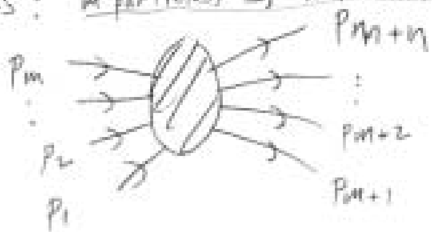
Physics 152/252

Lecture 8: Scattering and Feynman Graphs

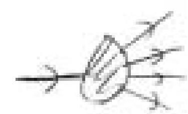
[See Griffiths, Chap. 6]

- Almost all particle physics information is obtained, in the end, from <u>scattering</u> processes: <u>m particles → n particles</u>



$$p_1 + p_2 + \cdots p_m = p_{m+1} + p_{m+2} + \cdots + p_{m+n}$$

- <u>$m=0$</u> is not kinematically allowed
- <u>$m=1$</u> describes <u>particle</u> <u>decay</u>



- <u>$m=2$</u> describes usual <u>particle</u> <u>collisions</u>

- <u>$m>2$</u> ≃ Impossible to arrange in laboratory, but can be useful to think about theoretically.

prob. of 3 particles close enough to interact = very small.

- These processes are quantum mechanical <u>transitions</u>, from states which are approximately superpositions of plane waves (wave <u>packets</u> really) $|p_1, \ldots, p_m\rangle$ in the "<u>infinite past</u>"

to similar superpositions ~~of~~ "out" states in the infinite future,

$$\langle p_{m+1}, \ldots, p_{m+n} |$$

- Thus there is a QM amplitude, or matrix element,

$$\underbrace{\mathcal{M}(p_1, \ldots, p_m, p_{m+1}, \ldots, p_{m+n})}_{\text{complex number}} = \langle p_{m+1}, \ldots, p_{m+n} | \mathcal{M} | p_1, \ldots, p_m \rangle$$

QM operator, the "scattering matrix" or "S matrix"

- As usual in QM, the probability (or rate) of a transition is proportional to ~~the~~ norm of the complex amplitude, $|\mathcal{M}|^2$.
- However, we also typically have to sum (integrate) over a continuous set of final-state momenta, with the right weight (integration measure) = "volume of phase space"
- And we need similar initial state "flux" normalization factors.

"Fermi's Golden Rule":
$$\boxed{\text{transition rate} = \frac{2\pi}{\hbar} |\mathcal{M}|^2 \times (\text{phase space})}$$

not derived here

In principle there is ambiguity in this separation, if state normalizations are changed. We will use ~~[illegible]~~ normalizations so $|\mathcal{M}|^2$ is ~~Lorentz~~ invariant.

(M=1)

Particle Decay

Recall that prob.^{n} of an unstable particle which is present at time $t=0$, still being there at time t_0, is independent of $t_c < 0$ when it was created. I.e., it only depends on t_0

$p = P(t_0)$.

~~If we have~~ Taking t_0 infinitesimal,

$$P(t_0) = 1 - t_0 \cdot \Gamma$$

$\boxed{\Gamma \equiv -\frac{dP(t_0)}{dt_0}\Big|_{t_0=0}}$ is a constant, the prob./time of decay, or the decay rate (Γ)

- For a large collection of identical particles, say $N(0)$ at $t=0$, number left at time t is $N(t) = N(0) \cdot P(t)$

$$\Rightarrow \frac{dN(t)}{dt}\Big|_{t=0} = N(0)\frac{dP(t)}{dt}\Big|_{t=0} = -\Gamma N(0)$$

But $\frac{dN(t)}{dt}$ must be proportional to $N(t)$

$\Rightarrow \boxed{\frac{dN(t)}{dt} = -\Gamma N(t)}$ which integrates to $\boxed{N(t) = N(0)e^{-\Gamma t}}$

Since $\boxed{N(t) = N(0)\,e^{-t/\tau}}$ with τ the lifetime (time to go to $1/e$ of initial sample)

We have

$$\tau = \frac{1}{\Gamma}$$

(lifetime)

total decay rate or total "width"

(explain name later)

· Usually there are several decay <u>modes</u>, or combinations of final states, for a given unstable particle.

For example, $\pi^0 \to \gamma\gamma$

$\qquad\qquad\qquad\to e^+e^-\gamma$

$K^+ \to \mu^+\nu_\mu$

$\qquad\quad \pi^+\pi^0$

$\qquad\quad \pi^+\pi^+\pi^-$ etc.

<u>Total decay rate:</u> $\Gamma = \sum_{i=1}^{n} \Gamma_i$ (decay to ith mode)

$\left(dN = -\sum_{i=1}^{n} \Gamma_i \, dt\right)$

(prob. of "losing" particle = sum of prob's of losing it ~~to~~ ith mode)

$\Gamma \equiv \Gamma_{tot}$

· Probability of decay of "X" particle going into ith mode is called <u>branching ratio</u>:

$$\boxed{Br(X \to i) \equiv \frac{\Gamma_i}{\Gamma}}$$

· How do we calculate Γ_i from $\mathcal{M}(X \to i)$?

Let $1 \to \underbrace{2 + 3 + 4 + \cdots + n}$ $\qquad p_j^\mu = (E_j, \vec{p}_j)$

(1) $$d\Gamma_i = |\mathcal{M}|^2 \frac{S}{2m_1} \prod_{j=2}^{n} \frac{d^3\vec{p}_j}{(2\pi)^3 2E_j} \times (2\pi)^4 \delta^4(p_1 - p_2 - \cdots - p_n)$$

energy-mom. conservation

<u>differential</u> decay rate, for particle 2 with momentum in d^3p_2 about $\vec{p}_2$, etc.

· Usually want to integrate over all final state momenta (Γ_i) or all except one angle or energy, or ...

To make eq. (1) look more Lorentz invariant, note that $\displaystyle\int \frac{d^3\vec{p}_j}{(2\pi)^3\, 2E_j} = \int \frac{d^4 p_j}{(2\pi)^3}\, \delta(p_j^2 - m_j^2)$

since $\displaystyle\int dE_j\, \delta(E_j^2 - \vec{p}_j^{\,2} - m_j^2)$

$$= \frac{1}{\underbrace{\dfrac{\partial(E_j^2 - \vec{p}_j^{\,2} - m_j^2)}{\partial E_j}}} = \frac{1}{2E_j}$$

change of variables:

$$\int_0^1 dx\, \delta(f(x)) = \int_0^1 \frac{df(x)}{\left(\frac{df}{dx}\right)}\, \delta(f(x)) = \frac{1}{\frac{df}{dx}}$$

S is a product of $\frac{1}{j!}$ for each group of j identical particles in final state, enforces Fermi/Dirac + Bose/Einstein statistics.

2-body decay: $1 \to 2 + 3$

$$\Gamma = \frac{S}{m_1}\left(\frac{1}{4\pi}\right)^2 \frac{1}{2} \int d^3\vec{p}_2\, d^3\vec{p}_3\, \underbrace{\delta^4(p_1 - p_2 - p_3)}_{\delta(E_1 - E_2 - E_3)\, \delta^3(\vec{p}_1 - \vec{p}_2 - \vec{p}_3)}\, \frac{|\mathcal{M}|^2}{E_2 E_3}$$

• For unpol. 2-body decay, $\mathcal{M}(\vec{p}_2, \vec{p}_3) \to \mathcal{M}(\vec{p}_2)$ only depends on $|\vec{p}_2|$, ~~and~~ $d\Omega_2$

Pretty, only depends on masses, ∴

First do $\int d^3 p_3$ (easy because $\vec{p}_3$ appears linearly here)

$$\Rightarrow \Gamma = \frac{S}{2m_1}\left(\frac{1}{4\pi}\right)^2 \int \underbrace{d^3\vec{p}_2}\, \delta(E_1 - E_2 - E_3)\, \frac{|\mathcal{M}|^2}{E_2 E_3}$$

$$\int d|\vec{p}_2|\, |\vec{p}_2|^2\, d\Omega_2 = \int dE_2\, E_2\, |\vec{p}_2|\, d\Omega_2$$

$$E_2^2 = |\vec{p}_2|^2 + m^2 \Rightarrow d|\vec{p}_2|\, |\vec{p}_2| = dE_2\, E_2$$

$$\Rightarrow \Gamma = \frac{S}{2m_1}\frac{\left(\int d\Omega_2\right)}{(4\pi)^2}\int dE_2 \frac{|\vec{p}_2|}{E_3} |\mathcal{M}|^2 \delta(E_1 - E_2 - E_3)$$

Now E_3 depends on E_2, because

$$E_3^2 = \underbrace{|\vec{p}_3|^2 + m_3^2 = |\vec{p}_2|^2 + m_3^2} = E_2^2 - m_2^2 + m_3^2$$

$$\Rightarrow 2E_3 \frac{\partial E_3}{\partial E_2} = 2E_2 \quad , \quad \frac{\partial E_3}{\partial E_2} = \frac{E_2}{E_3}$$

Also $\int d\Omega_2 = \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi = 4\pi$

$$\Rightarrow \Gamma = \frac{S}{8\pi m_1} \frac{|\vec{p}_2|}{\underbrace{E_3 \left(1 + \frac{\partial E_3}{\partial E_2}\frac{E_2}{E_3}\right)}} |\mathcal{M}|^2$$

$$E_3\left(\frac{E_2 + E_3}{E_3}\right) = E_2 + E_3 = E_1 = m_1$$

$$\Rightarrow \boxed{\Gamma = \frac{S|\vec{p}_2|}{8\pi m_1^2} |\mathcal{M}|^2}$$

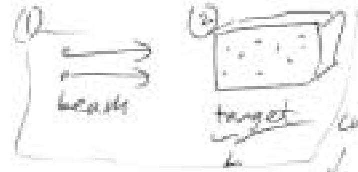
· Only complicated part (for general masses) is formula for $|\vec{p}_2|$ in terms of m_1, m_2, m_3:

$$\boxed{|\vec{p}| = \frac{\sqrt{\lambda(m_1^2, m_2^2, m_3^2)}}{2m_1}}$$

where

$$\boxed{\lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz}$$

(Exercise 3.16)



<u>Scattering</u>:

$$\text{Recall } N_{events} = N_{beam} \cdot \overset{cm^{-3}}{n_{target}} \cdot \overset{(cm^2)}{\sigma_i} \cdot \overset{cm}{L}$$

σ_i is an <u>area</u>, hence "cross-section" for $\underset{beam}{1} + \underset{target}{2} \to$ (final state i)

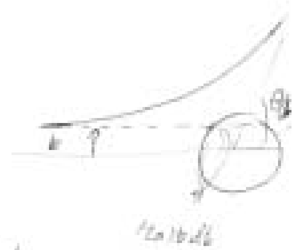
$$\sigma_{tot} = \sum_{i=1}^{n} \sigma_i$$

[<u>note</u>: this is often <u>infinite</u>]

: Quite often interested in <u>differential cross section</u>, e.g.

$\dfrac{d\sigma_i}{d\cos\theta_{cm}}$ for $2 \to 2$ process

- Griffiths discusses "<u>deterministic</u>" scattering where particle comes in at a definite <u>impact parameter</u> b and goes out at angle $\theta(b)$



$$\Rightarrow \quad d\sigma = |b\, db\, d\phi|$$

$$d\Omega = d(\cos\theta)\, d\phi = d\theta \sin\theta\, d\phi$$

$$\Rightarrow \quad \boxed{\frac{d\sigma}{d\Omega} = \left|\frac{b}{\sin\theta} \frac{db}{d\theta}\right|}$$

- But this is not too useful for normal, probabilistic QM collisions (1 $b \to$ distribution of θ's)

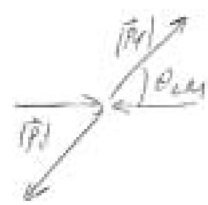
<u>Golden rule for scattering</u> $1+2 \to 3+4+\dots+n$

- Final state factors same as before.
  Only difference is that initial state "$\frac{1}{2m_1}$"
  for <u>decay</u> is replaced by $\dfrac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}$

$\left(= \dfrac{1}{2s} \quad \text{if } m_1, m_2 \ll \sqrt{s} \right)$

$$\Rightarrow \boxed{d\sigma = |\mathcal{M}|^2 \frac{S}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \prod_{j=3}^{n} \frac{d^3\vec{p}_j}{(2\pi)^3 2E_j} \cdot (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4-\dots-p_n)}$$

<u>2 → 2 scattering:</u> $1+2 \to 3+4$, in CM frame.



- Using $\vec{p}_2 = -\vec{p}_1$, $p_1\cdot p_2 = E_1E_2 + |\vec{p}_1|^2$,
  can show (Prob. 6.7)

$$\boxed{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2} = (E_1+E_2)\,|\vec{p}_1|}$$

$|\mathcal{M}|^2$ really only depends on s, θ_{cm}.

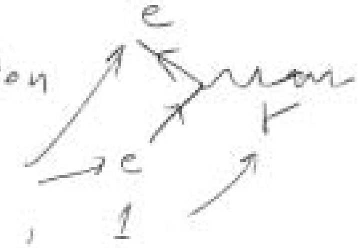
- Final state phase space integration is same as before, <u>except</u> $\frac{1}{E_2+E_3} = \frac{1}{m_1} \Rightarrow \frac{1}{E_3+E_4} = \frac{1}{E_1+E_2}$
  And we should not do the $\int d\Omega$ this time

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{S\,|\vec{p}_f|\,|\mathcal{M}|^2}{(4\pi)^2 \cancel{2m_1}\,(E_1+E_2)\,|\vec{p}_i|\,4(E_1+E_2)}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2}\frac{S\,|\mathcal{M}|^2}{s}\frac{|\vec{p}_f|}{|\vec{p}_i|}} \quad \text{or} \quad \boxed{\frac{d\sigma}{d\cos\theta_{cm}} = \frac{1}{32\pi}\frac{S\,|\mathcal{M}|^2}{s}\frac{|\vec{p}_f|}{|\vec{p}_i|}}$$

Feynman rules for $\mathcal{M}$

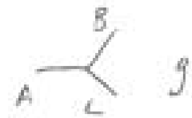
QED: electrons + photons
— only 1 interaction



Complications due to spin $\frac{1}{2}$, 1

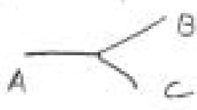
∴ First look at toy theory with only scalars, spin 0. A, B, C (types).
Each is own antiparticle.

· Suppose they have a Feynman vertex rule

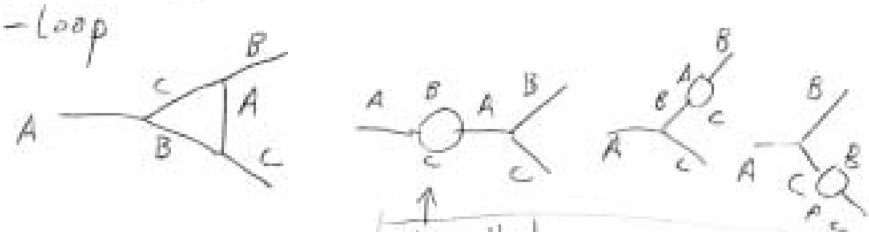


· There are also "propagators" — lines ~~[illegible]~~ connecting vertices to each other

· Feynman says draw all diagrams for a given process, like $\boxed{A \to B + C}$ (assume $m_A > m_B + m_C$) with a given number of closed loops, starting with 0 loops [tree graphs]



Then 1-loop

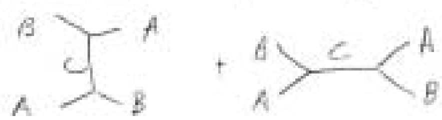


Note that

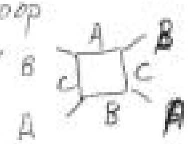
is the same diagram

(8.10)

Another process: $A+B \to A+B$



+ 1-loop

( $+ \ldots$)

The Feynman Rules

(1) Notation: $p_1, \ldots p_n$ for external momenta

$q_1, \ldots, q_n$ ~~for~~ internal momenta

arrow on each line for direction of momentum.

(2) Coupling constant factors: $-ig$ ← (strength of interaction)

For each vertex (here there is only 1 kind) here ~ mass^1 (normally dimensionless)

(3) Propagator factor:

$\dfrac{i}{q_j^2 - m_j^2}$ for each internal line.

collapse to $\int$ over undetermined (loop) momenta

(4) Energy-momentum cons. $\Rightarrow (2\pi)^4 \delta^4(k_1+k_2+k_3)$ at each vertex / all incoming or all outgoing – watch signs!

(5) Integration over each internal momenta $\displaystyle\int \frac{d^4 q_j}{(2\pi)^4}$

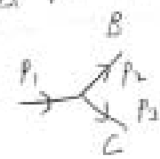
(6) Cancel an overall EM δ-function, $(2\pi)^4 \delta^4(p_1+p_2+\cdots p_n)$

What is left is $-i\mathcal{M}$.

Tree-level $A \to B + C$

(8.11)

① - Simplest case: (no internal momenta)



$$-ig \; \cancel{(2\pi)^4 \delta^4(p_1 - p_2 - p_3)} = -i\mathcal{M}$$

erased

$$\Rightarrow \boxed{\mathcal{M} = g}$$

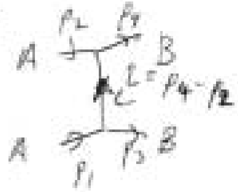
~ mass^2

$$\Rightarrow \Gamma = \frac{S\,|\mathcal{M}|^2\,|\vec{p}|}{8\pi\, m_A^2} = \frac{g^2\,|\vec{p}|}{8\pi\, m_A^2}$$

~ mass^1 ✓

$$\tau = \frac{1}{\Gamma} = \frac{8\pi\, m_A^2}{g^2\,|\vec{p}|}$$

② Tree-level $A + A \to B + B$

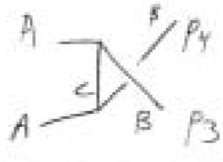


$$\Rightarrow (-ig)^2 \frac{i}{q^2 - M_C^2} \int (2\pi)^4 \delta^4(p_1 - p_3 - q)\,(2\pi)^4 \delta^4(p_2 + q - p_4) \frac{1}{(2\pi)^4} d^4q$$

$-(p_1 - p_2)$

$$= -i g^2 \frac{\cancel{(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}}{(p_4 - p_2)^2 - M_C^2} = \frac{-ig^2}{(p_4 - p_2)^2 - M_C^2} = -i\mathcal{M}$$

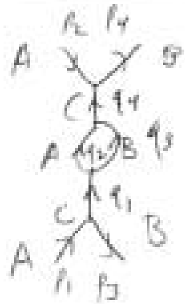
2nd diagram



Use $p_3 \Leftrightarrow p_4$

$$\Rightarrow \boxed{\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - M_C^2} + \frac{g^2}{(p_3 - p_2)^2 - M_C^2}} \quad \ldots$$

Look at one 1-loop diagram for $A + A \to B + B$:



$(q \equiv q_3)$

$$\ldots \Rightarrow \mathcal{M} = \frac{i\, g^4}{\left((p_1 - p_3)^2 - m_C^2\right)^2} \underbrace{\int \frac{d^4 q}{(2\pi)^4} \frac{1}{\left((p_1 - p_3 - q)^2 - m_A^2\right)} \cdot \frac{1}{q^2 - m_B^2}}$$

typical Feynman integral.

· Note large q "ultraviolet" behavior:

$$\sim \int \frac{d^4 q}{q^4} = \int dq \frac{q^3}{q^4} = \int \frac{dq}{q} = \ln q \Big|^{\infty} = \infty$$

Need to regularize the integral, e.g. by inserting $\frac{-M^2}{(q^2 - M^2)}$ with M very large

or by letting $d = 4 - 2\epsilon$! (with $\epsilon \ll 1$.)

· Divergences can be absorbed into redefinitions of masses ($m_{A,B,C}$ here) and coupling constants; e.g. $m_A^2 = (m_A^0)^2 + \delta m_A^2(g, m_{A,B,C})$

RENORMALIZATION finite as measured $\to$ $g_a = g_a^0 + \delta g_a$ (divergent as $M \to \infty$)