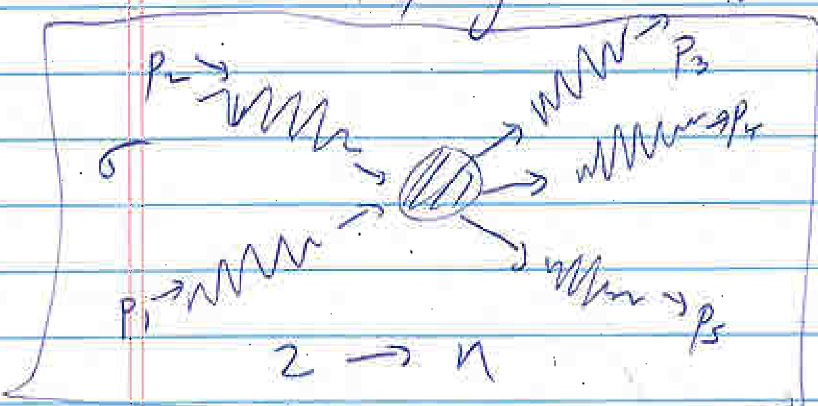
5.1

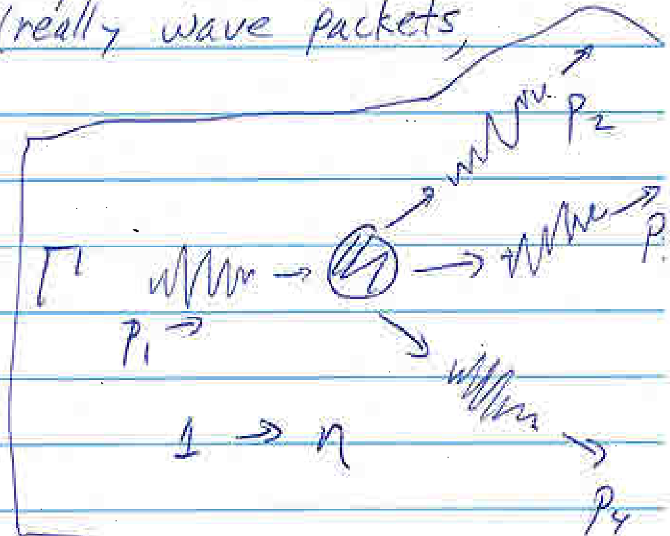
Ph152A/252A Lecture 5

Cross Sections, Decays, and Feynman Rules ~~All That~~

- Bettini Ch. 1.6, 2.5, 5.3-5.7
- Griffiths Ch. 1.4, 2.2, Ch 6

Cross sections (σ) and decay rates (Γ)
are quantum processes
computed as squares of amplitudes
between plane waves (really wave packets, but usually ignore this):





$\mathcal{M}(p_1, p_2; p_3, p_4, p_5, \ldots)$ ← (complex number)
$= \langle p_1, p_2 | \mathcal{M} | p_3, p_4, p_5, \ldots \rangle$

$\langle p_1, p_2|$: state in infinite past; $|p_3, p_4, p_5, \ldots\rangle$: ∞ future

$\mathcal{M}(p_1; p_2, p_3, p_4, \ldots)$
$= \langle p_1 | \mathcal{M} | p_2, p_3, p_4, \ldots \rangle$

$\langle p_1|$: ∞ past; $|p_2, p_3, p_4, \ldots\rangle$: ∞ future

$\mathcal{M}$: QM operator, the "scattering matrix" or "S matrix"

"Fermi's Golden Rule":
Transition Rate $= \frac{2\pi}{\hbar} |\mathcal{M}(p_i)|^2 \times$ (phase space)

Ambiguity in separation depending on how wave functions are normalized.
Use this to make $|\mathcal{M}|^2$ Lorentz invariant.

Particle Decay

- Probability of particle at time $t=0$ to be there at time t is <u>independent</u> of time $\underline{t_0 < 0}$ when it was created: $P = P(t)$

- For "very small" t, $P(t) \simeq 1 - t\Gamma$
(constant)

$\Rightarrow \left.\dfrac{dP(t)}{dt}\right|_{t=0} = -\Gamma$ where $\boxed{\Gamma \equiv \text{total decay width (rate)}}$

- If we have $N(0)$ identical particles at $t=0$
$N(t)$ " " " time t,

$$\left.\frac{dN(t)}{dt}\right|_{t=0} = N(0)\left.\frac{dP(t)}{dt}\right|_{t=0} = -\Gamma N(0)$$

- But $\frac{dN(t)}{dt}$ must be proportional to $N(t)$

$$\Rightarrow \frac{dN(t)}{dt} = -\Gamma N(t) \quad \Rightarrow \quad \boxed{N(t) = N(0)\, e^{-\Gamma t}}$$

But $\boxed{N(t) = N(0)\, e^{-t/\tau}}$ with τ the <u>lifetime</u> (time to go to $\underline{1/e}$)

$$\Rightarrow \boxed{\tau = \frac{\hbar}{\Gamma}}$$

(Γ in eV, τ in sec)

Partial Widths and Branching Ratios

Usually there are several decay modes (sometimes 100's ⇒ see the PDG Book ⇒ order it!)

For example: $\pi^0 \to \gamma\gamma$
$\to e^+e^-\gamma$
...

$K^+ \to \mu^+\nu_\mu$
$\to \pi^+\pi^0$
$\to \pi^+\pi^+\pi^-$
...

Total decay width $\Gamma = \sum_{i=1}^{n} \Gamma_i$ [partial width for decay to i^{th} "mode"]

$$dN = -\underbrace{\left(\sum_{i=1}^{n} \Gamma_i\right)}dt$$

prob. of losing particle = sum of ways of losing it

"Branching Ratio" $Br(X \to i) \equiv \dfrac{\Gamma_i}{\Gamma}$

= prob. of decay to i^{th} final state

$$\sum_{i=1}^{n} Br(X \to i) = 1$$

Rule for calculating Γ_i for $1 \to 2+3+\dots+n$

(1) $$d\Gamma_i = |\mathcal{M}(q_1; p_2, \dots, p_n)|^2 \underbrace{\frac{S}{2m_1}}_{\text{"flux factor" in particle 1 rest frame}} \underbrace{\left(\prod_{j=2}^{n} \frac{d^3\vec{p}_j}{(2\pi)^3\, 2E_j}\right)}_{\text{relativistically invariant phase-space}} \underbrace{(2\pi)^4 \delta^4(p_1 - p_2 - \dots - p_n)}_{\text{(energy-momentum conservation)}}$$

differential decay rate for particle i with momentum in
$(p_2^x, p_2^x + dp_2^x)$
$(p_2^y, p_2^y + dp_2^y)$
$(p_2^z, p_2^z + dp_2^z)$
etc.

$p_j^\mu = (E_j, \vec{p}_j)$

- Usually want to integrate over all, or most, final-state momenta.

To make eq. (1) look more Lorentz invariant, recall that
$$\int_a^b dx\, \delta(f(x)) = \int \frac{df}{df/dx}\, \delta(f)$$

$$\Rightarrow \boxed{\int dE_j\, \delta(p_j^2 - m_j^2)} = \int \frac{dp_j^2\, \delta(p_j^2 - m_j^2)}{\partial p_j^2/\partial E_j} = \frac{1}{2E_j}$$

$$p_j^2 = E_j^2 - \vec{p}_j^{\,2} = m_j^2 \Rightarrow \frac{\partial p_j^2}{\partial E_j} = 2E_j$$

$$\Rightarrow \int \frac{d^3\vec{p}_j}{(2\pi)^3\, 2E_j} = \int \frac{d^4 p_j}{(2\pi)^3}\, \delta(p_j^2 - m_j^2)$$

- "Symmetry Factor" S'
is a product of $\frac{1}{j!}$ for each group of
j identical particles in final state
– enforces Fermi/Dirac + Bose/Einstein statistics.

Example: 2-body decay, $1 \to 2+3$

(index of angles for spinless decay)

$$\Gamma = \frac{S'}{2m_1} \frac{1}{(4\pi)^2} \int d^3\vec{p}_2 \int d^3\vec{p}_3 \, \delta^3(\vec{p}_1 - \vec{p}_2 - \vec{p}_3) \, \delta(E_1 - E_2 - E_3) \frac{|\mathcal{M}|^2}{E_2 E_3}$$

integral easy, gives 1

$$\Gamma = \frac{S'}{2m_1} \frac{1}{(4\pi)^2} \int d|\vec{p}_2| \, |\vec{p}_2|^2 \, d\Omega_2 \, \delta(E_1 - E_2 - E_3) \frac{|\mathcal{M}|^2}{E_2 E_3}$$

$$E_2^2 = |\vec{p}_2|^2 + m_2^2$$

$$\Rightarrow dE_2 \, E_2 = d|\vec{p}_2| \, |\vec{p}_2|$$

But E_3 also depends on $|\vec{p}_2|$

because $E_3^2 = |\vec{p}_3|^2 + m_3^2 = |\vec{p}_2|^2 + m_3^2$

$\Rightarrow dE_3 E_3 = dE_2 E_2$, or $\frac{\partial E_3}{\partial E_2} = \frac{E_2}{E_3}$

$$\Rightarrow \Gamma = \frac{S'}{2m_1} \frac{\int d\Omega_2}{(4\pi)^2} \int dE_2 \frac{|\vec{p}_2|}{E_3} |\mathcal{M}|^2 \delta(E_1 - E_2 - E_3(E_2))$$

$$= \frac{S'}{2m_1} \frac{1}{4\pi} \frac{|\vec{p}_2|}{E_3} \frac{|\mathcal{M}|^2}{\left(1 + \frac{\partial E_3}{\partial E_2}\right)}$$

$$E_3\left(1 + \frac{E_2}{E_3}\right) = E_2 + E_3 = m_1$$

$$\boxed{\Gamma = \frac{S' |\vec{p}_2|}{8\pi m_1^2} |\mathcal{M}|^2}$$

using $\int d\Omega_2 = \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi = 4\pi$

with spin: $\Gamma = \frac{S'}{8\pi m_1^2} |\vec{p}_2| \frac{1}{(2s_1+1)} \sum_{s_1, s_2, \ldots} |\mathcal{M}|^2$

Only complicated part of formula is formula for $|\vec{p}_2|$, in terms of m_1, m_2, m_3:

$$|\vec{p}_2| = \frac{\sqrt{\lambda(m_1^2, m_2^2, m_3^2)}}{2m_1}$$

where $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2(xy + xz + yz)$

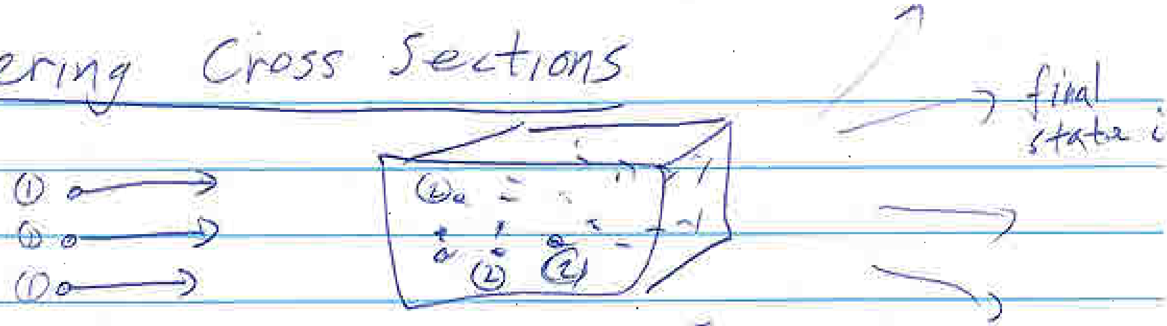
Note: When m_3 (or m_2) can be neglected,

$\sqrt{\lambda(x, y, z=0)} = \sqrt{(x-y)^2} = x - y$

$\Rightarrow |\vec{p}_2| = \frac{m_1^2 - m_2^2}{2m_1}$ $(m_3 = 0)$

If m_2 can also be neglected, $|\vec{p}_2| = \frac{m_1}{2}$ $(m_{2,3} = 0)$

Scattering Cross Sections



$$\boxed{N_{events,i} = \underbrace{N_{beam}}_{(1)} \cdot \underbrace{n_{target}}_{(cm^{-3})} \cdot \sigma_i \cdot \underbrace{L}_{(cm)}}$$

(1)

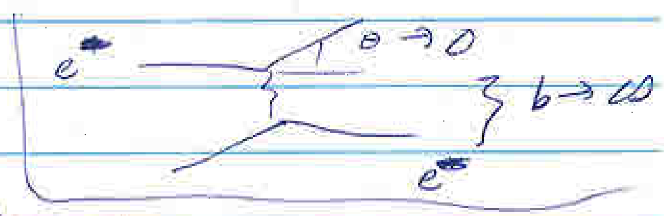
must be cm^2 area

σ_i is an area, hence "cross section"

for $1+2 \to$ (final state i)

$$\boxed{\sigma_{\substack{tot\\(1+2)}} = \sum_{i=1}^{n_f} \sigma_i}$$

Note: σ_{tot} is often infinite due to long-range interactions like electromagnetism



- Golden rule for scattering same as for decay, except initial state $\frac{1}{2m_1}$ ~~xxxxx~~ is replaced by

$$\frac{1}{\underbrace{(2E_1)(2E_2)}_{\text{initial state wave functions}}\underbrace{|\vec{\beta}_1 - \vec{\beta}_2|}_{\text{to take into account that rate also involves how fast beam + target are closing on each other}}} = \underbrace{\frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}}_{\text{relativistically invariant form}}$$

(Usually, $|\vec{p}_1 - \vec{p}_2| = \begin{cases} 1 & \text{fixed target, rel. beam} \\ 2 & \text{rel. colliding beams} \end{cases}$)

$$\boxed{d\sigma_i = \frac{S}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \prod_{j=3}^{n} \frac{d^3\vec{p}_j}{(2\pi)^3\, 2E_j} |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - \cdots - p_n)}$$

Example: $2 \to 2$ scattering

$1 + 2 \to 3 + 4$ in CM frame

Using $\vec{p}_2 = -\vec{p}_1$,

$p_1 \cdot p_2 = E_1 E_2 + |\vec{p}_1|^2$

can show $\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2} = (E_1 + E_2)|\vec{p}_1|$

$\boxed{s = (E_1 + E_2)^2}$

$|\mathcal{M}|^2$ really only depends on s, t or s, θ_{cm}

Final-state phase-space integration as before,

except $\frac{1}{E_2 + E_3} = \frac{1}{E_1} \to \frac{1}{E_3 + E_4} = \frac{1}{E_1 + E_2}$

and we don't do $\int d\Omega$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{S\,|\vec{p}_f|\,|\mathcal{M}|^2}{(4\pi)^2\, 4(E_1 + E_2)|\vec{p}_i|\,(E_1 + E_2)} = \frac{1}{(8\pi)^2} \frac{S|\mathcal{M}|^2}{s} \frac{|\vec{p}_f|}{|\vec{p}_i|}}$$

$d\Omega = 2\pi\, d\cos\theta_{cm}$

$$\Rightarrow \boxed{\frac{d\sigma}{d(\cos\theta_{cm})} = \frac{1}{32\pi} \frac{S|\mathcal{M}|^2}{s} \frac{|\vec{p}_f|}{|\vec{p}_i|}}$$

with spin:

$$|\mathcal{M}|^2 \to \frac{1}{(2s_1+1)(2s_2+1)} \underbrace{\sum}_{\text{initial, final spins}} |\mathcal{M}|^2$$

(Bettini, 2.3)

Example: Spin of π^+ using assumption of time reversal invariance

$$\frac{d\sigma}{d\Omega}(\pi^+ d \to \underbrace{pp}_{\text{identical}}) \propto \frac{p_p}{p_\pi}\frac{1}{(2s_\pi+1)(2s_d+1)}\frac{1}{2}\sum_{i,f}|\mathcal{M}_{fi}|^2$$

$\frac{1}{2}$: S

$$\frac{d\sigma}{d\Omega}(pp \to \pi^+ d) \propto \frac{p_\pi}{p_p}\frac{1}{\underbrace{(2s_p+1)^2}_{(2\cdot\frac{1}{2}+1)^2=4}}\sum_{i,f}|\mathcal{M}_{if}|^2$$

(equal by T invariance)

$$\Rightarrow \boxed{\underbrace{\frac{\sigma(\pi^+ d \to pp)}{\sigma(pp \to \pi^+ d)}}_{\text{(same cm energy, } \theta_{cm})} = \frac{(2s_p+1)^2}{2\underbrace{(2s_\pi+1)}_{?}\underbrace{(2s_d+1)}_{2\cdot 1+1=3}}\frac{p_p^2}{p_\pi^2} = \frac{2}{3(2s_\pi+1)}\frac{p_p^2}{p_\pi^2}}$$

numerator measured by Durbin et al, Clark et al (1951) $T_\pi = 24$ MeV

denominator by Cartwright et al. (1953) $T_1 = 341$ MeV

ratio $\Rightarrow$ $2s_\pi + 1 = 0.97 \pm 0.31$ $\Rightarrow$ $s_\pi = 0$

Feynman Rules (Griffiths, ch. 6) (5.9)

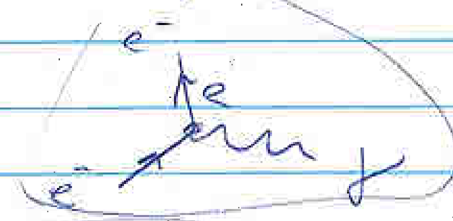
- How do we compute $\mathcal{M}$?
- sometimes it is very hard,

e.g. for strong interactions
at low energy $\pi^- + p \to \pi^0 + n$
few hundred MeV

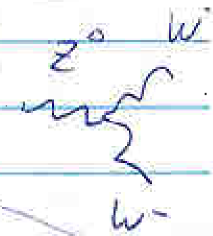
Then maybe we can only use symmetries
or approximate models.

- But sometimes we get "lucky" and
we can do a perturbative expansion
in a small parameter:

QED $\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137.036\ldots}$



Electroweak: $\alpha_w = \frac{g_w^2}{4\pi} \approx \frac{1}{30}$



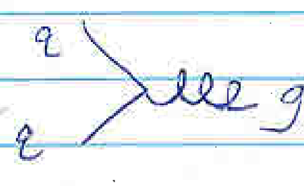
short distance QCD:

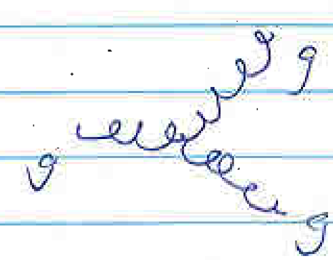
$\alpha_s = \frac{g_s^2}{4\pi}$ depends strongly on scale.

Only at $d << 0.1$ fm $\Delta p >> 1$ GeV
is $\alpha_s << 1$. [At $M_Z = 90$ GeV, $\alpha_s(M_Z) = 0.12$.]

$\mathcal{M} = \mathcal{M}^{(0)} + \frac{\alpha}{\pi}\mathcal{M}^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 \mathcal{M}^{(2)} + \ldots$

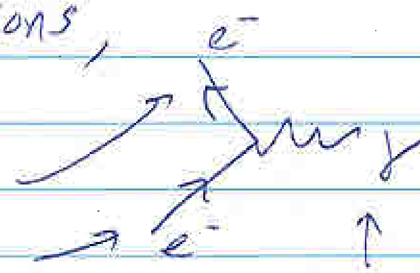
"usually good enough"





Then we can evaluate $\mathcal{M}$
using Feynman diagrams — covariant perturbation theory
- relatively small number of diagrams needed to get first approx. to $\mathcal{M}$

QED: electrons + photons, only 1 interaction.



spin $\frac{1}{2}$ → e^- ↑ spin 1

- But spin complicates things.
- So first we look at a toy theory, only scalar particles (spin 0).

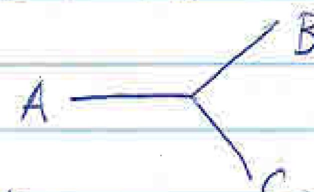
lines

<u>Feynman rules are graphical</u> ⇒ vertices + propagator

- Assume we have 3 different types of scalars A, B, C.

$\frac{i}{q_j^2 - M_j^2}$ for momentum q_j

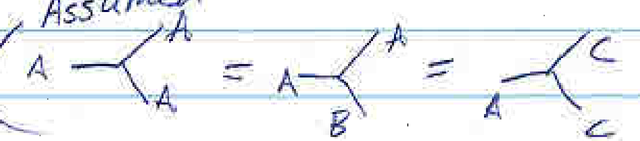
- Also assume only one basic interaction (vertex)



$= -ig$

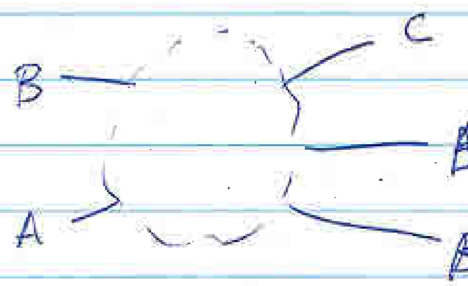
(In principle this could also depend on $p_i \cdot p_j$)

Assumed



$= \ldots = 0$

- <u>Feynman says:</u>
- For a given process, e.g.



draw all diagrams using the nonzero vertices & propagators.

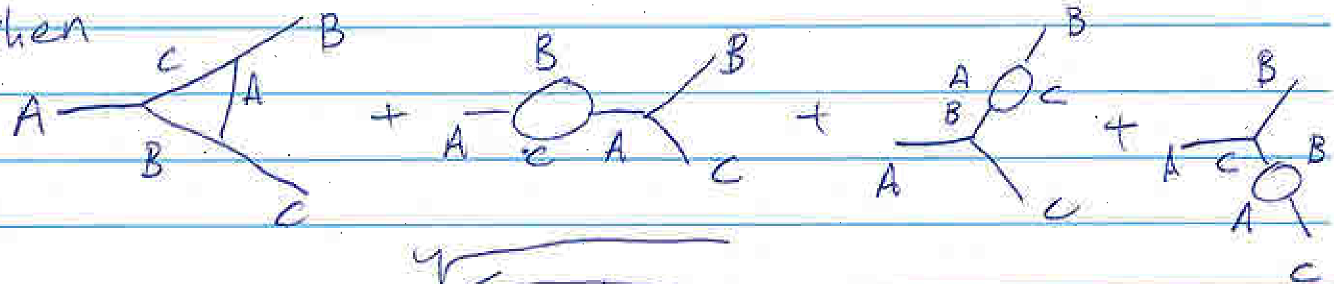
[But diagrams with more than the minimal number of vertices can be neglected at first.]

For example, for $A \to B+C$ we draw

tree graph
(only 1 in this case)

then

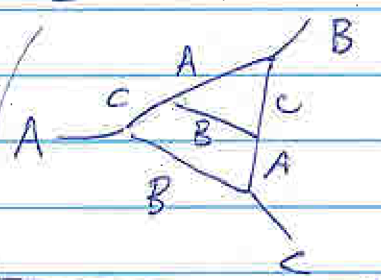
1-loop



note that

is the same graph

\+ 2-loops



\+ ...

Feynman rules assign a complex number to each graph

(1) $p_i \equiv$ external momenta $q_i \equiv$ internal momenta

Number is a product of:

(2) Coupling constant ~~(vertex)~~ factors: $-ig$ (in this case) for each vertex

(3) propagator factors $\frac{i}{q_j^2 - m_j^2}$ for each internal line (scalars)

(4) energy-momentum cons. at each vertex
$(2\pi)^4 \delta^4(k_1+k_2+k_3)$ [if all incoming/outgoing]

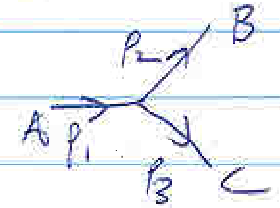
(5) integration over each internal momentum $\int \frac{d^4 q_j}{(2\pi)^4}$

(6) Remove an overall δ fn., $(2\pi)^4 \delta(p_1+p_2+\dots+p_n)$
What's left is $-i\mathcal{M}$

(5.12)

① simplest example: $A \to B + C$ $\quad (m_A > m_B + m_C)$

Tree $\qquad p_1 = p_2 + p_3$



$$\underbrace{-ig(2\pi)^4 \delta^4(p_1 - p_2 - p_3)}_{\text{erase}} = -i\mathcal{M}$$

$$\Rightarrow \boxed{\mathcal{M} = g}$$

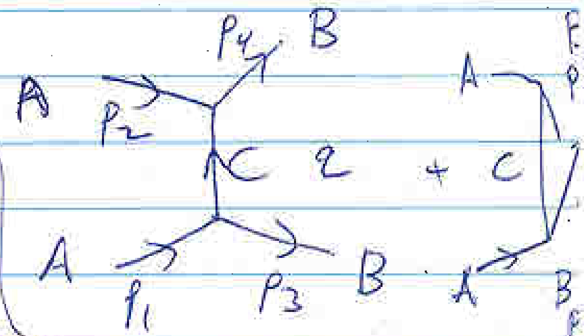
$$\Rightarrow \Gamma = \underset{\substack{\uparrow \\ 1 \\ (B \neq C)}}{S} \frac{|\mathcal{M}|^2 |\vec{p}|}{8\pi m_A^2} \Rightarrow \boxed{\Gamma_A = \frac{g^2 |\vec{p}|}{8\pi m_A^2}}$$

at "tree level" (Born level)

Note that $\Gamma \sim \text{GeV}$ $\Rightarrow g \sim \text{GeV}$ in this case

$$\tau = \frac{1}{\Gamma} = \frac{8\pi m_A^2}{g^2 |\vec{p}|}$$

(2) Tree-level $A + A \to B + B$



$$\Rightarrow (-ig)^2 \frac{i}{q^2 - m_C^2} \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 - p_3 - q) (2\pi)^4 \delta^4(p_2 + q - p_4)$$

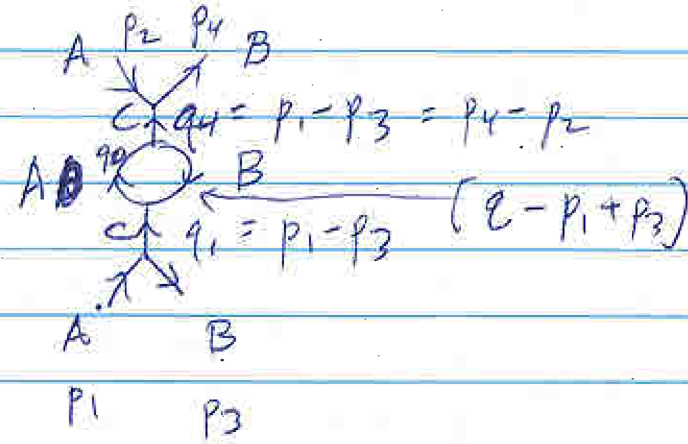
$(p_3 \leftrightarrow p_4)$

$$= \frac{-ig^2}{(p_4 - p_2)^2 - m_C^2} \underbrace{(2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)}_{\text{(erase)}} \Rightarrow \boxed{\mathcal{M} = \frac{g^2}{(p_4 - p_2) - m_C^2} + \frac{g^2}{(p_3 - p_2)^2 - m_C^2}}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\frac{1}{2} |\vec{p}_B| |\mathcal{M}|^2}{(8\pi)^2 |\vec{p}_A| s_{AA}}}$$

(S) at tree level

Let's look at <u>one</u> of the <u>many</u> 1-loop diagrams for $A + A \to B + B$



$$\ldots \Rightarrow \mathcal{M} = \frac{i g^4}{((p_1-p_3)^2 - M_C^2)^2} \underbrace{\int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 - M_A^2)((q-p_1+p_3)^2 - M_B^2)}}_{\text{typical Feynman loop integral}}$$

- Sometimes they are <u>finite</u>
- But sometimes (like here) they <u>diverge</u> in the "ultraviolet region" $q \to \infty$
- There, the integral is

$$\approx \int^{\Lambda} \frac{d^4 q}{q^4} \approx \int^{\Lambda} dq \, \frac{q^3}{q^4} = \int^{\Lambda} \frac{dq}{q} \approx \ln \Lambda$$

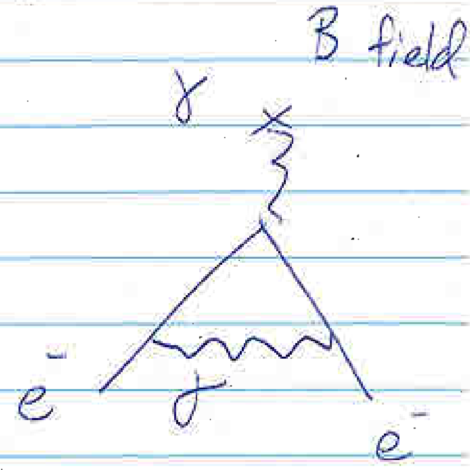
$\to \infty$ as $\Lambda \to \infty$.

- In this case (usually), the divergences can be <u>reabsorbed</u> into redefinitions of <u>masses</u> $(m_{A,B,C}, \ldots)$ and couplings $(g, \ldots)$

e.g. $\underset{\text{(finite)}}{g} = \underset{\text{(divergent)}}{g_0(\Lambda)} + (\#) g^3 \ln \Lambda$ $\Rightarrow$ $\boxed{\text{RENORMALIZATION}}$

• Although ~~It might seem that~~ one can "define away" the infinities, the finite parts remain, and produce small, loop-level shifts in other quantities.

• For example in QED, the coupling of the electron to the photon is modified subtly so that the precession of the e^- spin in a B field changes "anomalous magnetic dipole moment of e^-"

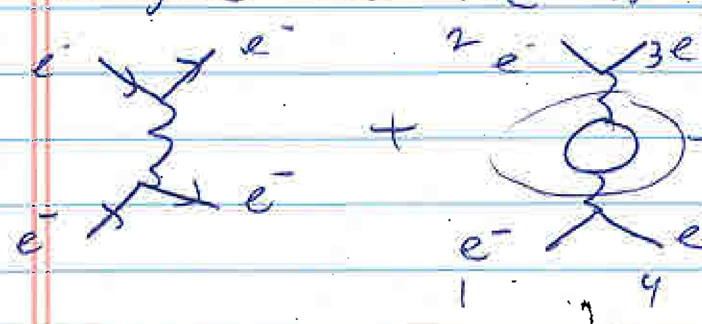


$$\frac{g_e - 2}{2} = \underbrace{\frac{\alpha}{2\pi}}_{0.0011614} + \mathcal{O}\left(\left(\frac{\alpha}{2\pi}\right)^2\right)$$

0.0011614 → 5921...

(exp't + theory agree to ≈ 7 or 8 digits)

• Also, the scattering of electrons at energies >> m_e is modified.



$$\longrightarrow \sim \frac{\alpha}{\pi} \ln\left(\frac{\Lambda^2}{(p_1 - p_4)^2}\right)$$

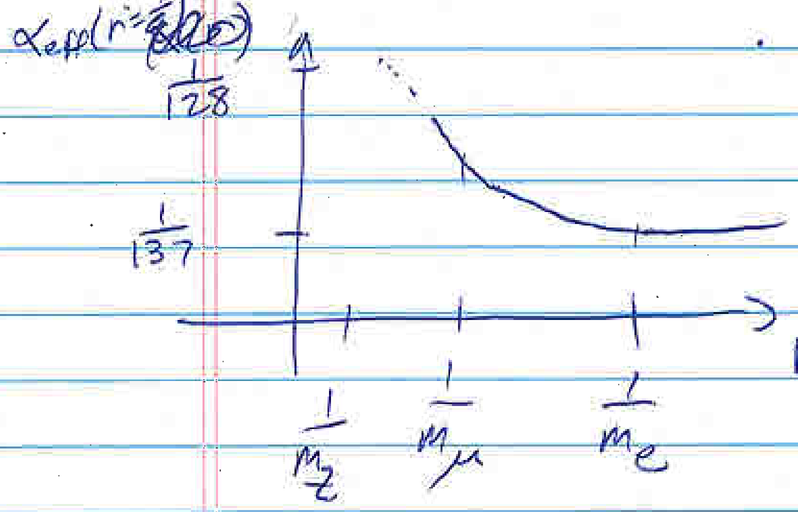
remove by renormalization

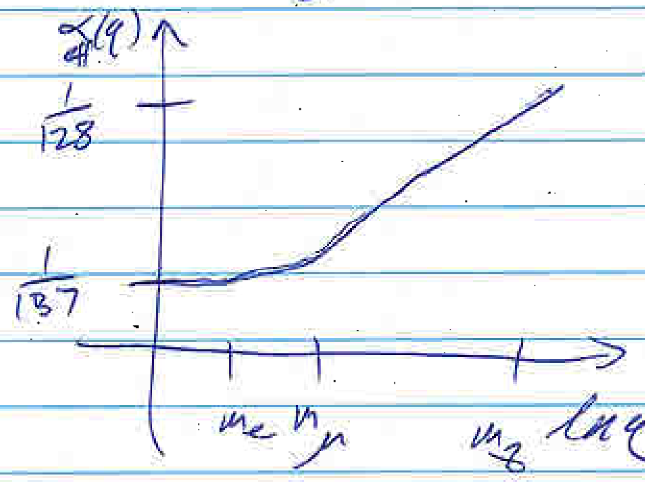
QED

Thus

$$\alpha \Rightarrow \boxed{\alpha_{eff}(q^2) = \alpha + \frac{\alpha^2}{3\pi} \ln\left(q^2/m_e^2\right)}$$

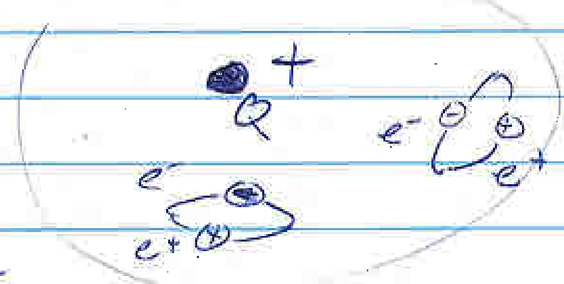
⇒ Coupling constant gets stronger at shorter distances $r \sim \frac{1}{q}$





• We say that virtual e^+e^- pairs polarize the vacuum

like normal dielectric:
smaller electric field seen out here



⇒ charge Q looks smaller at larger distances

⇔ $\alpha \downarrow$ as $r \uparrow$

• In QCD, quarks do same thing

But gluon effect is opposite in sign (and bigger) ⇒ asymptotic freedom