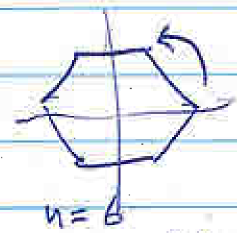
Lecture 10 – Symmetries

(Bettini Ch 3
Griffiths Ch 4)

General types of symmetries

EXAMPLES	Discrete (multiplicative) $\frac{2\pi}{3}$	Continuous (any θ)
Spacetime	Parity P (±) Charge Conjugation C(±) Time Reversal T(±) [Theorem: CPT is a good symmetry]	3d Rotations SO(3) + Lorentz Boosts Lorentz Group SO(1,3)
Internal "Global"	· No good examples in standard Model. · Models of dark matter often have an internal discrete sym. particles even/odd ⇒ Lightest odd particle stable.	"Abelian" U(1) ~ 1d rotation ⇔ "Discrete additive" of Bettini · electric charge · baryon (quark) number · lepton number · strangeness · charm (violated by weak int.)
~~?~~	(even fewer examples	Nonabelian Isospin u⇔d SU(2) 3-flavor generalization u ⇔ d, d ⇔ s, u ⇔ s SU(3) "flavor" (contains "strangeness")
Internal "Local" or "Gauge"		All the elementary particle forces: electromagnetism $U(1)_{EM}$ weak interactions $SU(2)_{weak}$ QCD $SU(3)_{color}$

• Examples of "Discrete Multiplicative" Groups provided by symmetries of n-sided polygon



symmetric under rotation by $\theta = \frac{2\pi}{n}$

$\Leftrightarrow$ Group element $g = e^{\frac{2\pi}{n} i}$ under complex multiplication

. Group elements $G = \{1, g, g^2, \ldots, g^{n-1}\}$ $\quad g^n = 1$

$G = \mathbb{Z}_n$ $\quad g^{n+k} = g^k$, $k \in [0, \ldots, n-1]$

$\mathbb{Z}_n$ is "abelian" (commutative)

$g_1 g_2 = g_2 g_1$ for all g_1, g_2.

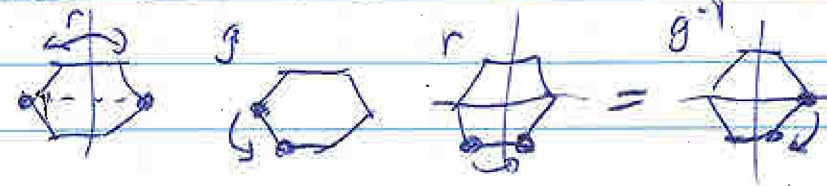
• By definition, all groups are:
• associative $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
• have identity $1 \cdot g = g \cdot 1 = g$.
• have inverse: For all $g \in G$, $\exists\, g^{-1} \in G$ such that $g g^{-1} = g^{-1} g = 1$

. If any two elements do not commute, i.e. if $g_1 g_2 \neq g_2 g_1$ then group is nonabelian

. Example of a nonabelian discrete group:

* All the symmetries of an n-sided polygon
– this includes a "flip" (reflection) symmetry, r which doesn't commute with rotations:

$r^2 = 1$ $\quad r^{-1} = r \rightarrow r g r^{-1} = g^{-1}$



$G = \{1, g, g^2, \ldots, g^{n-1}, r\}$
($G = D_n$ (dihedral group))

- Nonabelian groups can usually be represented by matrix multiplication.
- Our examples of D_n:

General 2d rotation:
$$\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$
$$x'^2 + y'^2 = x^2 + y^2$$

$$g = \begin{pmatrix}\cos\frac{2\pi}{n} & \sin\frac{2\pi}{n}\\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n}\end{pmatrix} \equiv \begin{pmatrix}c & s\\ -s & c\end{pmatrix}$$

$$r = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

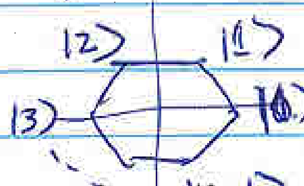
$$rgr^{-1} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}c & s\\ -s & c\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}c & -s\\ +s & c\end{pmatrix}$$
$$= \begin{pmatrix}\cos(-\frac{2\pi}{n}) & \sin(-\frac{2\pi}{n})\\ -\sin(-\frac{2\pi}{n}) & \cos(-\frac{2\pi}{n})\end{pmatrix}$$

✓ $rgr^{-1} = g^{-1}$

$$rg = g^{-1}r \neq gr$$

~~Subgroup of O(2) 2×2~~

- Abelian groups have simple 1-dimensional representations



Take linear combinations in order to Build eigenstates of g

Under g,
$$|1\rangle \to |2\rangle \to \dots \to |n-1\rangle \to |0\rangle$$

$$\sum_{k=0}^{n-1} e^{-2\pi i\frac{km}{n}}|k\rangle \to \sum_{k=0}^{n-1} e^{-2\pi i\frac{km}{n}}|k+1\rangle$$
$$= e^{+2\pi i\frac{m}{n}}\left[\sum_{k'=0}^{n-1} e^{-2\pi i\frac{k'm}{n}}|k'\rangle\right] \qquad (k' \equiv k+1)$$

possible eigenvalues of g

Note that $G = Z_n \Leftrightarrow$ addition modulo n

$m = 0, 1, 2, \ldots, n-1$

are n possible "charges".

Two important special cases:

multiplication

$n=2$: $G = \{1, g\}$, $e^{2\pi i m/n} = \begin{cases} +1 & \text{"even"} \\ -1 & \text{"odd"} \end{cases}$

Examples

Parity P, C, T: $g^2 = 1$ $\Leftrightarrow$ (weak)

$\boxed{\text{Addition mod 2}}$

$n \to \infty$: $g = e^{i\theta}$ any θ "$U(1)$"

- Charges obey Addition with no restriction
  - electric charge Q $\qquad \sum_i Q_i = \sum_i Q_f$
  - baryon number B
  - strangeness S ← (turning off weak interactions)

- For nonabelian groups, ~~all~~ most representations are multi-dimensional.
- For example, r mixes the $e^{\frac{2\pi i m}{n}}$ and $e^{-\frac{2\pi i m}{n}}$ eigenvectors of g, because $rg \neq gr$. (Not simultaneously diagonlizable.)

(3d) Rotation group: $[J_i, J_j] = i\epsilon_{ijk} J_k$ $\quad (SU(2))$

- Diagonalize J_3, form raising/lowering operators $\vec{J}$ $\quad [J^{\pm}, J_3] = \pm J$
- Spin j state:

$J^{\pm}$

$|-j\rangle \quad |-j+1\rangle \cdots\cdots |j-1\rangle \quad |+j\rangle$ $\qquad J^+|j\rangle = 0$

The most complicated non-abelian group we need for the Standard Model is $SU(3)$

$U \in SU(3)$ special, unitary, 3×3 complex matrices

if $U^\dagger U = \underline{1}$

$(U^{-1} = U^\dagger)$

$2\times(3\times3) = 18$ parameters

9 relations

and $\det U = \underline{1}$

1 relation

$\Rightarrow 18 - 9 - 1 = 8$ parameters (vs. 3 for $SU(2)$ rotation group)

$\Rightarrow$ 8 generators of the symmetry

2 can be diagonalized at the same time.

• We call them I_3 ← 3rd component of isospin

and Y "hypercharge" - related simply to strangeness by

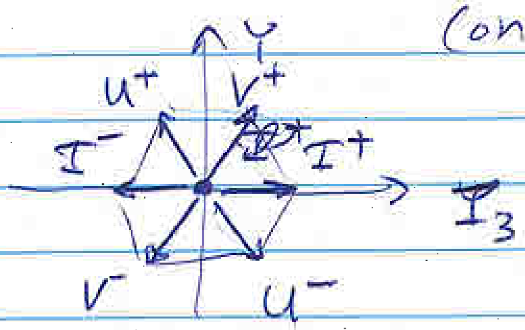
$Y = B + S$

(baryon #) (strangeness)

• Representations of $SU(3)$ fill out points in a 2-dimensional space of $(I_3, \hat{Y})$ eigenvalues (on a hexagonal lattice)

$\boxed{\hat{Y} \equiv \frac{\sqrt{3}}{2} Y}$

$Y = \begin{pmatrix} \frac{1}{3} & & \\ & \frac{1}{3} & \\ & & -\frac{2}{3} \end{pmatrix}$



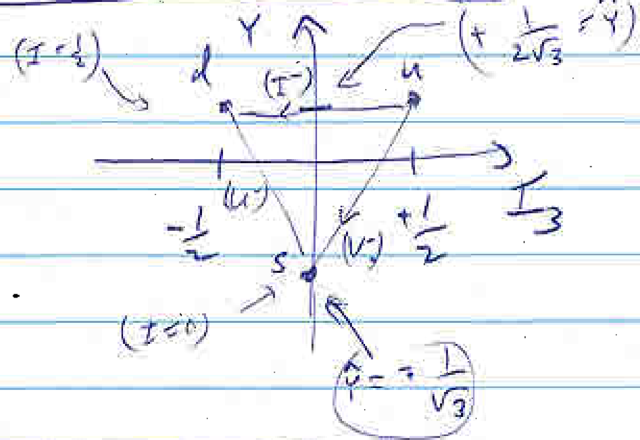
$I_3 = \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$

$I^+ = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ raises I_3 leaves Y alo

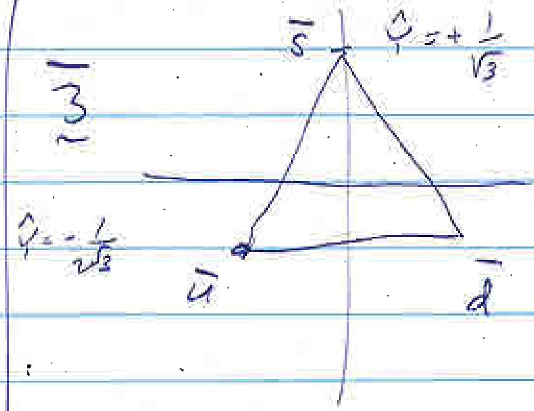
$\hat{Y} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$ $V^+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $U^+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

simplest nontrivial rep's

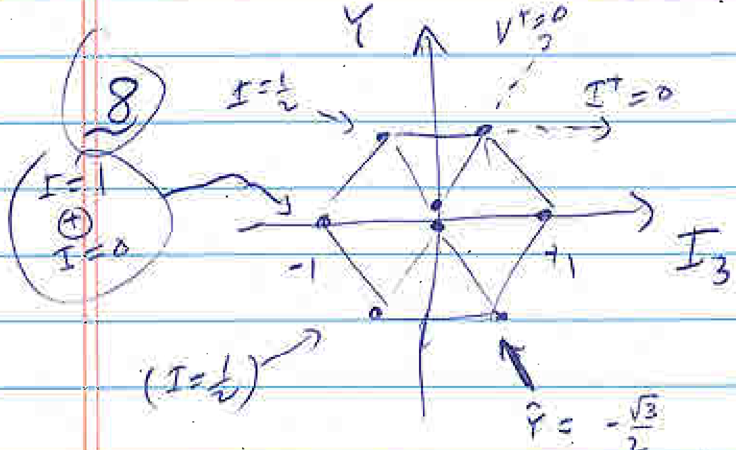
$\underset{\sim}{3}$



$\underset{\sim}{\bar{3}}$



$\underset{\sim}{8}$

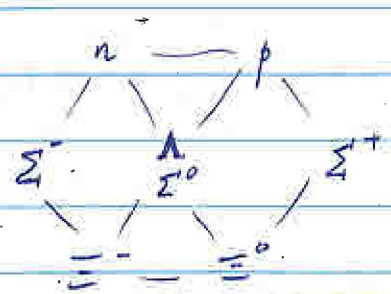


$$\underset{\sim}{8} = \underset{\sim}{3} \otimes \underset{\sim}{\bar{3}} - \underset{\sim}{1}$$

~~K^0~~ K^0, K^+

mesons $|q_i \bar{q}_j\rangle \to \pi^\pm \pi^0, K^\pm, \{\eta, \eta'\}$ π^-, π^0, π^+

spin $\frac{1}{2}$ Baryons $|q_i q_j q_k\rangle$ $\bar{K}^-$ $\bar{K}^0$



Now, examples in more detail

$\boxed{\text{Parity } P}$ $\quad P^2 = 1$

Violated only by weak interactions

$\vec{p} \to -\vec{p}$ 3d vector

$\vec{L} = \vec{r} \times \vec{p} \to +\vec{L}$ 3d axial vector

$\vec{s} \to +\vec{s}$

$(\vec{p} = \vec{0})$

- A single particle at rest can have a definite (intrinsic) parity, $P = \pm 1$. ← [We "turn off" weak decays for this purpose.]
- Unambiguous for bosons.
- For fermions, always make in pairs ($\vec{J}$ conservation)

so we can only define relative parities, with respect to proton.

By convention $\boxed{P(p) = +1}$.

- For photon, produced by $\vec{A}$, $P: \vec{A} \to -\vec{A}$

$\Rightarrow \boxed{P(\gamma) = -1}$

- Anti-proton: Can make $p\bar{p}$ from virtual photon through parity conserving process $\gamma^* \to p\, \bar{p}$

$P: \quad -1 \quad +1 \quad (-1)$

$\Rightarrow \boxed{P(\bar{p}) = -1}$

- Similarly, at quark level we will take $P(q) = +1$, and then $\gamma^* \to q\bar{q} \Rightarrow \boxed{P(\bar{q}) = -1}$
- What about a 2-particle system?
- Best to use states of definite relative angular momentum $|l, m\rangle$:

$|\vec{p}, -\vec{p}\rangle = |\vec{p}, \theta, \phi\rangle$

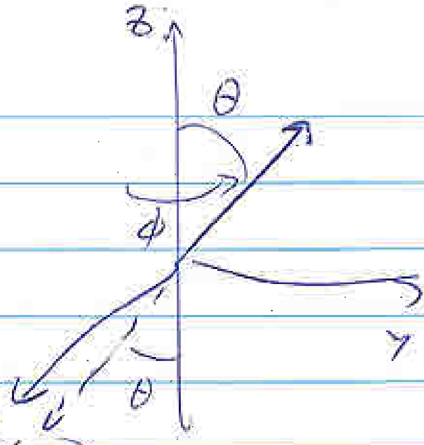
$$|p, l, m\rangle = \sum_{\theta,\phi} |p, \theta, \phi\rangle\langle p, \theta, \phi | p, l, m\rangle = \sum_{\theta,\phi} \underbrace{Y_l^{*m}(\theta, \phi)}_{\text{spherical harmonics}} |\vec{p}, -\vec{p}\rangle$$

(10.8)

under P: $\vec{x} \to -\vec{x}$

$\theta \to \pi - \theta$

$\phi \to \phi + \pi$



and

$$Y_\ell^{m}(\theta,\phi) \to \underbrace{Y_\ell^{m}(\pi-\theta, \phi+\pi) = (-1)^\ell}\, Y_\ell^{m}(\theta,\phi)$$

can look this up, or use the fact that Y_ℓ^m's come from polynomials of degree ℓ in (x,y,z) (after dividing out $r^\ell = (\sqrt{x^2+y^2+z^2})^\ell$)

$$P_\ell(x,y,z) \to P_\ell(-x,-y,-z) = (-1)^\ell P_\ell(x,y,z)$$

$$\therefore \; P|p,\ell,m\rangle = \boxed{(-1)^\ell \cdot P_1 P_2}\, |p,\ell,m\rangle \qquad (*)$$

intrinsic parities of the two particles.

- Since P is symmetry of strong interactions, and $\pi^- \underset{I^\pm}{\longleftrightarrow} \pi^0 \underset{I^\pm}{\longleftrightarrow} \pi^+$ are in the same isospin multiplet,

we should have $P(\pi^-) = P(\pi^0) = P(\pi^+)$

$\Rightarrow$ Any 2 pion state, according to $(*)$ has parity $(-1)^\ell (P(\pi))^2 = (-1)^\ell$ (ℓ = rel. ang. mom.)

- Since π's are spin-less $(S=0)$,

$\vec{J} = \vec{L} + \vec{S} \Rightarrow J = \ell$. $\Rightarrow$ $J^P = 0^+, 1^-, 2^+, 3^-, \ldots$

If both π's have same charge, Bose statistics $\Rightarrow$ ℓ even $\Rightarrow$ $J^P = 0^+, 2^+, \ldots$

- For fermion anti-fermion pair, opposite intrinsic parities $\Rightarrow$ $\boxed{P(f\bar{f}) = (-1)^{\ell+1}}$

- If f has spin $\frac{1}{2}$, $\quad \frac{1}{2} \oplus \frac{1}{2}$

then for $\ell = 0$, $J = S = 0$ or 1

$\Rightarrow J^P = 0^-$ for ${}^1S_0$ ← spectroscopic notation

$J^P = 1^-$ for ${}^3S_1$

most light, narrow mesons

also lowest $c\bar{c}$ states η_c, J/ψ

$\boxed{{}^{2S+1}L_J}$

Mnemonic for L

$L=0$	Sober
1	Physicists
2	Don't
3	Find
4	Giraffes
:	Hiding
:	In
:	Kitchens
:	Like
:	Mine

for $\ell = 1$ $\quad S = 0$ or 1 couples to $\ell = 1$

$\Rightarrow J = 1, \; 0, 1, 2$

"h_c" $\Rightarrow J^P = 1^+ \quad {}^1P_1$

χ_c $\begin{cases} J^P = 0^+ & {}^3P_0 \\ J^P = 1^+ & {}^3P_1 \\ J^P = 2^+ & {}^3P_2 \end{cases}$

more $c\bar{c}$ examples)

Can test for (limit) parity non-conservation in strong interactions by observing (or rather not observing

$$p + {}^{19}F \rightarrow [{}^{20}Ne^*] \rightarrow {}^{16}O + \alpha \quad \leftarrow {}^4He$$

$$\underbrace{\tfrac{1}{2}^+ \quad \tfrac{1}{2}^+}_{\ell=0 \;\Rightarrow\; P=+} \quad \uparrow \text{allowed by } P \qquad 1^+ \;(p=+) \quad \longrightarrow \quad \underbrace{0^+ \quad 0^+}_{\ell=1,\; S=0} \Rightarrow \boxed{P = (-)(+)(+) = (-)}$$

forbidden by P

$$\Rightarrow |M_{\not{P}} / M_{P\text{-cons}}|^2 \lesssim 10^{-8}$$

Evidence that $P(\pi) = -1$

Capture in ~~liquid~~ deuterium, observe

$$\pi^- \; d \rightarrow n + n$$

$$0^{P_\pi} \quad 1^+ \quad \uparrow \quad 1^+ \quad 1^+$$

strong interactions

2 nucleons in $L=0$ state ($S=1$) ($J=1$)

Using Fermi symmetry

$S=1 \longleftrightarrow L=1, 3, \ldots$ (sym, antisym)

$S=0 \longleftrightarrow L=0, 2, 4, \ldots$ (antisym, sym)

Assume for the moment that $L=0$

then $J_{initial} = 1$, $P_{initial} = P_\pi$

$\Rightarrow J_{final} = 1$. Only way to make is using $S=1$, $L=1$

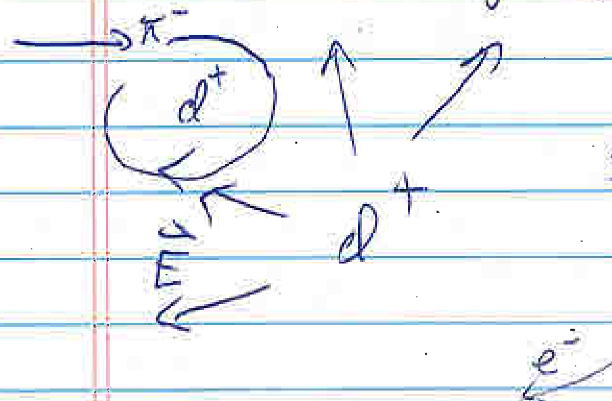
$\Rightarrow P_{final} = (-1)^\ell (P_n)^2 = -1$

$\Rightarrow \underline{P_\pi = -1}$

Why is $L=0$ for $\pi^- d$ annihilation?

· Because of "Stark Mixing"

· Starts at high (n, ℓ). ~~Stark mixin~~



E field near another d nucleus mixes S, P, D, F, ... states

S has good overlap with nucleus

→ capture dominated by S-wave.

Charge conjugation C

- In the problem set, you saw how $C: \psi \to i\gamma^2\psi^*$ exchanges particle $\longleftrightarrow$ antiparticle ($u \Leftrightarrow v$)
- Changes sign of all charges: $Q, B, S, \ldots$

$C^2 = 1 \Rightarrow$ e.v.'s $C = \pm 1$

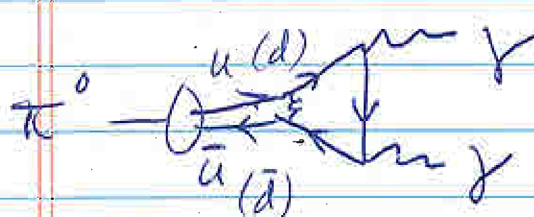
- We also argued that $C(\gamma) = -1$

$$C(n\gamma) = (-1)^n$$

- Note that if n is odd, this is another argument for Furry's Thm in QED:
- Amplitude for odd number of photons (only) must vanish (violates C conservation)

C: $(+1)$ $\quad(-1)(-1)$

- Now $\pi^0 \to \gamma\gamma$ (main decay mode, electromagnetic)



analog of $e^+e^- \to \gamma\gamma$ but from $q\bar{q}$ bound state

$\Rightarrow C(\pi^0) = +1 \qquad C|\pi^0\rangle = +|\pi^0\rangle$

$\pi^\pm$ exchange under C: $C|\pi^\pm\rangle = +|\pi^\mp\rangle$

infer from isospin
$(u\bar{u}) \to +(u\bar{u})$
$\to u\bar{d} \to +(\bar{u}d)$

- The other light, non-strange meson is the η(548 MeV)

$\eta \to \gamma\gamma$ also, so $C|\eta\rangle = +|\eta\rangle$

$J^{PC} = 0^{-+}$ for light neutral mesons (pseudoscalar)