Physics 152A Lecture 2003

Relativistic Kinematics

· First, a word about units.

· In particle physics, it is traditional to set speed of light $c = 1$ and (reduced Planck's constant $\hbar \equiv \frac{h}{2\pi} = 1$.

· This is the same as converting times and distances into energies, for which the standard unit is

$1\ \text{GeV} = 10^9$ electron volts.

(Convenient because the rest energy of the proton is $m_p c^2 = 0.938$ GeV.)

· Times → distances using $c \simeq 3\times 10^{10}\ \text{cm s}^{-1} = 1\ \text{foot ns}^{-1} = 30\ \text{cm ns}^{-1}$

Example: Several unstable particles (B's, D's, ...) have lifetimes $\tau \simeq 1$ ps

$\Rightarrow c\tau \simeq 0.3$ mm ← (very approximate flight distance before decay, if $v \approx c$)

~~Distances~~ · Microscopic distances:

· Atomic physicists often use $1 Å = 10^{-10}\ \text{m} = 10^{-8}\ \text{cm}$ ($\approx r_{atom}$)

· Nuclear/particle physicists use $1\ \text{fermi} = 1\ \text{fm} \equiv 1$ femtometer $= 10^{-15}\ \text{m} = 10^{-13}\ \text{cm} \approx r_{proton}$

· Distances → Energies using

$$\boxed{1} = \hbar c = 197\ \text{MeV fm} \quad \text{(nuclear)}$$
$$= 0.197\ \text{GeV fm} \quad \text{(particle)}$$

$[L] = [T] = \frac{1}{[E]}$

· "Cross sections" ≡ area ≃ probability of interaction

measured by nuclear physicists in "barns" $1\ b = 10^{-24}\ \text{cm}^2$.

Typical particle physics cross sections: $1\ \text{nb} = 10^{-33}\ \text{cm}^2$, $1\ \text{pb} = 10^{-36}\ \text{cm}^2$, $1\ \text{fb} = 10^{-39}$

$1 = (\hbar c)^2 = 3.89 \times 10^8\ \text{GeV}^2\text{pb}$ ← Use to convert $1/\text{GeV}^2 \to$ pb

Special Relativity

· Laws of physics apply in any reference frame, provided that we make appropriate transformations of coordinates (and other quantities).

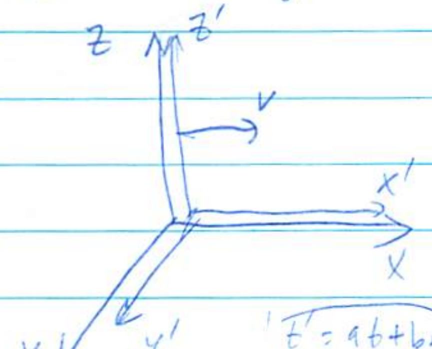
· Speed of light = constant, c in all frames.

(Michelson-Morley experiment)

"Galilean transformations"
$$\begin{cases} t' = t \\ x' = x - vt \end{cases}$$
2 frames with rel. velocity v in x direction

no good
- leads to wrong velocity addition formula, $u = u' + v$
- For $u = c \Rightarrow u' = c + v \neq c$

· Need to generalize to
$$t' = at + bx$$
$$x' = dt + ex$$



$$\begin{cases} y' = y \\ z' = z \end{cases}$$

to avoid paradox! does ball get through cylinder or not?

Light in $\pm x$ direction

$t = x \Rightarrow t' = x'$

$\Rightarrow a + b = d + e$

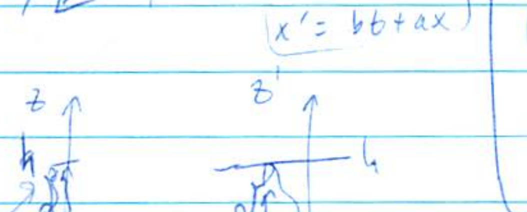
$t = -x \Rightarrow t' = -x'$

$\Rightarrow -a + b = d - e$

$\Rightarrow \boxed{d = b,\ e = a}$

$t' = at + bx$
$x' = bt + ax$

$\boxed{\beta \equiv \frac{v}{c} \text{ often}}$



$t = h$

$t' = ah$

$x' = bh$

$c = 1 = \frac{\sqrt{x'^2 + h^2}}{t'} = \frac{\sqrt{b^2+1}}{a} \Rightarrow \sqrt{b^2+1} = a$

Satisfied by $a = \gamma \equiv \frac{1}{\sqrt{1-v^2}}$, $b = -v\gamma$

$\sqrt{1 + v^2\gamma^2} = \sqrt{1 + \frac{v^2}{1-v^2}} = \frac{1}{\sqrt{1-v^2}} = \gamma$ ✓

$$\Rightarrow \boxed{\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(-vt + x) \\ y' &= y \\ z' &= z \end{aligned}}$$

[reduces to Galilean for $v \ll c$.]

$\boxed{\text{Inverse transformation} \iff v \leftrightarrow -v}$

Consequences:

(1) $\Delta t' = \gamma(\Delta t - v\Delta x)$

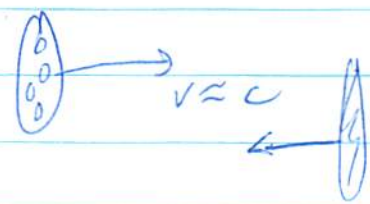
so $\Delta t = 0$ can $\Rightarrow$ $\Delta t' \neq 0$ "relativity of simultaneity"

(2) Lorentz contraction

moving stick,
length L in rest frame
looks like length L/γ.

$(t, x) = (0, \frac{L}{\gamma})$ $\leftarrow$ using $x' = \gamma(x - vt)$
$L \quad 0$

$\Rightarrow$ Protons & nuclei "pancake" at high energies.



(3) Time dilation.

$(t', x') = (0,0) \to (t', x') = (\tau, 0)$
$= (t, x)$

$\Delta t = \gamma(t' + vx')$

$\Rightarrow \boxed{\Delta t = \gamma\tau}$

· Particle lifetimes are longer by γ.

· Average flight length is $\Delta x = \beta\Delta t = \boxed{\beta\gamma c\tau}$

(can be $\gg c\tau$ for $\beta \simeq 1$)

(e.g. B mesons with $c\tau \simeq 0.3\,\text{mm}$ can fly several mm before decaying.)

Velocity addition formula becomes $v = \dfrac{v_1 + v_2}{1 + v_1 v_2}$

$v_2 = c = 1 \Rightarrow v = \dfrac{1 + v_1}{1 + v_1} = 1$ ✓

4-vectors

$x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z$

- Write x^μ

Lorentz transformation as

(*) $\boxed{x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu}$ implicit $\sum_{\nu=0}^{3}$

where for v in x-direction,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Anything transforming like x^μ in (*) is called a ~~co~~contravariant 4-vector.

- Recall proper time interval $(\Delta\tau)^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$

(time elapsed on clock travelling along from $(0,0,0,0) \to (\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3)$)

$= \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$

where $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is Minkowski metric

Used to raise/lower indices, e.g. $x_\mu \equiv \eta_{\mu\nu} x^\nu$ (called a covariant 4-vector)

$\cancel{A^\mu B_\mu = A^0 B^0 - \sum_{i=1}^{3} A^i B^i}$

$\eta_{\mu\nu}$ is an invariant tensor,

$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$

"=" $\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & +\beta\gamma \\ -\beta\gamma & -\gamma \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- True for all Λ, in fact it defines the group of Lorentz transformations: $SO(1,3)$ (real matrices) preserves $(+ - - -)$

4-vectors can be combined into Lorentz invariant quantities (Lorentz scalars) using $\eta_{\mu\nu}$ (or $\eta^{\mu\nu} = \eta^{-1}_{\mu\nu} = \eta_{\mu\nu}$)

$$\boxed{a \cdot b \equiv a^\mu b_\mu = \eta_{\mu\nu} a^\mu b^\nu \xrightarrow{\text{Lorentz}} \eta_{\mu\nu} \Lambda^\mu{}_\alpha a^\alpha \Lambda^\nu{}_\beta b^\beta = \eta_{\alpha\beta} a^\alpha b^\beta = a \cdot b}$$

$$\boxed{a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}}$$

$$\boxed{a^2 = (a^0)^2 - \vec{a}^2 \begin{cases} > 0 & \text{timelike} \\ = 0 & \text{lightlike} \\ < 0 & \text{spacelike} \end{cases}}$$

- Tensors transform with more indices

$$\boxed{T^{\mu\nu} \to \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}}, \ \ldots$$

Energy & momentum of particles

- Ordinary velocity $\vec{v} \equiv \frac{d\vec{x}}{dt} = \frac{d\vec{x}}{dx^0}$ does not have simple Lorentz transformation properties because of x^0 in denominator.

Better to put proper time interval $d\tau$ in denominator. $\boxed{dt = \gamma d\tau}$ (time dilation)

$\Rightarrow \ \eta^i \equiv \frac{dx^i}{d\tau} = \gamma \frac{dx^i}{dt}$ will transform like 3 of the 4-components of a 4-vector.

The ~~4th~~ 0th component is

$$\eta^0 \equiv \frac{dx^0}{d\tau} = \frac{dt}{d\tau} = \gamma .$$

$\Rightarrow$ 4-velocity $\boxed{\eta^\mu \equiv \gamma(1, v_x, v_y, v_z)}$

Lorentz invariant magnitude of 4-velocity: $\boxed{\eta^\mu \eta_\mu = \gamma^2(1-v^2) = 1}$!

Momentum of a particle

- Because $\vec{v} = \frac{d\vec{x}}{dt}$ does not have simple transformation properties, neither does $\vec{p} = m\vec{v}$.
- Better to use $\vec{p} = m\vec{\eta} = m\frac{d\vec{x}}{d\tau} = m\gamma\vec{v}$.

Agrees with old $\vec{p}$ for $v \ll 1$.

- The full 4-vector is (*) $\boxed{p^\mu = m\eta^\mu = m\gamma(1, v_x, v_y, v_z)}$
- The time component $\boxed{p^0 \equiv E}$ $= m\gamma$ ← (total energy of particle: $\boxed{\text{rest energy}}$ + kinetic energy) ($E_{rest} = mc^2$)

$$\boxed{p^\mu = (E, p_x, p_y, p_z)}$$

$p^\mu p_\mu$ is Lorentz invariant. Evaluate in particle rest frame $\Rightarrow p^\mu = (m, 0, 0, 0)$ (rest energy only)

or directly from (*).

$\Rightarrow$ $\boxed{p^2 = p^\mu p_\mu = m^2}$ ← "on-shell condition" (mass)

- Thus $\boxed{E^2 - \vec{p}^2 = m^2}$ total kinetic energy $T \equiv E - m$

$\boxed{\text{Can also expand for small } v}$ $\Rightarrow E = m\gamma = \frac{m}{\sqrt{1-v^2}} = m + \frac{1}{2}mv^2 + \frac{3}{8}mv^4 + \ldots$

rest energy (nonrelativistic kinetic energy) (relativistic correction)

- As $v \to c$, m becomes inconvenient and irrelevant. $\gamma \to \infty$

$\vec{p} = m\vec{v}\gamma \gg m$

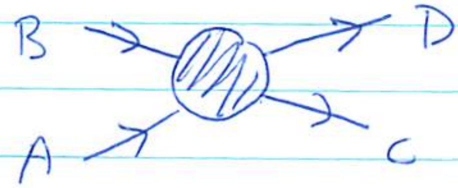
$E = m\gamma \gg m$.

$E^2 - \vec{p}^2 \simeq 0$, or $\boxed{E \simeq |\vec{p}|}$ ← (massless (γ) or high energy particle)

- Difference between $E=$ 2 eV and 3 eV photon is not the mass, but (via Quantum mechanics) the frequency, $E = h\nu$, or wavelength, $E = hc/\lambda$

Collisions

$A + B \to C + D$ "$2 \to 2$" (most frequent)



"free particles at infinity" $\Rightarrow$ p_i = constants at ∞
$i = A, B, C, D$

- Collision takes place over a very short time/distance interval $\Rightarrow$ ignore all <u>external</u> forces & fields.
- Then energy-momentum conservation

$$\Rightarrow \quad \boxed{p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu} \qquad (1)$$

<u>0th component</u>: $E_A + E_B = E_C + E_D \qquad (2)$

<u>total</u> energy is conserved.

- But this includes rest energy, which can change if $m_A + m_B \neq m_C + m_D$
- So total kinetic energy is <u>not</u> conserved.
- An extreme example: $e^+e^- \to \tau^+\tau^-$ at <u>threshold</u>.
  with $E_{e^+} = E_{e^-} = 1.8$ GeV $\qquad m_{\tau^\pm} = 1.777$ GeV

$\left(\text{Eq. } (2) + \text{symmetry}\right) \Rightarrow E_{\tau^+} = E_{\tau^-} = 1.8\text{ GeV} = \gamma_\tau \cdot 1.777\text{ GeV}$

$$\Rightarrow \gamma_\tau = 1.01 = \frac{1}{\sqrt{1 - v_\tau^2}} \Rightarrow \boxed{v_\tau = \sqrt{1 - \frac{1}{1.01^2}} = 0.16c}$$

τ's are nonrelativistic,

$\boxed{(K.E.)_\tau = \frac{1}{2} m_\tau v_\tau^2 = 22 \text{ MeV} \ll m_\tau}$

while
$(K.E.)_e \simeq E_e = 1.8$ GeV

· How many kinematic quantities characterize the "process" or "reaction" $A + B \to C + D$?

~~Start with p_A^μ~~

· Not counting $m_A, \ldots, m_D$ which are attributes of the particles, not the process.

· Start with $4 \times 4 = 16$ quantities,

$$p_A^\mu \quad p_B^\mu \quad p_C^\mu \quad p_D^\mu$$

· But $p_i^2 = m_i^2$ removes 4 $(i = A, B, C, D)$

$p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu$ removes 4 more

$16 - 4 - 4 = \underline{8}$.

Still not done. There are $\underline{3}$ boosts (along x, y, z axes)

and $\underline{3}$ rotations (around x, y, z axes)

which only change the frame, not the essential attributes of the reaction.

omit

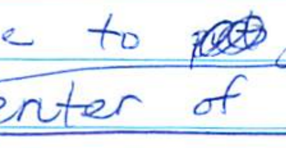
$$8 - 3 - 3 = \boxed{2}$$ ← true number of kinematic quantities

· What are they?

· Let's use ~~boost~~ invariance to go to ~~A & B along z axis~~ center of mass frame

$\vec{p}_{TOT} = \vec{p}_A + \vec{p}_B = 0$

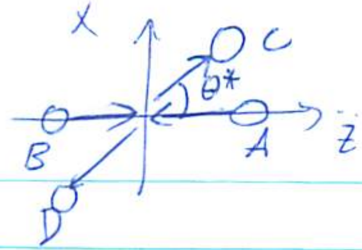
$(= \vec{p}_C + \vec{p}_D)$



· Then use rotation invariance to line $\vec{p}_A$ & $\vec{p}_B$ up (the "beamline") along z direction.

A O ⟶ ⟵ O B

· Rotate around z axis until C & D are in (say) xz plane



$\boxed{\theta^* \equiv \theta_{CM} \text{ is center of mass scattering angle.}}$

The other quantity is just the

$\boxed{\text{center-of-mass energy, } E_{cm}}$

- Let's count a second way (Mandelstam) construct Lorentz invariants $(p_i \pm p_j)^2$. $\boxed{\text{problem 3.22 of Griffiths}}$
- How many are there?

  $i = j$ gives 0 or $(2p_i)^2 = 4m_i^2$ (we don't count this)

- So take $i \neq j$.

Since $(p_i + p_j)^2 + (p_i - p_j)^2 = 2(p_i^2 + p_j^2) = 2(m_i^2 + m_j^2)$,

we can take $+$ or $-$ but not both.

Mandelstam chose

$$\boxed{\begin{aligned} s &\equiv (p_A + p_B)^2 = (p_C + p_D)^2 \\ t &\equiv (p_A - p_C)^2 = (p_B - p_D)^2 \\ u &\equiv (p_A - p_D)^2 = (p_B - p_C)^2 \end{aligned}}$$

with one constraint eqn:

$$0 = (p_A + p_B - p_C - p_D)^2$$

$$= \left(\sum_{i=1}^{4} m_i^2\right) + 2(p_A \cdot p_B + p_C \cdot p_D) - 2(p_A \cdot p_C + p_B \cdot p_D) - 2(p_A \cdot p_D + p_B \cdot p_C)$$

$$= \left(\sum_{i=1}^{4} m_i^2\right) + 2s - \sum_{i=1}^{4} m_i^2 + 2t - \sum_{i=1}^{4} m_i^2 + 2u - \sum_{i=1}^{4} m_i^2$$

$$\Rightarrow \boxed{s + t + u = \sum_{i=1}^{4} m_i^2}$$

Thus u (say) can be eliminated.

$\Rightarrow$ 2 variables: $\boxed{s, t}$

Now, in CM frame $s = (\underbrace{E_A + E_B}_{E_{cm}}, \vec{0})^2$

$$\Rightarrow \boxed{s = E_{cm}^2}$$

t must be a function of E_{cm} and θ_{cm}

$$\boxed{t = f(E_{cm}, \theta_{cm})}$$

- In the ~~...~~ case where ~~...~~ all particles are massless, this relation is particularly simple:

$$p_A = E(1, 0, 0, 1)$$
$$p_B = E(1, 0, 0, -1)$$
$$p_C = E(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$p_D = E(1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta)$$
$$(t \quad x \quad y \quad z)$$

$E = \frac{E_{cm}}{2}$

$$\Rightarrow t = (p_A - p_C)^2 = -2 p_A \cdot p_C = -2E^2(1-\cos\theta)$$
$$= -2\left(\frac{E_{cm}}{2}\right)^2 (1-\cos\theta)$$

$$\Rightarrow \boxed{t = -\tfrac{1}{2} E_{cm}^2 (1-\cos\theta) = -\tfrac{1}{2} s (1-\cos\theta)}$$

$$\boxed{u = -\tfrac{1}{2} s (1+\cos\theta)}$$

2 Body Decay is Simpler

π at rest

Ex. 3.3 of Griffiths, $\pi^+ \to \mu^+ \nu_\mu$

$m_{\nu_\mu} \simeq 0$. What is velocity of muon? (Also momentum)

Use 4-vectors, $p_{\nu}^2 = m_\nu^2$

$\Rightarrow$ rewrite $p_\pi = p_\mu + p_\nu$ as $p_\nu = p_\pi - p_\mu$, square it

$$\Rightarrow \quad 0 = m_\nu^2 = (p_\pi - p_\mu)^2 = m_\pi^2 - 2 p_\pi \cdot p_\mu + m_\mu^2$$

$p_\pi = (m_\pi, \vec{0})$
$p_\mu = (E_\mu, \vec{p}_\mu)$ $\Rightarrow$ $(p_\pi \cdot p_\mu = m_\pi E_\mu)$

$$\Rightarrow \quad \boxed{E_\mu = \frac{m_\pi^2 + m_\mu^2 - m_\nu^2}{2 m_\pi}}$$

$$p_\mu = \sqrt{E_\mu^2 - m_\mu^2} = \sqrt{\frac{(m_\pi^2 + m_\mu^2 - m_\nu^2)^2 - 4 m_\pi^2 m_\mu^2}{4 m_\pi^2}}$$

for $m_\nu = 0$

$$\Rightarrow \quad p_\mu = \frac{m_\pi^2 - m_\mu^2}{2 m_\pi}$$ ← (measure with $\vec{B}$ field in detector)

$$v_\mu = p_\mu / E_\mu = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}$$

$m_\pi = 140$ MeV
$m_\mu = 106$ MeV

$\Downarrow$

$v_\mu = 0.271$ ← (semi-relativistic)

~~For case where $m_\nu^2 \neq 0$ see Problems Intro. Chap. 1 of Bettini~~

- Note that v_μ, p_μ, E_μ are all <u>fixed</u> by the particle masses in a <u>2-body decay</u>. "Mono-energetic muon".
- In a <u>3-body decay</u>, like $^{60}Co \to {}^{60}Ni + e^- + \bar{\nu}_e$ ($N \to N' + e^- + \bar{\nu}_e$) this is not true: electron can have a <u>spectrum of different energies</u>. Key to Pauli's proposal of the neutrino in 1930.

Use of CM frame

Ex. 3.4 of Griffiths

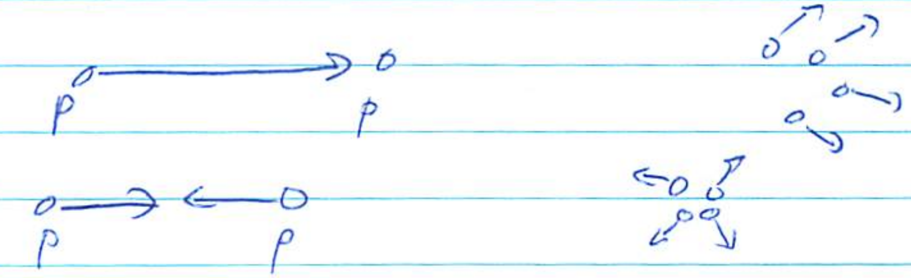
Bevatron at Berkeley built to produce antiprotons.

$$\text{(beam)} \rightarrow p + \underset{\uparrow \text{rest}}{p} \rightarrow p + p + p + \bar{p}$$

What is minimum beam energy E^{min} required?

Lab frame

CM frame



- Minimum beam energy $\Longleftrightarrow$ minimum CM energy

$\Longleftrightarrow$ all particles at rest in CM (no wasted K.E.)

$\Rightarrow E_{cm}^{min} = 4m_p$

Now we compute $s = E_{cm}^2$ before collision, in lab frame.

$$s = \left((E,\vec{p}) + (m,\vec{0})\right)^2 = (E+m,\vec{p})^2 = (E+m)^2 - p^2$$

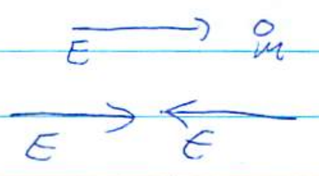
$$= \cancel{E^2} + 2m_pE + m_p^2 - \cancel{p^2}(\cancel{E^2} - m_p^2) = 2m_p(E+m_p)$$

So, we have $2m_p(E^{min} + m_p) = 16m_p^2$

$\Rightarrow \boxed{E^{min} = 7m_p}$

- For $E \gg m$, $s \simeq 2E_{beam}m_{target}$ for fixed target expt.
- In colliding beams, ~~[illegible]~~

$\boxed{s = 4E_{beam}^2} \gg 2E_{beam}m_{target}$



e.g. Tevatron 1 TeV + 1 TeV $\Longleftrightarrow$ 2000 TeV fixed target !!!

Remark:

Transverse momenta (with respect to z) are boost-invariant for a boost along z:

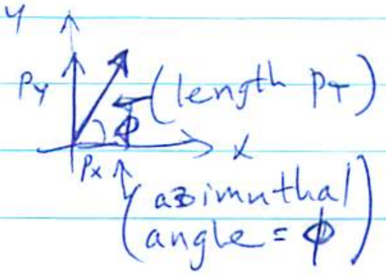
$$E' = \gamma(E - \beta p_z)$$

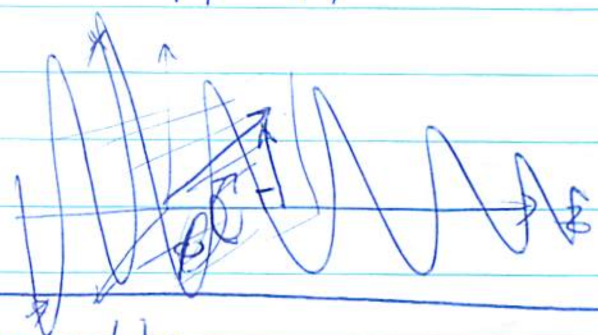
$$p_z' = \gamma(-\beta E + p_z)$$

$$p_x' = p_x$$

$$p_y' = p_y$$

$$\boxed{p_T \equiv \sqrt{p_x^2 + p_y^2}}$$





• A better longitudinal variable is sometimes the rapidity

$$y = \frac{1}{2} \ln\left(\frac{E+p_z}{E-p_z}\right)$$

under a boost

$$y \to y' = \frac{1}{2}\ln\left(\frac{E'+p_z'}{E'-p_z'}\right) = \frac{1}{2}\ln\left(\frac{\gamma(1-\beta)(E+p_z)}{\gamma(1+\beta)(E-p_z)}\right)$$

$$\boxed{y' = y + \frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right)}$$

(p_T, η, ϕ) Used often in hadron colliders, where you don't know the CM frame of the quark collisions