Physics 152/252
Lecture 9: Quantum Electrodynamics (QED)
[See Griffiths, Chap. 7]

Wave equations

Nonrelativistic particle: Classically, $E = K.E. + P.E. = \frac{1}{2}mv^2 + V$

$$\boxed{E = \frac{\vec{p}^2}{2m} + V}$$

In QM, $[x, p] = i\hbar$, or $\boxed{\vec{p} = \frac{\hbar}{i}\vec{\nabla}}$

and $E \to i\hbar\frac{\partial}{\partial t}$

$\therefore$ Obtain $\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi}$

as Schrödinger eq. for nonrel. particle moving in a potential.

Klein-Gordon eq. for relativistic particle with spin 0:

- Derived in similar way from relativistic energy-mom relation: $E^2 - \vec{p}^2 = m^2$
- Don't want square-roots, so don't solve for $E = \sqrt{p^2+m^2}$.
- Above substitutions ($\hbar=1$) $\Rightarrow$ $\boxed{-\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = m^2\psi}$

note resemblance to eqn for $\vec{E}(t, x)$ with no matter in free space

[Klein-Gordon eq.]

or $\boxed{-\partial^\mu\partial_\mu\psi - m^2\psi = 0}$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

- Ironically, Schrödinger rejected this equation as incompatible with $|\psi|^2_{(x,y,z,t)}$ being prob. of finding particle at x, y, z, t

$\Rightarrow \int d^3x\,|\psi|^2$ const. Only $J_\mu = \psi^*\partial_\mu\psi - (\partial_\mu\psi^*)\psi$ is "conserved", $J^0 = \psi^*\partial_0\psi - (\partial_0\psi^*)\psi$, $\partial_0\int d^3x\,J_0 = 0$

$\boxed{\text{Aside}}$ "Lagrangians"

(1) Lorentz-invariant functions of fields.
(2) Classical eqn's of motion (wave equations) derived by minimizing with respect to field variations (calculus of variations)
(3) Can read Feynman rules directly off Lagrangian
(actual derivation proceeds via path integral)

Simplest ex: Nonrelativistic particle

Hamiltonian (Energy) $H = \frac{1}{2}m\dot{x}^2 + V(x) = K.E. + P.E.$

$\dot{x} \equiv \frac{dx}{dt}$ $\quad H(p,x) = \frac{p^2}{2m} + V(x) \longleftarrow \boxed{p \equiv \frac{\partial H}{\partial \dot{x}} = m\dot{x}}$

Lagrangian
$L(x,\dot{x}) \equiv p\dot{x} - H = \frac{1}{2}m\dot{x}^2 - V(x)$

Principle of least action: Minimize Action $\boxed{S \equiv \int dt\, L}$

$$0 = \int dt\, \delta L = \int dt \Big[\underbrace{\tfrac{1}{2} m \cdot 2 \cdot \dot{x}\ \delta\dot{x}}_{\frac{d}{dt}(m\dot{x}\,\delta x) - m\ddot{x}\,\delta x} - V'(x)\,\delta x \Big]$$

$$= \int dt\, [-m\ddot{x} - V'(x)]\,\delta x \qquad \text{for } \underline{\text{any}}\ \delta x(t).$$

$\Rightarrow$ must have $\boxed{m\ddot{x} = \underbrace{-V'(x)}_{F}}$

✓ proper equations of motion.

Now let's try it for a scalar field.
Let $L = \int d^3x\, \mathcal{L}$, $\boxed{S = \int d^4x\, \mathcal{L}}$ $\mathcal{L} \equiv$ Lagrangian density (often omit this term)

- We want $\mathcal{L}(\phi, \partial_\mu \phi)$ to be Lorentz invariant, $\phi = \phi(x) \to \phi(x')$ under L.T.

Any $V(\phi(x))$ will be Lorentz invariant.

- Derivatives should appear in pairs, so we can contract tensor indices to make a scalar.
- If there are only 2 derivatives, only 2 possibilities: $\phi \partial_\mu \partial^\mu \phi$ and $\partial_\mu \phi \partial^\mu \phi$
- Actually these are equivalent, up to a total derivative ← (usually innocuous, won't affect eqn's of motion at finite x^μ)

$$\partial_\mu (\phi \partial^\mu \phi) = \partial_\mu \phi \partial^\mu \phi + \phi \partial_\mu \partial^\mu \phi.$$

So let's write down $\boxed{\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)}$

(conventional normalization) see

$$\boxed{\begin{aligned} &\text{so} \quad \mathcal{H} = \frac{\delta \mathcal{L}}{\delta \dot\phi} \dot\phi - \mathcal{L} \\ &\mathcal{H} = \underbrace{\tfrac{1}{2} \dot\phi^2}_{\text{K.E.}} + \underbrace{(\vec\nabla \phi)^2}_{\text{gradient energy}} + \underbrace{V(\phi)}_{\text{potential energy of scalar field}} \end{aligned}}$$

$V(\phi)$ has right sign

Then

$$0 = \int d^4x\, \delta \mathcal{L}$$

$$= \int d^4x \left[\partial_\mu \phi\, \partial^\mu (\delta \phi) - V'(\phi) \delta \phi \right]$$

$$= \int d^4x \left[-\partial_\mu \partial^\mu \phi - V'(\phi) \right] \delta \phi \quad + \text{surface term}$$

drop

$$\Rightarrow \quad \boxed{-\partial_\mu \partial^\mu \phi - V'(\phi) = 0}$$

For $V(\phi) = \frac{1}{2} m^2 \phi^2$ ← (standard mass term)

we recover the K-G eqn, $-\partial_\mu \partial^\mu \phi - m^2 \phi = 0$

NOTE: Lagrangian for "ABC" theory of Chap. 6 is

$$\boxed{\begin{aligned} \mathcal{L} = \tfrac{1}{2} \big(&\partial_\mu \phi_A \partial^\mu \phi_A + \partial_\mu \phi_B \partial^\mu \phi_B + \partial_\mu \phi_C \partial^\mu \phi_C \\ &- m_A^2 \phi_A^2 - m_B^2 \phi_B^2 - m_C^2 \phi_C^2 - g\, \phi_A \phi_B \phi_C \big) \end{aligned}}$$

$\rho_0 = \psi^\dagger\psi$ for $v \ll 1$, $E_0 = m$.

- But relativistically, pair creation and annihilation is possible, so $\int d^3x\, \psi^\dagger\psi$ does not have to be constant.

- Problem was traced to 2nd order $\frac{\partial^2}{\partial t^2}$ in Klein-Gordon eqn.

- Motivated Dirac to look for a 1st order relativistic eqn. for electron.

- He tried to factor $\quad p_\mu p^\mu - m^2 \stackrel{?}{=} (\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m)$

$$= \beta^\kappa \gamma^\lambda p_\kappa p_\lambda + m(\beta^\kappa - \gamma^\kappa)p_\kappa - m^2$$

$\therefore$ Take $\beta^\kappa = \gamma^\kappa$

$\therefore$ Require also

$$\gamma^\kappa \gamma^\lambda p_\kappa p_\lambda = p^\mu p_\mu$$

or
$$(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = (\gamma^0)^2 (p^0)^2 + \cdots + (\gamma^3)^2 (p^3)^2$$
$$+ (\gamma^0\gamma^1 + \gamma^1\gamma^0) p_0 p_1$$
$$+ \cdots + (\gamma^2\gamma^3 + \gamma^3\gamma^2) p_2 p_3$$

- Different γ^μ's must anticommute with each other.

[Recall $[A,B] = AB - BA$ is commutator of A, B
$\{A,B\} = AB + BA$ is anticommutator]

- Cannot be numbers, must be matrices, satisfying $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$ "Dirac algebra"

- One representation of this algebra, by 4×4 matrices, is

(2x2 identity matrix)

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

2x2 <u>Pauli</u> matrices

$$\sigma_1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

· <u>Dirac's eqn:</u> is now one of the 2 factors (doesn't matter which), say, $\gamma^\mu p_\mu - m = 0$

or, as a wave eqn, $\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0}$

$m \cdot I$

4x4

Since γ^μ is 4x4 matrix, ψ must be a 4-element column vector

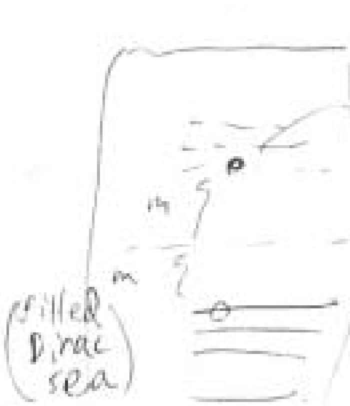
$$\boxed{\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}}$$

"Dirac spinor" – not a 4-vector

<u>Solutions to Dirac eqn:</u>

$\vec{p} = 0$: $\Leftrightarrow \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0 \Rightarrow i\gamma^0 \frac{\partial\psi}{\partial t} = m\psi$

or $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial\psi_A/\partial t \\ \partial\psi_B/\partial t \end{pmatrix} = -im \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$



$$\Rightarrow \boxed{\begin{aligned} \psi_A(t) &= e^{-imt}\psi_A(0) \\ \psi_B(t) &= e^{+imt}\psi_B(0) \end{aligned}}$$

← "normal" $E = m > 0$ solution
interpret as electrons

← [$E = -m < 0$ solution ??]
Dirac interpreted as positrons

Really 4 indep. solutions:

$\psi^{(1)} = e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $(e^- \uparrow)$ ($\uparrow$ (z direction)) $\psi^{(2)} = e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $(e^- \downarrow)$

$\psi^{(3)} = e^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $(e^+ \uparrow)$ $\psi^{(4)} = e^{imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ $(e^+ \downarrow)$

- Next look for $\vec{p} \neq 0$ solutions, of form

$$\psi(x) = a\, e^{-ip_\mu x^\mu} u(p)$$

$$i\partial_\mu e^{-ip_\mu x^\mu} = p_\mu e^{-ip_\mu x^\mu}$$

so Dirac eq $\Rightarrow \boxed{(\gamma^\mu p_\mu - m)\, u(p) = 0}$

$$\Leftrightarrow 0 = \begin{pmatrix} E-m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E-m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} (E-m)u_A - \vec{p}\cdot\vec{\sigma}\, u_B \\ \vec{p}\cdot\vec{\sigma}\, u_A - (E+m)\, u_B \end{pmatrix}$$

$\therefore$ $u_A = \frac{p\cdot\sigma}{E-m} u_B$, $\quad u_B = \frac{p\cdot\sigma}{E+m} u_A$ $\qquad \vec{S} = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix}$ spin up only if $p_x = p_y = 0$!

Find 4 solutions as follows:

$E = \sqrt{p^2+m^2}$
(1) Pick $u_A = \begin{pmatrix}1\\0\end{pmatrix}$ $\Rightarrow u_B = \left(\frac{p\cdot\sigma}{E+m}\right)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{E+m}\begin{pmatrix}p_z\\p_x+ip_y\end{pmatrix}$ $(e^-\uparrow)$

(2) Pick $u_A = \begin{pmatrix}0\\1\end{pmatrix}$ $\Rightarrow u_B = \left(\frac{p\cdot\sigma}{E+m}\right)\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{E+m}\begin{pmatrix}p_x-ip_y\\-p_z\end{pmatrix}$ $(e^-\downarrow)$

(3) Pick $u_B = \begin{pmatrix}1\\0\end{pmatrix}$ $\Rightarrow u_A = \frac{p\cdot\sigma}{E-m}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{E-m}\begin{pmatrix}p_z\\p_x+ip_y\end{pmatrix}$ $(e^+\uparrow)$

$E = -\sqrt{p^2+m^2}$ (so $\not p \to 0$ nonsingular)
(4) Pick $u_B = \begin{pmatrix}0\\1\end{pmatrix}$ $\Rightarrow u_A = \frac{p\cdot\sigma}{E-m}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{E-m}\begin{pmatrix}p_x-ip_y\\-p_z\end{pmatrix}$ $(e^+\downarrow)$

$u \equiv \begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}$ $\quad u^\dagger = \underline{\alpha^*\ \beta^*\ \gamma^*\ \delta^*}$ $\quad \boxed{u^\dagger u = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 2|E|}$ normalization convention

. The last steps are to:

(1) include $N \equiv \sqrt{|E|+m}$ to normalize u's

(2) relabel $u^{(3)}, u^{(4)}$ solutions to $v^{(1)}(E,\vec{p}) = u^{(4)}(-E,\vec{p})$

$v^{(2)}(E,\vec{p}) = -u^{(3)}(-E,-\vec{p})$

so they now have

$\underline{E>0}$, correspond to $\underline{\text{positrons}}$ ~~opposite charges~~

Dirac eq, $\gamma^0 E + \gamma^i p_i - m = 0$ requires flipping $\vec{p}$ too. (Like letting $m \to -m$)

. Finally, the basic plane wave states with $E = +\sqrt{p^2+m^2}$ are:

$$
\boxed{
\begin{array}{lll}
(e^-) & u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} & u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \\[2ex]
(e^+) & v^{(1)} = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} & v^{(2)} = -N \begin{pmatrix} p_z/(E+m) \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}
\end{array}
}
$$

Reason for this scup: charge conjugation C turns out to take $e^- \uparrow \leftrightarrow e^+ \downarrow$ so we can have $C: u^{(1)} \leftrightarrow v^{(1)}$, $u^{(2)} \leftrightarrow v^{(2)}$

Let $\boxed{\not{p} \equiv \gamma^\mu p_\mu}$

Note $(\not{p}-m)u^{(1)} = 0$, but $(\not{p}+m)v^{(1)} = 0$

. To build Lagrangian, we want Lorentz invariants, or at least terms with nice covariant transformation properties. Since $\psi \sim$ spin $\frac{1}{2}$, we are probably going to need $\underline{\text{pairs}}$ of ψ's.

ψ itself transforms as the "spinor representation" of the Lorentz group. $\underline{\text{Infinitesimal}}$ transformations ($\ll 1$) corresponding to $x^\mu \to \Lambda^\mu{}_\nu x^\nu = x^\mu + \varepsilon^\mu{}_\nu x^\nu$ are given by $\psi \to \psi + i\varepsilon^{\mu}{}_{\nu}\, \sigma_\mu{}^\nu \psi$

$\boxed{\text{can show } [\sigma_{\mu\nu}, \sigma_{\alpha\beta}] = \left(\text{same form as } [L^{\alpha\beta}_{\mu\nu}, L_{\alpha\beta}]\right) \quad (L_{\mu\nu} \equiv x_\mu \partial_\nu - x_\nu \partial_\mu)}$ (1.6)

where $\boxed{\sigma_{\mu\nu} \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = i\gamma^\mu\gamma^\nu \quad (\mu \neq \nu)}$

- Boost in x direction corresponds to $\varepsilon_{01} \neq 0$

$\Rightarrow \psi \to \psi + i\varepsilon_{01}\sigma_{01}\psi = \psi + (-1)\varepsilon_{01}\gamma^0\gamma^1\psi$

- Griffiths gives finite form of this transformation,

$S_x = a_+ + a_-\gamma^0\gamma^1 \qquad a_\pm = \sqrt{\frac{\gamma \pm 1}{2}}$

- Is $|\psi|^2 = \psi^\dagger\psi = \sum_{i=1}^{4}|\psi_i|^2$ a scalar?

- No, ~~cause~~ $\psi^\dagger\psi \to \psi^\dagger S^\dagger S\psi$

$\boxed{\begin{array}{l}\gamma^{0\dagger} = \gamma^0 \\ \gamma^{i\dagger} = -\gamma^i\end{array}}$

and $S^\dagger S = (a_+ + a_-\gamma^0\gamma^1)^\dagger(a_+ + a_-\gamma^0\gamma^1)$

$= (a_+ - a_-\gamma^1\gamma^0)(a_+ + a_-\gamma^0\gamma^1)$

$= \underbrace{(a_+^2 + a_-^2)}_{\gamma} + \underbrace{2a_+a_-}_{\sqrt{\gamma^2-1}\, = \, v\gamma}\gamma^0\gamma^1 \qquad v = \sqrt{1 - \frac{1}{\gamma^2}}$

$S^\dagger S \neq 1$

$\psi^\dagger\psi$ not a scalar

$S^\dagger S = \gamma(1 + v\gamma^0\gamma^1)$

Looks a little more like 0^{th} component of 4-vector — and indeed it is

- Instead, the $\boxed{\text{scalar is } \bar{\psi}\psi}$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0 = (\psi_1^* \ \psi_2^* \ -\psi_3^* \ -\psi_4^*)$ is the adjoint spinor.

For example, $S^\dagger\gamma^0 S = (a_+ - a_-\gamma^1\gamma^0)\gamma^0(a_+ + a_-\gamma^0\gamma^1)$

$= (a_+^2 - a_-^2)\gamma^0 + 0 \cdot a_+a_-\gamma^0\gamma^1$

$= \gamma^0$

~~A~~ 4-vector is provided by "the current" $\rightarrow$ $\boxed{\bar{\psi}\gamma^\mu\psi} = \psi^\dagger\gamma^0\gamma^\mu\psi = (\psi^\dagger\psi,\ \psi^\dagger\gamma^0\gamma^1\psi,\ \psi^\dagger\gamma^0\gamma^2\psi,\ \psi^\dagger\gamma^0\gamma^3\psi)$