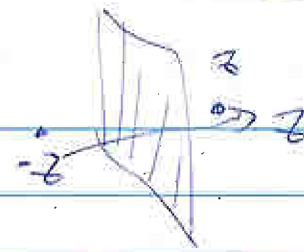
Lecture 7 QED (CONT.)



- Parity: Physics looks same "in a mirror" $t \to t,\ x \to x,\ y \to y,\ z \to -z$
- Or equivalently, under total spatial inversion

assuming rotational invariance

parity

$$\boxed{P:\ t \to t,\ x \to -x,\ y \to -y,\ z \to -z}$$

Rotation by $180^\circ = \pi$ in xy-plane takes $x \to -x,\ y \to -y,\ z \to z$

- Parity invariance is an observational fact for QED, QCD (but not for weak interactions)
- How does EM field transform?

capacitor

3d vector $\to$ $\boxed{\vec{E}(t,\vec{x}) \to -\vec{E}(t,\vec{x})}$

$E_i = -\partial_i \phi - \partial_t \vec{A} \Rightarrow \phi(t,\vec{x}) \to +\phi(t,-\vec{x})$

$\partial_i \to -\partial_i$ $\quad$ $\vec{A} \to -\vec{A}$

current carrying wires

3d pseudo vector $\to$ $\vec{B}(t,\vec{x}) \to +\vec{B}(t,\vec{x})$

$B_i = \epsilon_{ijk}\, \partial_j A_k \Rightarrow$ $\cancel{\vec{A}(t,-\vec{x})}$

$\partial_j \to -\partial_j$ $\quad$ $\vec{A}(t,\vec{x}) \to -\vec{A}(t,\vec{x})$

(3d vector)

$\therefore A_\mu = (\phi, \vec{A})$ is a true 4d vector (not a pseudovector)

$$\boxed{A_\mu = (\phi, \vec{A}) \to (+\phi, -\vec{A}) \text{ under } P}$$

- How does P act on ψ?

Claim: $\boxed{\psi \to \psi' = \gamma^0 \psi}$

($\gamma^{0\dagger} = \gamma^0$)

Then $\bar\psi = \psi^\dagger \gamma^0 \to (\psi\gamma^0)^\dagger \gamma^0 = \psi^\dagger (\gamma^0)^2$

$$\boxed{\bar\psi \to \bar\psi \gamma^0} \qquad = \bar\psi \gamma^0$$

And

$$\boxed{\bar\psi \gamma^\mu \psi \to \bar\psi \underbrace{\gamma^0 \gamma^\mu \gamma^0}_{\begin{cases} +\gamma^0 & \mu = 0 \\ -\gamma^i & \mu = i \end{cases}} \psi}$$

- Thus $\bar\psi\gamma^\mu\psi$ transforms just like A^μ under P, i.e. like a true (polarized) vector.

In particular, the EM interaction,

$$\boxed{e\bar\psi\gamma^\mu\psi A_\mu \to +\bar\psi\gamma^\mu\psi A_\mu \text{ is invariant under } P.}$$

What about other "fermion bilinears"?

For example, $\bar\psi\psi \to \bar\psi \underbrace{\gamma^0\gamma^0}_{(\gamma^0)^2 = 1} \psi = \bar\psi\psi$ is a true scalar.

- To build some other bilinears, we need to introduce $\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ (in Dirac rep.)

- Important because it anticommutes with all γ^μ.

$$\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right\} = \begin{pmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{pmatrix} + \begin{pmatrix} +\sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = 0, \text{ etc.} \Rightarrow \boxed{\{\gamma^\mu, \gamma^5\} = 0}$$

- This also means that it commutes with Lorentz ~~boosts~~ transformations generated by $\sigma^{\mu\nu} = i\gamma^\mu\gamma^\nu$:

$$[\gamma^\mu\gamma^\nu, \gamma^5] = \gamma^\mu\gamma^\nu\gamma^5 - \gamma^5\gamma^\mu\gamma^\nu$$
$$= -\gamma^\mu\gamma^5\gamma^\nu - \gamma^5\gamma^\mu\gamma^\nu$$
$$= +\gamma^5\gamma^\mu\gamma^\nu - \gamma^5\gamma^\mu\gamma^\nu = 0$$

$$\Rightarrow \boxed{[\sigma^{\mu\nu}, \gamma^5] = 0}$$

$\Rightarrow$ $\bar\Psi\gamma^5\Psi$ is also a Lorentz scalar, but under parity,

$$P: \bar\Psi\gamma^5\Psi \to \bar\Psi\gamma^0\gamma^5\gamma^0\Psi = -\bar\Psi\gamma^5\Psi$$

$\Rightarrow$ $\boxed{\bar\Psi\gamma^5\Psi \text{ is a pseudo-scalar}}$

- Similarly, $\boxed{\bar\Psi\gamma^\mu\gamma^5\Psi \text{ is a pseudovector}}$ (or axial vector)

$$\bar\Psi\gamma^0\gamma^5\Psi \to -\bar\Psi\gamma^0\gamma^5\Psi$$
$$\bar\Psi\gamma^i\gamma^5\Psi \to +\bar\Psi\gamma^i\gamma^5\Psi$$

- If an interaction couples a vector field to both $\bar\Psi\gamma^\mu\Psi$ and $\bar\Psi\gamma^\mu\gamma^5\Psi$, it must violate parity.
- This is what happens in the weak interactions.
- For example,

$$(g_V\bar\Psi\gamma^\mu\Psi + g_A\bar\Psi\gamma^\mu\gamma^5\Psi)Z_\mu$$

$$\xrightarrow{P} (g_V\bar\Psi\gamma^\mu\Psi - g_A\bar\Psi\gamma^\mu\gamma^5\Psi)Z_\mu$$

$g_A \neq 0 \Rightarrow$ parity violation

axial vector coupling

Note that $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$\Rightarrow$ $\sigma^{\mu\nu}$ and $\sigma^{\mu\nu}\gamma^5$ are not independent

e.g. $\sigma^{01} = \pm\sigma^{23}\gamma^5$

Also $\sigma^{\mu\nu}$ is antisym. $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$.

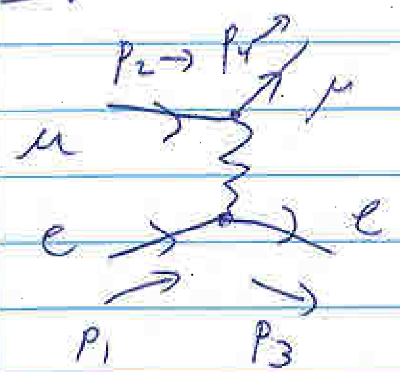
$\Rightarrow$ $\binom{4}{2} = \frac{4\cdot 3}{1\cdot 2} = 6$ components.

$\underset{(4)}{\psi_i^*}\ \underset{(4)}{\psi_j}$ has $4\times 4 = 16$ independent components.

The combinations with nice Lorentz transf. properties are:

$\bar\psi\psi$	scalar	1	component
$\bar\psi\gamma^\mu\psi$	vector	4	"
$\bar\psi\sigma^{\mu\nu}\psi$	antisym. tensor	6	"
$\bar\psi\gamma^\mu\gamma^5\psi$	pseudo vector	4	"
$\bar\psi\gamma^5\psi$	pseudo scalar	1	"

Simplest QED Feynman diagram



$$\Rightarrow (2\pi)^4 \int d^4q \, \left[\bar{u}^{(s_3)}(p_3)(ie\gamma^\mu)u^{(s_1)}(p_1)\right]$$

$$\times \frac{-ig_{\mu\nu}}{q^2}\left[\bar{u}^{(s_4)}(p_4)(ie\gamma^\nu)u^{(s_2)}(p_2)\right]$$

$$\times \delta^{(4)}(p_1-p_3-q)\,\delta^{(4)}(p_2+q-p_4)$$

$$\Rightarrow \boxed{\mathcal{M} = \frac{i\cdot i\cancel{(-i)}}{\cancel{(-i)}}\frac{e^2}{(p_1-p_3)^2}\left[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)\right] \times \left[\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)\right]}$$

What ~~to~~ do next?

- One could plug in formulas for the spinor $u^{(s_i)}(p_i)$, $\bar{u}^{(s_j)}(p_j)$, and evaluate the matrix (γ^μ) "sandwiches" explicitly for each choice of $s_i = \pm\frac{1}{2}$ along some axis

$\rightsquigarrow$ "(Helicity) amplitude approach."

OR

- For unpolarized cross sections, usually don't evaluate $\mathcal{M}$ for each $\{s_i\}$, but go directly to $\frac{1}{\prod_{\text{initial}}(2s_i+1)}\sum_{\text{spins}}|\mathcal{M}|^2 = \langle|\mathcal{M}|^2\rangle$.

And we use "Casimir's trick"

Casimir's trick

From the fermion line

in ~~one~~ every Feynman diagram for $\mathcal{M}$,
we get a term: $\bar{u}(a)\,\Gamma_1\,u(b)$
(a string of γ matrices)

And in $\mathcal{M}^*$, there are terms like

$$[\bar{u}(a)\,\Gamma_2\,u(b)]^*$$

(might come from another Feynman diagram, so $\Gamma_2 \neq \Gamma_1$)

$$\therefore \langle|\mathcal{M}|^2\rangle \supset \sum_{s_a, s_b} [\bar{u}(a)\,\Gamma_1\,u(b)]\,\underbrace{[\bar{u}(a)\,\Gamma_2\,u(b)]^*}$$

$$u^\dagger(b)\,\Gamma_2^\dagger\,\gamma^{0\dagger}\,u(a)$$

$$= \bar{u}(b)\,\gamma^0\,\Gamma_2^\dagger\,\gamma^0\,u(a)$$

$$= \bar{u}(b)\,\bar{\Gamma}_2\,u(a)$$

where $\boxed{\bar{\Gamma}_2 \equiv \gamma^0\,\Gamma_2^\dagger\,\gamma^0}$

$\Rightarrow$ need

$$\sum_{s_a}\sum_{s_b} \bar{u}(a)\,\Gamma_1\,\underbrace{u(b)\,\bar{u}(b)}\,\bar{\Gamma}_2\,u(a)$$

use completeness relation

$$\boxed{\sum_{s=1,2} u^{(s)}\,\bar{u}^{(s)} = \not{p} + m} \leftarrow \text{"positive energy projector"}$$

You can work this out explicitly, or note that it should be $\propto \alpha\not{p} + \beta m$

use $(\not{p} - m)\sum u\bar{u} = 0 \Rightarrow b = -\alpha$. Normalize at $p^i = 0$

(for e^+)

For the $v^{(s)}$ spinors, need

$$\boxed{\sum_{s=1,2} v^{(s)} \bar{v}^{(s)} = \not{p} - m}$$ "neg. energy projector"

And for photons, (for later) $\sum'_{s=1,2} (\epsilon_{(s)})_i (\epsilon^*_{(s)})_j = \delta_{ij} - \hat{p}_i \hat{p}_j$ (Coulomb gauge)

or $\sum_{s=1,2} \epsilon_{(s)\mu} (\epsilon^*_{(s)})_\nu = \eta_{\mu\nu} - \frac{r_\mu q_\nu + q_\mu r_\nu}{r \cdot q}$

~~....~~ axial gauge

$q^\mu \equiv$ photon 4-momenta

r^μ = "gauge vector" (arbitrary)

So,

$$\sum'_{s_a} \bar{u}(a) \Gamma_1 \sum_{s_b} u(b) \bar{u}(b) \bar{\Gamma}_2 u(a)$$

$$= \sum'_{a} \bar{u}(a) \Gamma_1 (\not{p}_b + m_b) \bar{\Gamma}_2 u(a)$$

$$= \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b) \bar{\Gamma}_2 \sum_{s_a} u(a) \bar{u}(a) \right]$$

$$= \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b) \bar{\Gamma}_2 (\not{p}_a + m_b) \right]$$

· For e-μ scattering, we apply trick twice once on e line, once on μ line $\boxed{\bar{\gamma}^\mu = \gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu}$

$$\Rightarrow \langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \frac{e^4}{((p_1 - p_3)^2)^2} \mathrm{Tr}\left[\gamma^\mu (\not{p}_1 + m_e) \gamma^\nu (\not{p}_3 + m_e) \right]$$

$$\times \mathrm{Tr}\left[\gamma_\mu (\not{p}_2 + m_\mu) \gamma_\nu (\not{p}_4 + m_\mu) \right]$$

$$\boxed{\frac{1}{(2s_e+1)(2s_\mu+1)} = \left(\frac{1}{2}\right)^2}$$

Now we need some <u>Trace rules</u>

<u>Linearity</u>: $\mathrm{Tr}[\alpha A + \beta B] = \alpha \mathrm{Tr}(A) + \beta \mathrm{Tr}\, B$

(α, β = numbers)

<u>Cyclicity</u>: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

Use $\gamma^\mu \gamma^\nu = 2\eta^{\mu\nu} - \gamma^\nu \gamma^\mu$ to move γ's around

$\Leftrightarrow \quad \not{a}\not{b} = 2a\cdot b - \not{b}\not{a}$

$\gamma_\mu \gamma^\mu = 4$

$\gamma_\mu \gamma^\nu \gamma^\mu = 2\eta_\mu{}^\nu \gamma^\mu - 4\gamma^\nu = -2\gamma^\mu$

$\Rightarrow \quad \gamma_\mu \not{a} \gamma^\mu = -2\not{a}$

$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4\eta^{\nu\lambda} \quad \Longleftrightarrow \quad \gamma_\mu \not{a}\not{b} \gamma^\mu = 4a\cdot b$

<u>Most useful</u>:

$$\boxed{\begin{aligned} &\mathrm{Tr}(\mathbb{1}) = 4 \\ &\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu} \\ &\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\lambda\sigma} - \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda}) \end{aligned}}$$

7.9 ~~W.3~~ ~~W.10~~

$m_e \equiv m$
$m_\mu \equiv M$

Apply trace rules to $e\mu$ scattering

$$\mathrm{Tr}[\gamma^\mu(\not p_1+m)\gamma^\nu(\not p_3+m)] = \mathrm{Tr}[\gamma^\mu \not p_1 \gamma^\nu \not p_3] + m^2\,\mathrm{Tr}[\gamma^\mu\gamma^\nu]$$
$$= 4(p_1^\mu p_3^\nu - g^{\mu\nu} p_1\cdot p_3 + p_3^\mu p_1^\nu + m^2 g^{\mu\nu})$$

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = \frac{4e^4}{((p_1-p_3)^2)^2}\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2 - p_1\cdot p_3)\right]\left[p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + g_{\mu\nu}(M^2 - p_2\cdot p_4)\right]$$

$((-4+4)\, p_1\cdot p_3\, p_2\cdot p_4 \to 0)$

$$\langle|\mathcal{M}|^2\rangle = \frac{8e^4}{((p_1-p_3)^2)^2}\left[p_1\cdot p_2\; p_3\cdot p_4 + p_1\cdot p_4\; p_2\cdot p_3 - M^2(p_1\cdot p_3) - m^2\, p_2\cdot p_4 + 2m^2M^2\right]$$

skip in class

Ex: $e\,\mu \to e\,\mu$ (or $e\,p \to e\,p$)

with $E_e \ll M$ ($m \lesssim$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi M)^2}\langle|\mathcal{M}|^2\rangle$$

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{e^2 M}{\vec p^{\,2}\sin^2\frac{\theta}{2}}\right)^2 \left[E^2 - \cancel{\tfrac{m^2}{2}} - \vec p^{\,2}\sin^2\tfrac{\theta}{2} - \cancel{\tfrac{m^2}{2}} + m^2\right]$$

($e^2 \to 4\pi\alpha$; $E^2 = m^2 + \vec p^{\,2}$)

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{2\vec p^{\,2}\sin^2\frac{\theta}{2}}\right)^2\left[m^2 + \vec p^{\,2}\cos^2\tfrac{\theta}{2}\right]}$$

Mott formula

$p_1 = (E, \vec p_1)$
$p_2 = (M, \vec 0)$
$p_3 = (E, \vec p_3)$
$p_4 = (M, \vec 0)$ (approx)

$$(p_1-p_3)^2 = -(\vec p_1 - \vec p_3)^2 = -2\vec p^{\,2}(1-\cos\theta) = -4\vec p^{\,2}\sin^2\tfrac{\theta}{2}$$

$$p_1\cdot p_3 = \tfrac{1}{2}\left[p_1^2 + p_3^2 - (p_1-p_3)^2\right] = m_e^2 + 2\vec p^{\,2}\sin^2\tfrac{\theta}{2}$$

$$p_1\cdot p_2\; p_3\cdot p_4 = p_1\cdot p_4\; p_2\cdot p_3 = ME^2$$

$p_2\cdot p_4 = M^2$

HE limit of $e\mu$ scattering

$$p_1 \cdot p_2 = p_3 \cdot p_4 = s/2$$

$$p_1 \cdot p_4 = p_2 \cdot p_3 = -t/2$$

$$(p_1 - p_3)^2 = u$$

$$\Rightarrow \langle |\mathcal{M}|^2 \rangle = 2e^4 \left[\left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2 \right]$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2 s} 2e^4 \left[\left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2 \right] \qquad (e^2 = 4\pi\alpha)$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \frac{s^2 + t^2}{u^2} \qquad t = -s/2(1+\cos\theta) \quad u = -s/2(1-\cos\theta)$$