Lecture 11 (see Lecture 10 too)

Consequences of Nonabelian Continuous Symmetries:

1) (Quasi) Degenerate Multiplets – if representation is nontrivial.

If $[H, \vec{I}] = 0$, and $H|\psi\rangle = E|\psi\rangle$
$= m|\psi\rangle$ (for $\vec{p}=0$ particle state)

then $H\vec{I}|\psi\rangle = \vec{I}H|\psi\rangle$
$= m\vec{I}|\psi\rangle$

So $\vec{I}|\psi\rangle$ and $|\psi\rangle$ are states with same mass

Suppose $\vec{I}$ = SU(2) generators (isospin)

Then $\{|-I\rangle, |-I+1\rangle, \ldots, |+I\rangle\}$

is an isospin I multiplet of (almost) degenerate states. E.g pions

(symmetry not exact)

$\pi^-(139.6)$ $\quad$ $\pi^0(134.98)$ $\quad$ $\pi^+(139.6)$

$|I=1, I_3=-1\rangle$ $\quad$ $|I=1, I_3=0\rangle$ $\quad$ $|I=1, I_3=+1\rangle$

$|\bar{u}d\rangle$ $\quad$ $\frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$ $\quad$ $|u\bar{d}\rangle$

kaons

$K^0(497.61)$ $\quad$ $K^+(493.7)$

$|I=\frac{1}{2}, I_3=-\frac{1}{2}\rangle$ $\quad$ $|I=\frac{1}{2}, I_3=+\frac{1}{2}\rangle$

$|d\bar{s}\rangle$ $\quad$ $|u\bar{s}\rangle$

n-p baryons

$n(939.6)$ $\quad$ $p(938.3)$

$|udd\rangle$ $\quad$ $|uud\rangle$

- Symmetry really due to $\underset{|d\rangle}{|I=\frac{1}{2}, I_3=-\frac{1}{2}\rangle}$ $\underset{|u\rangle}{|I=\frac{1}{2}, I_3=+\frac{1}{2}\rangle}$ at quark level
- Broken by $m_d > m_u$ $\Rightarrow$ $H = H_0 + H_1$

(i.e. isospin preserving) $\quad$ (small, $\propto I_3$)

(11.2)

(total ang. mom.) $\rightarrow$ $\boxed{\vec{F} = \vec{J} + \vec{S}_N = \vec{L} + \vec{S}_e + \vec{S}_N}$

Analogy to atomic physics:

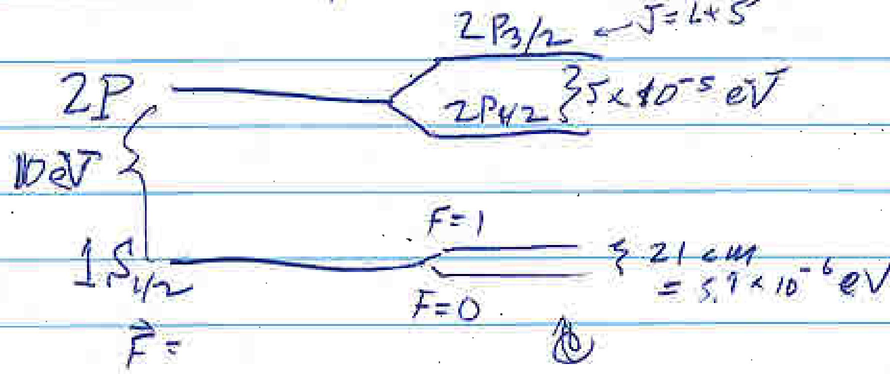
$[H, \vec{F}] = 0$, so all atomic states are in $|F, F_3\rangle$ multiplets.

But $[H, \vec{S}] \neq 0 \neq [H, \vec{L}]$,

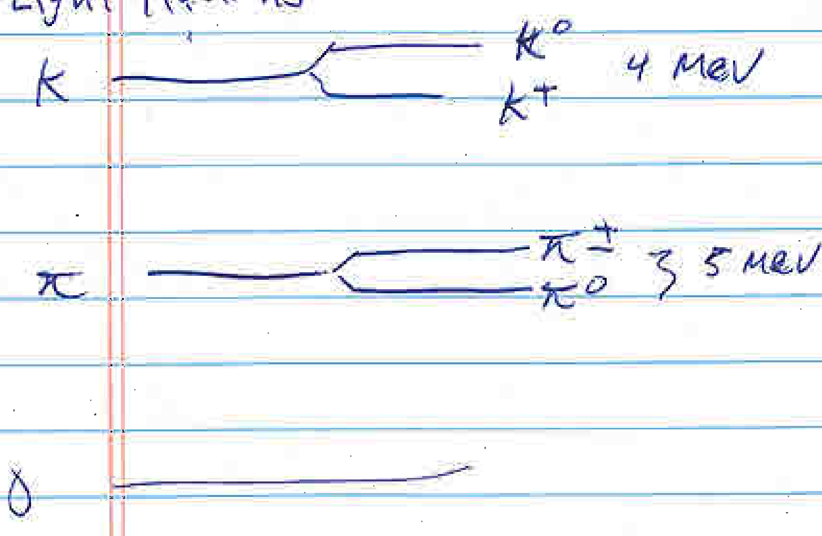
due to $\overset{\text{small}}{\lambda}\, \vec{L}_e \cdot \vec{S}_e$ terms (fine structure)

& even smaller $\vec{S}_N \cdot \vec{S}_e$ terms (hyperfine structure)

Hydrogen:



The Light Hadrons



2) (a) Selection Rules — what transitions are allowed with large rates,

(b) Relations between rates

[≈Wigner-Eckart thm. for SU(2) rotations]

$$\langle jm | \underbrace{T^k_q}_{\text{rank } k \text{ sph. tensor}} | j'm' \rangle = \underbrace{\langle j \| T^k \| j' \rangle}_{\substack{\text{reduced matrix element}\\ \text{- same for all } m,\, m'}} \underbrace{C^{jm}_{kq\, j'm'}}_{\langle j'm';\, kq | jm \rangle}$$

$\Rightarrow$ $J = k$ operator, $J_3 = q$

Clebsch-Gordan group theory coeff. for adding $j' \oplus k \to j$

- Zero unless the addition of angular mom. j' $\oplus$ ang. mom. k contains ang. mom. j.
- Recall rules for adding $\vec{J} = \vec{J}_1 + \vec{J}_2$

$\vec{J}_1$: $|J_3 = j_1\rangle$, $\vec{J}_2$: $|J = j_2\rangle$

get

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, (j_1 + j_2) - 1, j_1 + j_2$$

~~Note that # of states in tensor prod~~

Example: Low lying mesons $(q\bar{q})$ have $L = 0$

$$q\,(\text{spin } \tfrac{1}{2}) + \bar{q}\,(\text{spin } \tfrac{1}{2}) \Rightarrow \tfrac{1}{2} \oplus \tfrac{1}{2} = 0 \text{ or } 1$$

total # of states: $2 \times 2 = 1 + 3$

Spin $S = 0 \Rightarrow \pi, K, \eta \Leftrightarrow J^{P(C)} = 0^{-(+)}$ pseudoscalar mesons

Spin $S = 1 \Rightarrow \rho, K^*, \omega \Leftrightarrow J^{P(C)} = 1^{-(-)}$ vector mesons ← (opposite behavior under $q \leftrightarrow \bar{q}$)

Baryons (qqq) with $L=0$

$$\Rightarrow \left(\tfrac{1}{2}+\tfrac{1}{2}\right)+\tfrac{1}{2} = (0+1)+\tfrac{1}{2}$$

$$= \tfrac{3}{2} + \tfrac{1}{2} + \tfrac{1}{2}$$

$M=1232$ MeV

Δ^- Δ^0 Δ^+ Δ^{++} all have $J=\tfrac{3}{2}$

n p Σ^- Λ Σ^0 Σ^+ Ξ^- Ξ^0 have $J=\tfrac{1}{2}$

(not actually allowed, due to Fermi statistics + color)

might think it shouldn't be allowed, by Fermi statistics:

Looks like a symmetric state of 3 fermions.

$$\Delta^{++}\left(J=\tfrac{3}{2},\ J_3=+\tfrac{3}{2}\right) = |u^\uparrow u^\uparrow u^\uparrow; L=0\rangle$$

The "secret" is that quarks also have color, which can take on any of 3 values

$$\Rightarrow \Delta^{++} = |\epsilon^{ijk} u_i^\uparrow u_j^\uparrow u_k^\uparrow;\ L=0\rangle$$

$i,j,k = 1,2,3$

$$\epsilon^{ijk} = \begin{cases} +1 & \text{even perm's of } 1,2,3 \\ -1 & \text{odd perm's " "} \end{cases}$$

totally antisymmetric in the hidden color index

- Clebsch-Gordon coefficients in Griffiths notation:

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{j=|j_1-j_2|}^{(j_1+j_2)} C^{j\,j_1\,j_2}_{m\,m_1\,m_2} |j\,m\rangle \qquad m=m_1+m_2$$

Tabulated.

Can also compute, starting from "highest weights"

Ex: $\tfrac{1}{2}+\tfrac{1}{2}$

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle |\tfrac{1}{2}\tfrac{1}{2}\rangle = |1\,1\rangle$$

$J=J_1+J_2$ $\Rightarrow \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + |\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = |1\,0\rangle$

$$|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle = |1\,{-1}\rangle$$

And $\frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - |\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) = |0\,0\rangle$

• Same exact techniques work for isospin.
(Heisenberg)

$$\begin{cases} p = |I=\tfrac{1}{2}\ I_3=\tfrac{1}{2}\rangle & \text{isospin} \uparrow \\ n = |I=\tfrac{1}{2}\ I_3=-\tfrac{1}{2}\rangle & \text{isospin} \downarrow \end{cases}$$

etc.

• We can relate reaction rates using the iso-analog of the Wigner-Eckart Theorem, and the same [SU(2)] Clebsch-Gordon coeff's.
• Can do the same for SU(3) as well.

Ex: $p + p \to d + \pi^+$ $\qquad \sigma$

$p + n \to d + \pi^0$ $\qquad \frac{1}{2}\sigma$

$n + n \to d + \pi^-$ $\qquad \sigma$

$\underbrace{I=\tfrac{1}{2}\quad I=\tfrac{1}{2}}$ $\quad I=0 \quad I=1 \quad \Rightarrow I_{final}=1$

must be in $I=1$ for reaction to go

$$|pn\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right)$$

I.e. $\langle \tfrac{1}{2}\,\tfrac{1}{2};\, \tfrac{1}{2}\, -\tfrac{1}{2};\, 1,0\rangle = \frac{1}{\sqrt{2}}$
in amplitude
– square to get $\frac{1}{2}$ in rate