

**Problem Set 2** — due April 30

1. From Maxwell's equations in free space, derive the wave equation for the electric field vector (for  $c = 1$ ),

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}.$$

Now consider an infinite, perfectly conducting, cylindrical waveguide of radius  $R$  (as in class). Using the form of the Laplacian in cylindrical coordinates, insert the  $TM_{0,1,0}$  ansatz  $E_z = E_0 f(r) e^{i(\omega t - kz)}$  into the equation, and obtain a differential equation for  $f(r)$ . Solve it in terms of the Bessel function  $J_n(u)$ , which obeys

$$\frac{d^2 J_n}{du^2} + \frac{1}{u} \frac{dJ_n}{du} + \left(1 - \frac{n^2}{u^2}\right) J_n = 0.$$

(The other solution to this equation,  $Y_n(u)$ , has  $Y_n(0) = \infty$  and so is unphysical.) Using the fact that the first zero of the Bessel function is at  $u = 2.405$ , find the relation between  $\omega$  and  $k$ , and from that, compute the phase velocity as a function of frequency,  $v_p(\omega) = \omega/k$ . Show that it is always greater than  $c$ ; so this geometry has to be modified for a particle accelerator. What is the minimum frequency  $\omega_c$  admitting a solution? What is the value of  $\nu_c \equiv \omega_c/(2\pi)$  in Hz if  $R = 5$  cm?

2. The PEP-II high energy ring (HER) contains electrons with  $E = 9$  GeV, 1658 bunches with  $2.7 \times 10^{10}$  particles per bunch. What should the energy of the low energy ring (LER) be, to obtain a center-of-mass energy equal to the  $\Upsilon(4S)$  resonance,  $M_{\Upsilon(4S)} = 10.56$  GeV? The LER also contains 1658 bunches, but with  $5.9 \times 10^{10}$  particles per bunch. The circumference of both rings is 2200 m. What is the current, in amps, flowing in each ring? Use the formula for energy lost per particle per revolution to synchrotron power,

$$\Delta E_{loss} = \frac{4\pi\alpha}{3} \frac{E^4}{m^4 r},$$

to compute this loss in MeV in the HER, and in the LER, assuming uniform bending around the ring. Now compute the total synchrotron power loss in each ring, in watts. (Actually, the bending in the LER is *lumped* into short magnets plus straight sections, so the power loss is larger than your estimate. Any idea why one would want to do that?)

Finally, work out the total synchrotron power loss in watts from the proton ring at the Tevatron (circumference 6300 m,  $E = 960$  GeV, 36 bunches,  $2.7 \times 10^{11}$  particles per bunch). If one could store the same number of electrons in the Tevatron, at the same magnetic field and hence the same beam energy, what would the total power loss be? (Maybe the units here should be nuclear power plants,  $\sim 1$  GW per plant!)

3. Show that the transfer matrix in the focusing plane of a quadrupole,

$$M_x = \begin{pmatrix} \cos(kl) & k^{-1} \sin(kl) \\ -k \sin(kl) & \cos(kl) \end{pmatrix},$$

can be written as a thin focusing lens,

$$M_{convergent} = \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix}$$

sandwiched between two drift regions,

$$M_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

Work to leading order in  $kl \ll 1$  for each term in the matrix. What are  $z_1$ ,  $f$ , and  $z_2$ ? Now repeat the exercise for the defocusing plane, with

$$M_y = \begin{pmatrix} \cosh(kl) & k^{-1} \sinh(kl) \\ k \sinh(kl) & \cosh(kl) \end{pmatrix}, \quad \text{and} \quad M_{divergent} = \begin{pmatrix} 1 & 0 \\ f^{-1} & 1 \end{pmatrix}.$$

Show that a focusing quad doublet focuses. That is, show that the product  $M_{divergent}M_zM_{convergent}$  is convergent, *i.e.* that it can be written as  $M_{z'}M_{convergent'}M_{z''}$  for some drifts  $z'$  and  $z''$ . Same problem for the other plane, which sees  $M_{convergent}M_zM_{divergent}$ . What is the new focal length  $f'$ , in terms of  $f$  and  $z$ ?

4. Problem 4.11 of Griffiths.
5. Problem 4.27 of Griffiths.
6. Problem 4.32 of Griffiths.