forward scattering $\theta \to 0$

  $\frac{1}{u}$ divergence due to photon propagator

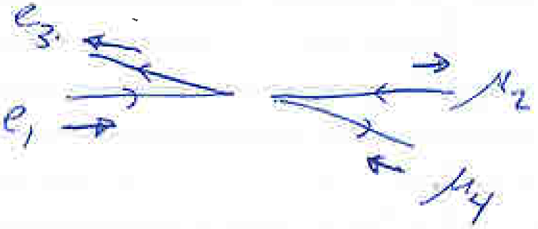
  no helicity suppression

  $J_z^{(i)} = +1 \longrightarrow J_z^{(f)} = +1$

  (similarly for other helicity configuration)

- $t \to 0$ for backward scattering $(\theta = \pi)$

  Now $J_z^{(i)} = +1 \to J_z^{(f)} = -1$



$\Rightarrow$ amplitude must vanish as $t \to 0$

And for $m=0$, it is a dimensionless function of t/u, with a $1/u$ pole

$\Rightarrow$ must be proportional to t/u (!) ✓

- Recover unpolarized $\langle|\mathcal{M}|^2\rangle$ by summing over helicities:

- There are $2^4 = 16$ helicity configurations, but only $2^2 = 4$ are nonzero for $m=0$. And 2 of these are related to the others by P.

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = \overset{\swarrow\text{(initial spin avg.)}}{\left(\frac{1}{2}\right)^2} \cdot \underset{\uparrow\text{(parity P)}}{2} \cdot \left[|\mathcal{M}|_+^2 + |\mathcal{M}|_-^2\right]$$

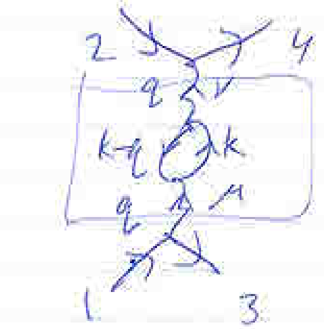
$$= 2e^4\left[\left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2\right]$$

[redacted]

9.9

QED divergences & renormalization

Consider 1-loop correction to $e\mu$ scattering, in particular



"vacuum polarization" due to virtual e^+e^- pairs

Neglecting m_e,

$$\mathcal{M} = +\frac{ie^2}{q^4}\bar{u}_3\gamma_\mu u_1 \underbrace{\left[e^2\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}[\gamma^\mu \not{k}\gamma^\nu(\not{k}-\not{q})]}{k^2(k-q)^2}\right]}_{\equiv I^{\mu\nu}(q)}\bar{u}_4\gamma_\nu u_2$$

- Note that $q_\mu I^{\mu\nu}(q) = e^2\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}[\overset{\not{k}-(\not{k}-\not{q})}{\not{q}}\not{k}\gamma^\nu(\not{k}-\not{q})]}{k^2(k-q)^2}$

$$= 4e^2\int\frac{d^4k}{(2\pi)^4}\left[\frac{(k-q)^\nu}{(k-q)^2}-\frac{k^\nu}{k^2}\right] = 0$$

at least, it should be if regulator is OK

Similarly,

$$\boxed{q_\nu I^{\mu\nu}(q) = 0}$$

$$\therefore\ I^{\mu\nu}(q) = (\underbrace{g^{\mu\nu}q^2}_{} - q^\mu q^\nu) I(q^2)$$

only this term contributes to $\mathcal{M}$

this term is easier to work out

Feynman's trick to combine denominators:

$$\frac{1}{k^2}\frac{1}{(k-q)^2} = \frac{1}{q^2-2q\cdot k}\left[\frac{1}{k^2}-\frac{1}{(k-q)^2}\right] = \int_0^1 dx\frac{1}{[k^2-2xk\cdot q+xq^2]^2}$$

$$= \int_0^1 dx\frac{1}{(\ell^2+x(1-x)q^2)^2} \quad \text{where} \begin{cases}\ell = k - xq\\ k = \ell + xq\end{cases}$$

So,

$$I^{\mu\nu} = e^2 \int \frac{d^4\ell}{\boxed{(2\pi)^4}} \int_0^1 dx \, \frac{\mathrm{Tr}\left[\gamma^\mu(\ell\!\!/ + x q\!\!/)\gamma^\nu(\ell\!\!/ - (1-x)q\!\!/)\right]}{\left[\ell^2 + x(1-x)q^2\right]^2}$$

$$\frac{1}{4}\mathrm{Tr}[\ldots] = (\ell^\mu + x q^\mu)(\ell^\nu - (1-x)q^\nu) + \mu \leftrightarrow \nu$$
$$- g^{\mu\nu}(\ell^2 + \boxed{(2x-1)}\,\ell\cdot\ell - x(1-x)q^2)$$

$$= 2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}x(1-x)q^2$$
$$+ \left[\text{linear in } \cancel{\ell^\nu, \ell^\mu}\right]$$

integrate to ϕ due to $\ell^\mu \leftrightarrow -\ell^\mu$ symmetry

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu\ell^\nu}{[\ell^2 + x(1-x)q^2]^2} = g^{\mu\nu}\cdot f(\ell^2)$$

We just want the $q^\mu q^\nu$ terms

$$\Rightarrow I(q^2) = e^2\cdot 2\cdot 4 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \, \frac{x(1-x)}{(\ell^2 + x(1-x)q^2)^2}$$

"UV cutoff"

$$\text{Now} \int^{\boxed{M}} \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 + x(1-x)q^2)^2} \approx \frac{\int d\Omega_4}{(2\pi)^4} \int_0^M \frac{d\ell\,\ell^3}{\ell^4}$$

(vol. of 4-sphere $= 2\pi^2$)

$$\approx \frac{2\pi^2}{(2\pi)^4} \cdot \frac{1}{2} \int_{\cancel{0}}^{M^2} \frac{d\ell^2}{\ell^2} = \frac{1}{16\pi^2}(\ln M^2 + \text{finite})$$

$$\text{So, } I(q^2) = \frac{e^2}{2\pi^2} \ln M^2 \underbrace{\int_0^1 dx \, x(1-x)}_{\frac{1}{2} - \frac{1}{3} = \frac{1}{6}} + \text{finite}$$

$$I(q^2) = \frac{e^2}{12\pi^2} \ln M^2 + \text{finite}$$

So, finally, ~~letting our~~ calling our original $e \to e_0(M)$ "the bare charge"

$$\boxed{\mathcal{M} = -e_0^2\, \bar{u}_3 \gamma^\mu u_1 \frac{g_{\mu\nu}}{q^2} \left\{ 1 - \frac{e_0^2}{12\pi^2} \left[\ln\left(\frac{M^2}{m_e^2}\right) - f\left(\frac{-q^2}{m_e^2}\right) \right] \right\} \times \bar{u}_4 \gamma^\mu u_1}$$

- To remove the divergence, we define the renormalized charge

$$\boxed{e_R = e_0(M) \sqrt{1 - \frac{e_0^2(M)}{12\pi^2} \ln\left(\frac{M^2}{m_e^2}\right)}}$$ such that e_R is finite as $M \to \infty$

Then

$$\boxed{\mathcal{M}_{(ren)} = -e_R^2\, \bar{u}_3 \gamma^\mu u_1 \frac{g_{\mu\nu}}{q^2} \left\{ 1 + \frac{e_R^2}{12\pi^2} f\left(-\frac{q^2}{m_e^2}\right) \right\} \bar{u}_4 \gamma^\nu u_2}$$

$\mathcal{M}_{(ren)}$ is finite as $M \to \infty$

For $q^2 >> m_e^2$, $f\left(-\frac{q^2}{m_e^2}\right) \simeq \ln\left(\frac{q^2}{m_e^2}\right)$

(to cancel m_e dependence in $\left[\ln\left(\frac{M^2}{m_e^2}\right) - f\left(-\frac{q^2}{m_e^2}\right)\right]$)

- We can ~~remove~~ absorb this dependence ~~from~~ in $\mathcal{M}$ too, in terms of a running coupling:

$$e_R(q^2) = e_R(0) \sqrt{1 + \frac{e_R(0)^2}{12\pi^2} f\left(-\frac{q^2}{m_e^2}\right)}$$

or

$$\boxed{\begin{aligned} \alpha(q^2) &= \alpha(0) \left[1 + \frac{\alpha(0)}{3\pi} f\left(-\frac{q^2}{m_e^2}\right) \right] \\ &\simeq \alpha(0) \left[1 + \frac{\alpha(0)}{3\pi} \ln\left(\frac{q^2}{m_e^2}\right) \right] \end{aligned}}$$ $|q| >> m_e$

(α increases logarithmically as q increases)