Lecture 6 Intro. to QED

Griffiths Ch 7
[Bettini, Ch 5]

- Before we can describe ~~Feynman~~ how to evaluate QED Feynman diagrams, we need to understand how to make a relativistic wave equation, first for spin 0, and then for spin $\frac{1}{2}$.

- Recall for a nonrel. particle,

$$E = \frac{\vec{p}^2}{2m} (+ V) \qquad \text{(energy-mom relation)}$$

$E = i\frac{\partial}{\partial t} \qquad \vec{p} = -i\vec{\partial}$
$(\partial_x, \partial_y, \partial_z)$

$$\Downarrow$$

$$i\frac{\partial}{\partial t}\psi = \left(-\frac{\vec{\partial}^2}{2m} (+ V)\right)\psi \qquad \text{Schrödinger Eq.}$$

- Relativistically, we use

$$E^2 = \vec{p}^2 + m^2 \qquad \text{(if no } V\text{)}$$
(no ext. EM field)

or $p^2 = m^2$

$$\Downarrow$$

$$-\Box\psi = m^2\psi \qquad \text{(Klein-Gordon eq.)}$$

where $\Box = \partial^\mu \partial_\mu = \underbrace{\partial_t^2 - \vec{\partial}^2}_{\text{D'Alembertian}}$

$\vec{\partial}^2 = \nabla^2 \equiv$ Laplacian

- But this equation is a bit different, because it is 2nd order in time
$\Rightarrow$ twice as many solutions.
In $(E, \vec{p})$ space, $E = \pm\sqrt{\vec{p}^2 + m^2}$

Suppose we have a "source" (for ψ, called) $J(x)$, and $\tilde{J}(p) \equiv \int d^4x\, J(x)\, e^{ip\cdot x}$ such that

$$(-\Box - m^2)\,\psi = J$$

or in p-space

$$(p^2 - m)\,\psi = \tilde{J};$$

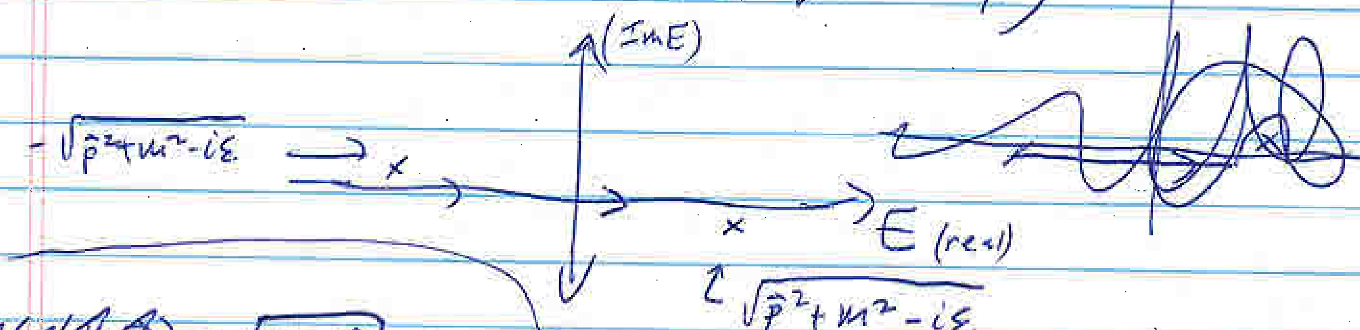
$$\psi = \frac{\tilde{J}}{p^2 - m^2} \leftarrow$$ This is the scalar propagator from last time, $\frac{+i}{p^2-m^2}$ $\xrightarrow[p]{A}$

$\boxed{\text{Aside}}$

Suppose we want to Fourier transform $\frac{-i}{p^2-m^2}$ back to the time domain, as

$$G(t,\vec{p}) \equiv \int_{-\infty}^{\infty} dE\, \frac{e^{-iEt}}{E^2 - \vec{p}^2 - m^2 + i\varepsilon}$$

(Feynman prescription, needed to combine two solutions in right way)



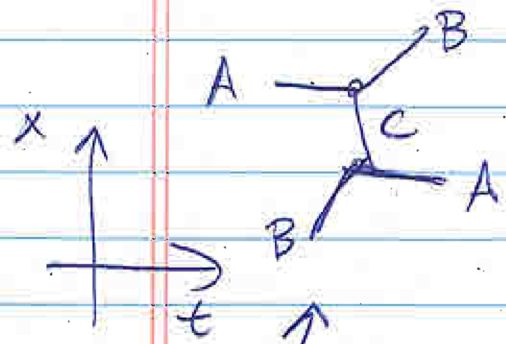
$\boxed{t>0} \Rightarrow e^{-iEt} \to 0$ for $\mathrm{Im}E \to -\infty$

$\Rightarrow$ close contour below, $\Rightarrow$ set $\boxed{E = \sqrt{\vec{p}^2+m^2}}$

$\boxed{t<0} \Rightarrow$ want $\mathrm{Im}E \to +\infty$, close contour above $\Rightarrow$ $E = -\sqrt{\vec{p}^2+m^2}$

$\therefore$ Negative energy solution $\Leftrightarrow$ propagating backwards in time

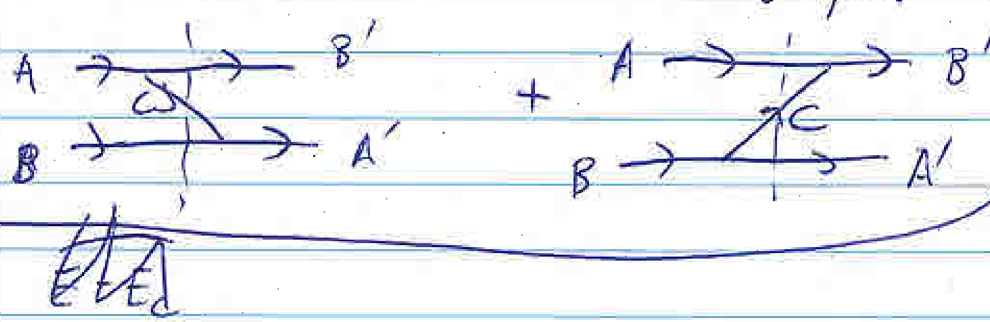
One covariant Feynman graph



Which way is C propagating?

relativistically, 3-vertices could "happen" in either time ordering. Both are represented by Feynman propagator $\frac{i}{p_c^2 - m_c^2}$

Sums up two time-ordered graphs!



Because role of negative energy solutions to 2nd order Klein-Gordon eqn. were not so clear, and because it did not include spin, Dirac looked for a 1st order equation for the electron. He came up with:

$$\boxed{\not{p} - m = 0} \leftarrow \boxed{\text{Dirac eq.}}$$

where $\not{p} \equiv p_\mu \gamma^\mu \leftarrow$ and γ^μ are the 4×4 Dirac γ matrices

· He also ~~wanted~~ needed $p^2 - m^2 = 0$ to hold, which he could arrange if

$$(\not{p} + m)(\not{p} - m) = 0$$

$$= \not{p}\not{p} + \cancel{m\not{p}} - \cancel{\not{p}m} - m^2 = 0$$

$\Rightarrow$ Needed $p_\mu p^\mu = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$

or $(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$

$$= \underbrace{(\gamma^0)^2}_{1}(p^0)^2 + \ldots + \underbrace{(\gamma^3)^2}_{(-1)}(p^3)^2$$

$$+ \underbrace{(\gamma^0\gamma^1 + \gamma^1\gamma^0)}_{0} p_0 p_1 + \ldots + \underbrace{(\gamma^2\gamma^3 + \gamma^3\gamma^2)}_{0} p_2 p_3$$

· Satisfied if $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$ $\leftarrow$ the Dirac algebra

where $\{A, B\} \equiv AB + BA$

and $\eta^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

· The minimal size of the $n \times n$ matrices solving $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ depends on the space-time dimension of $\mu = 0, 1, 2, \ldots D$

- For $D = 4$ it is 4×4.
- One choice in $D=4$ is the Dirac representation:

$$\gamma^0 \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

2x2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 2x2 matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

where σ^i are the 2x2 Pauli matrices used for spin ½ rep's of $SU(2)$ (rotation group; also isospin)

$$\left\{ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right.$$

$$\begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \qquad \begin{pmatrix} u \\ d \end{pmatrix} \qquad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

SU(2) generators: $J^i \equiv \frac{1}{2}\sigma^i, \quad [J^i, J^j] = \frac{i}{2}\epsilon^{ijk} J^k$

[$\epsilon^{123} = +1$, ϵ^{ijk} totally antisym. under pair exchange]

J^z eigenstates: $J^z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad J^z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Full Dirac Eq: $(\gamma^\mu p_\mu - m)\psi = 0$ p-space

(4 component "Dirac spinor")

or $(i\gamma^\mu \partial_\mu - m)\psi = 0$ x-space

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

4 types of solutions to Dirac eq.

E.g. for $\vec{p} = 0$, $\partial_x \psi = \partial_y \psi = \partial_z \psi = 0$

$$\Rightarrow \quad i\gamma^0 \frac{\partial}{\partial t}\psi = m\psi$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial\psi_A/\partial t \\ \partial\psi_B/\partial t \end{pmatrix} = -im \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\Rightarrow \quad \psi_A(t) = e^{-imt}\,\psi_A(0); \quad \psi_A(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ spin up}$$

"normal" $E = m > 0$ solution or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ spin down $\downarrow$

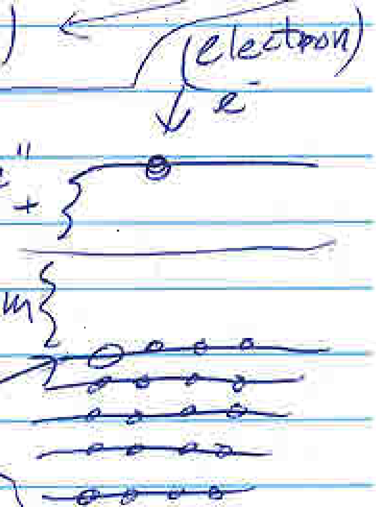
But also

$$\psi_B(t) = e^{+imt}\,\psi_B(0) \qquad \psi_B(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\uparrow$

$E = -m < 0$ solution

Dirac thought of as a "hole" in the "Dirac sea" of filled neg. energy states



hole $\Leftrightarrow$ positively charged $\Leftrightarrow$ positron, antiparticle of electron

Found by C. Anderson

Note: Spin (rotation) generators:

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$\uparrow$

explains why $\psi_B(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is "spin down"

4 indep. solutions:

$\psi^{(1)} = e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $(e^- \uparrow)$ ($\hat{z}$ direction) $\psi^{(2)} = e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $(e^- \downarrow)$

$\psi^{(3)} = e^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $(e^+ \downarrow)$ $\psi^{(4)} = e^{imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ $(e^+ \uparrow)$

Next look for $\vec{p} \neq 0$ solutions, of form

$$\psi(x) = a\, e^{-ip_\mu x^\mu} u(p)$$

$$i\partial_\mu e^{-ip_\mu x^\mu} = p_\mu e^{-ip_\mu x^\mu}$$

→ so Dirac eq $\Rightarrow$ $\boxed{(\gamma^\mu p_\mu - m)u(p) = 0}$

$$\Leftrightarrow 0 = \begin{pmatrix} E-m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E-m \end{pmatrix}\begin{pmatrix}u_A\\u_B\end{pmatrix} = \begin{pmatrix}(E-m)u_A - \vec{p}\cdot\vec{\sigma}\,u_B \\ \vec{p}\cdot\vec{\sigma}\,u_A - (E+m)u_B\end{pmatrix}$$

$\therefore$ $\boxed{u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E-m}u_B, \quad u_B = \frac{\vec{p}\cdot\vec{\sigma}}{E+m}u_A}$ $\boxed{\vec{S} = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0\\0 & \vec{\sigma}\end{pmatrix}}$ spin only if $p_x = p_y = 0$!

Find 4 solutions as follows:

$E = \sqrt{p^2+m^2}$

(1) Pick $u_A = \begin{pmatrix}1\\0\end{pmatrix}$ $\Rightarrow u_B = \left(\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{E+m}\begin{pmatrix}p_z\\p_x+ip_y\end{pmatrix}$ $(e^- \uparrow)$

(2) Pick $u_A = \begin{pmatrix}0\\1\end{pmatrix}$ $\Rightarrow u_B = \left(\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{E+m}\begin{pmatrix}p_x-ip_y\\-p_z\end{pmatrix}$ $(e^- \downarrow)$

(3) Pick $u_B = \begin{pmatrix}1\\0\end{pmatrix}$ $\Rightarrow u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E-m}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{E-m}\begin{pmatrix}p_z\\p_x+ip_y\end{pmatrix}$ $(e^+ \downarrow)$

$E = -\sqrt{p^2+m^2}$ so $\vec{p} \to 0$ nonsingular

(4) Pick $u_B = \begin{pmatrix}0\\1\end{pmatrix}$ $\Rightarrow u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E-m}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{E-m}\begin{pmatrix}p_x-ip_y\\-p_z\end{pmatrix}$ $(e^+ \uparrow)$

$u \equiv \begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}$ $u^\dagger = (\alpha^* \; \beta^* \; \gamma^* \; \delta^*)$ $\boxed{u^\dagger u = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 2|E|}$ normalization convention

- The last steps are to:
  (1) include $N \equiv \sqrt{|E|+m}$ to normalize u's
  (2) relabel $u^{(3)}, u^{(4)}$ solutions to $v^{(1)}(E,\vec{p}) = u^{(4)}(-E,-\vec{p})$
  $v^{(2)}(E,\vec{p}) = -u^{(3)}(-E,-\vec{p})$

  so they now have
  $\underline{E>0}$, correspond to positrons ~~opposite charge~~
  Dirac eq, $\gamma^0 E + \vec{\gamma}\cdot\vec{p} - m = 0$ requires flipping $\vec{p}$ too. (Like letting $m \leftrightarrow -m$)

- Finally, the basic plane wave states with $E = +\sqrt{p^2+m^2}$ are:

$$(e^-) \qquad u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$(e^+) \qquad v^{(1)} = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v^{(2)} = -N \begin{pmatrix} p_z/(E+m) \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Reason for this swap: charge conjugation C turns out to take $e^-\uparrow \leftrightarrow e^+\Uparrow$ so we can have $C: u^{(1)} \leftrightarrow v^{(1)}$, $u^{(2)} \leftrightarrow v^{(2)}$

Let $\not{p} \equiv \gamma^\mu p_\mu$

Note $(\not{p}-m)u^{(i)} = 0$, but $(\not{p}+m)v^{(i)} = 0$

- To build Lagrangian, we want Lorentz invariants, or at least terms with nice covariant transformation properties. Since $\psi \sim$ spin $\frac{1}{2}$, we are probably going to need pairs of ψ's.

ψ itself transforms as the "spinor representation" of the Lorentz group. Infinitesimal transformations corresponding to $x^\mu \to \Lambda^\mu{}_\nu x^\nu = x^\mu + \epsilon^\mu{}_\nu x^\nu$ ($\ll 1$)

can show $[\sigma_{\mu\nu}, \sigma_{\alpha\beta}]$ = (same form as $[L^{\mu\nu}_{\mu\nu}, L_{\alpha\beta}]$) ($L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$) (6.9)

where $\boxed{\sigma_{\mu\nu} \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = i\gamma^\mu\gamma^\nu \quad (\mu \neq \nu)}$

- Boost in x direction corresponds to $\varepsilon_{01} \neq 0$

$\Rightarrow \psi \to \psi + i\varepsilon_{01}\sigma_{01}\psi = \psi + (-)\varepsilon_{01}\gamma^0\gamma^1\psi$

- Griffiths gives finite form of this transformation,

$S_x = a_+ + a_-\gamma^0\gamma^1 \quad a_\pm = \sqrt{\frac{\gamma \pm 1}{2}}$

- Is $|\psi|^2 = \psi^\dagger\psi = \sum_{i=1}^{4} |\psi_i|^2$ a scalar?

- No, $\psi^\dagger\psi \to \psi^\dagger S^\dagger S\psi$

$\gamma^{0\dagger} = \gamma^0$
$\gamma^{i\dagger} = -\gamma^i$

and $S^\dagger S = (a_+ + a_-\gamma^0\gamma^1)^\dagger(a_+ + a_-\gamma^0\gamma^1)$
$= (a_+ - a_-\gamma^1\gamma^0)(a_+ + a_-\gamma^0\gamma^1)$
$= \underbrace{(a_+^2 + a_-^2)}_{\gamma} + \underbrace{2a_+a_-}_{\sqrt{\gamma^2-1} = v\gamma}\gamma^0\gamma^1 \quad v = \sqrt{1-\frac{1}{\gamma^2}}$

$S^\dagger S \neq 1$

$S^\dagger S = \gamma(1 + v\gamma^0\gamma^1)$

$\psi^\dagger\psi$ not a scalar

Looks a little more like 0^{th} component of 4-vector — and indeed it is

- Instead, the scalar is $\bar{\psi}\psi$
where $\bar{\psi} \equiv \psi^\dagger\gamma^0 = (\psi_1^* \ \psi_2^* \ -\psi_3^* \ -\psi_4^*)$ is the adjoint spinor.

For example, $S^\dagger\gamma^0 S = (a_+ - a_-\gamma^1\gamma^0)\gamma^0(a_+ + a_-\gamma^0\gamma^1)$
$= (a_+^2 - a_-^2)\gamma^0 + 0 \cdot a_+a_-\gamma^0\gamma^1$
$= \gamma^0$

A 4-vector is provided by $\boxed{\bar{\psi}\gamma^\mu\psi \doteq \psi^\dagger\gamma^0\gamma^\mu\psi = (\psi^\dagger\psi, \ \psi^\dagger\gamma^0\gamma^1\psi, \ \psi^\dagger\gamma^0\gamma^2\psi, \ \ldots)}$

"the current"

The u and v spinors, and their adjoint spinors $\bar{u}$ and $\bar{v}$, are wave-function factors that we will need for Feynman diagrams.

<u>Electrons</u>: Incoming $\rightarrow\bullet$ u
Outgoing $\bullet\rightarrow$ $\bar{u}$

<u>Positrons</u>: Incoming $\leftarrow\bullet$ $\bar{v}$
(note arrow is reversed!) Outgoing $\bullet\leftarrow$ v

<u>Internal electron line propagator:</u>

$$\boxed{\xrightarrow{q} = \frac{i}{\not{q}-m} = \frac{i}{(\not{q}-m)}\frac{(\not{q}+m)}{(\not{q}+m)} = \frac{i(\not{q}+m)}{q^2-m^2\ (+i\varepsilon)}}$$

<u>Photons</u>: <u>Governed by Maxwell's eq</u>

$$\nabla\cdot E = 4\pi\rho \qquad \nabla\cdot B = 0$$

$$\nabla\times E + \partial_t B = 0 \qquad \nabla\times B - \partial_t E = 4\pi J$$

$$\Longleftrightarrow \quad \partial_\mu F^{\mu\nu} = 4\pi J^\nu \qquad J^\mu = (\rho, \vec{J})$$

$$\cancel{\begin{pmatrix} 0 & E_x & -E_y & -E_z \end{pmatrix}}$$

$$\begin{cases} F^{0i} = -E_i \\ F^{ij} = -\epsilon^{ijk}B_k \end{cases}$$ covariant field strength

Usually we write $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

A^μ = 4-vector potential
$= (V, \vec{A})$, with $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\vec{\nabla}V - \partial_t\vec{A}$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ inserted into $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$

$$\Rightarrow \boxed{\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = 4\pi J^\nu} \qquad (1)$$

Note that $F^{\mu\nu}$ is unchanged under a <u>gauge transformation</u> $A_\mu \to A'_\mu = A_\mu + \partial_\mu \lambda$

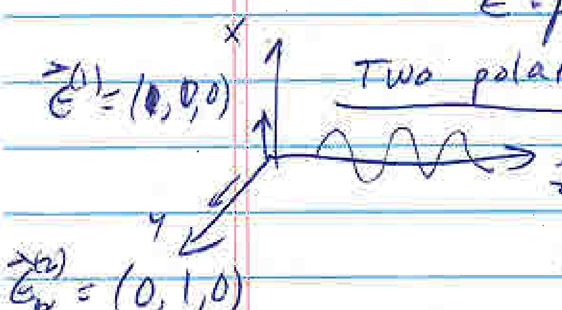
because $\partial_\mu \partial_\nu \lambda - \partial_\nu \partial_\mu \lambda = 0$

- However, $\partial_\mu A^\mu \to \partial_\mu A^\mu + \partial^2 \lambda$
- So we can <u>pick a gauge</u> with (for example) $\partial_\mu A^\mu = 0$ (Lorentz gauge)
- In <u>nonrel.</u> problems, $\vec{\nabla} \cdot \vec{A} = 0$ (Coulomb gauge) is useful. But not here.

- In Lorentz gauge, free wave eq. for A^μ looks just like Klein-Gordon with extra index μ: $\partial_\mu \partial^\mu A^\nu = 0$

$\Rightarrow$ solutions $A^\mu(x) = \epsilon^\mu(p)\, e^{-ip\cdot x}$ $\qquad p^2 = 0$

$\epsilon \cdot p = 0$ [from $\partial_\mu A^\mu = 0$]



Two polarizations

$\epsilon^{(1)*} \cdot \epsilon^{(2)} = 0$ orthogonality

$\underbrace{\epsilon^{(i)*\mu} \epsilon^{(i)}_\mu}_{\Rightarrow +1 \text{ for space-like components}} = -1$ unit spacelike vector

Propagator: $\underset{q\to}{\sim\!\sim\!\sim}$ $\quad \dfrac{-i\eta_{\mu\nu}}{q^2 + i\varepsilon}$

Incoming $\sim\!\sim$ ϵ_μ

Outgoing $\sim\!\sim$ ϵ^*_μ

The vertex:

$\Box A^\mu = 4\pi J^\mu$

comes from varying an action,

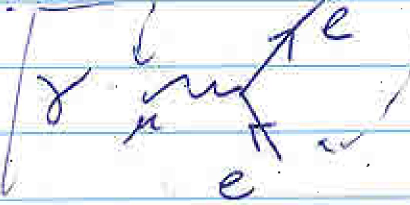
$-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - 4\pi A_\mu J^\mu$

The electromagnetic current $J^\mu = e\bar{\psi}\gamma^\mu\psi$

$\rho = J^0 = e\bar{\psi}\gamma^0\psi = e\underbrace{\psi^\dagger\psi}$ number density, etc.

The coupling $e A_\mu \bar{\psi}\gamma^\mu\psi$

leads to:



$= ie\gamma^\mu$

$\sqrt{4\pi\alpha}$

Griffiths uses g_e, but I use more standard particle physics notation (H-L units)

(we all agree in terms of $\alpha = \frac{1}{137\ldots}$)

- The rule for energy-mom. cons. at each vertex, $(2\pi)^4 \delta^4(k_1+k_2+k_3)$

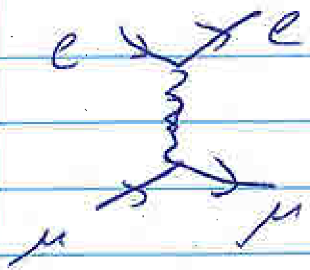
and
- Integrate over internal momenta $\int \frac{d^4q}{(2\pi)^4}$

and
- Cancel the overall $(2\pi)^4 \delta^4(p_1+p_2+\ldots+p_n)$

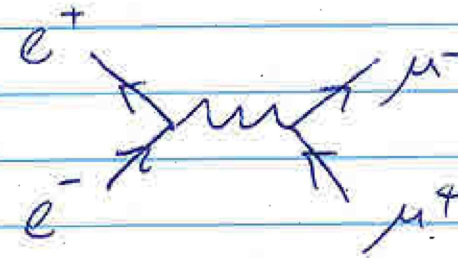
are exactly the same as before.

Basic QED processes

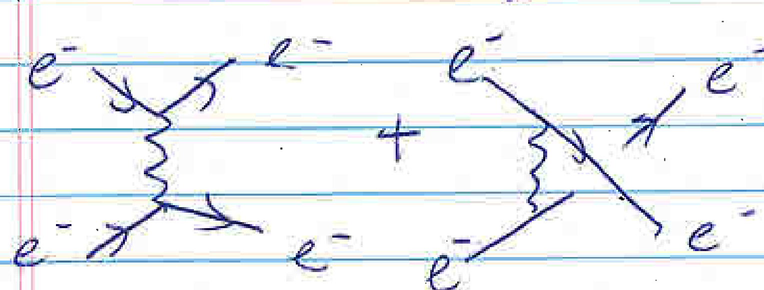
$e^- + \mu^- \to e^- + \mu^-$ $\Leftrightarrow$ $e^+ e^- \to \mu^+ \mu^-$



(crossing) $\Leftrightarrow$

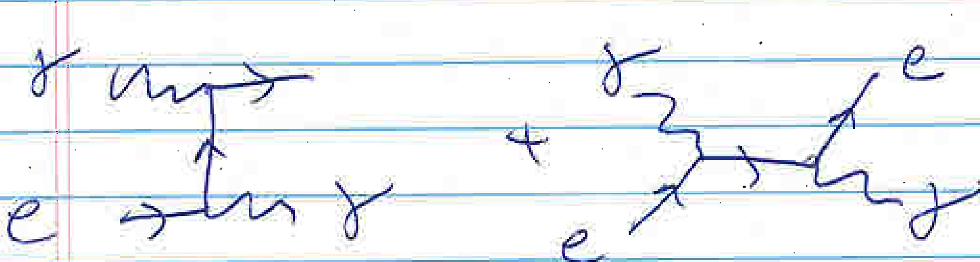


$e^- e^- \to e^- e^-$ [Møller scattering]



crossing $\Rightarrow$ $e^+ e^- \to e^+ e^-$ (Bhabha scattering)

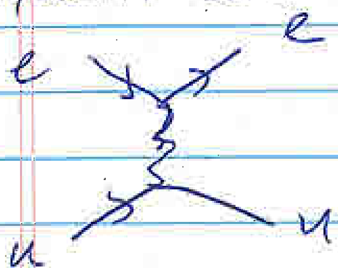
$\gamma e^- \to \gamma e^-$ (Compton scattering)



crossing $\Rightarrow$ $\gamma\gamma \to e^+ e^-$ pair production

$e^+ e^- \to \gamma\gamma$ annih.

e-quark scattering



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