Lecture 4b

4b.1

Relativistic
Charged particle in Electromagnetic Fields

- $\vec{E}$ looks naively like a vector.
- But recall that static charges make $\vec{E}$ fields, moving charges also make $\vec{B}$ fields.

$\therefore$ Lorentz transformations must mix $\vec{E}$ & $\vec{B}$.

Need a tensor with 6 indep. components.

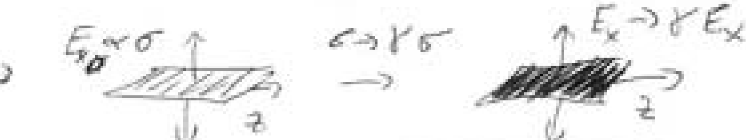
- Antisym. tensor $F^{\mu\nu}$ has " " "

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{pmatrix}$$

- Moving sheet of charge Lorentz contracts

$\Rightarrow$ $\perp$ components of $\vec{E}$ get a factor γ under a boost

$\boxed{\nabla\cdot E = 4\pi\rho}$



- Consistent with $\boxed{F_{0i} = E_i}$ $\boxed{F_{i0} = -E_i}$

$\boxed{\nabla\cdot B = 0}$

- Another 2 of Maxwell's Laws

$$\boxed{\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}}$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

~~is~~ $\cancel{\epsilon_{ijk}\partial_j E_k}$ $\cancel{[\partial_0 B_i}$ $\cancel{\partial_0 \epsilon_{ijk} F_{jk}]}$

$= \epsilon_{ijk}\partial_j F_{0k}$

- Try to write first one covariantly,

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

$$\underbrace{\vec{\nabla}\times\vec{B}}_{+\partial_j F_{ji}} \qquad \underbrace{\frac{1}{c}\frac{\partial\vec{E}}{\partial t}}_{-\partial_0 F_{0i}} \qquad \vec{J} \downarrow \; J_i^0 = q\frac{dx^j}{dt}g_{ij}$$

$$+\partial_j F_{ji} = \epsilon_{ijk}\partial_j B_k$$

$\Rightarrow$ Identify $\boxed{F_{ji} = \epsilon_{ijk}B_k}$ $\Rightarrow$

$$\boxed{\begin{aligned} F_{xy} &= -B_z = -F_{yx} \\ F_{yz} &= -B_x = -F_{zy} \\ F_{zx} &= -B_y = -F_{xz} \end{aligned}}$$

Invert

$$\epsilon_{ijl}\left(F_{ji} = \epsilon_{ijk}B_k\right) \Rightarrow \epsilon_{ijl}F_{ji} = \underbrace{\epsilon_{ijl}\epsilon_{ijk}}_{2\delta_{kl}}B_k = 2B_l$$

$$\cancel{\Rightarrow B_l = -\tfrac{1}{2}\epsilon_{ijl}F_{ij}} \quad \text{or} \quad B_l = -\frac{1}{2}\epsilon_{ijl}F_{ij}$$

$$\cancel{B_x = -F_{yz} = +F_{zy}}$$

$$\cancel{B_y}$$

So,

$$\boxed{F_{\mu\nu} = \begin{array}{c|cccc} & t & x & y & z \\ \hline t & 0 & E_x & E_y & E_z \\ x & -E_x & 0 & -B_z & +B_y \\ y & -E_y & B_z & 0 & -B_x \\ z & -E_z & -B_y & B_x & 0 \end{array}}$$

- Nonrelativistic Force Law: $F^i = ma^i = \dfrac{dp^i}{dt}$

$$= q\left(E^i + \epsilon^{ijk}v^j B^k\right)$$

$$= q\left(F^{i0} + v^j F^{ji}\right)$$

$$\frac{dp^i}{d\tau} = \gamma\frac{dp^i}{dt} = q\left(F^{i0}\gamma - F^{ij}(\gamma v^j)\right)$$

$$\frac{dp^i}{d\tau} = qF^{i\mu}\frac{dx_\mu}{d\tau} = \frac{q}{m}F^{i\mu}p_\mu$$

$\Rightarrow$ 4-vector relation $\boxed{\dfrac{dp^\mu}{d\tau} = \dfrac{q}{m}F^{\mu\nu}p_\nu}$

Let $B_z = B$, $B_x = B_y = \vec{E} = 0$

$$F_{xy} = -B = -F_{yx}$$

$$\Rightarrow \frac{dp^x}{d\tau} = \frac{-q}{m} F^{xy} p^y = \left(\frac{q}{m} B\right) p^y$$

$$\frac{dp^y}{d\tau} = -\frac{q}{m} F^{yx} p^x = -\left(\frac{qB}{m}\right) p^x$$

$$\Rightarrow \boxed{\frac{d^2 p^x}{d\tau^2} = -\left(\frac{qB}{m}\right)^2 p^x}$$

$$\Rightarrow p^x = |\vec{p}_\perp| \cos\left(\frac{qB_z}{m}\tau + \cancel{\phi_0}\right)$$

$$m\frac{dx}{d\tau} = |\vec{p}_\perp| \cos\left(\frac{qB}{m}\tau\right)$$

$$\Rightarrow x(\tau) = \frac{|\vec{p}_\perp|}{m} \int d\tau' \cos\left(\frac{qB}{m}\tau'\right)$$

$$= \frac{|\vec{p}_\perp|}{m}\left(\frac{m}{qB}\right) \sin\left(\frac{qB}{m}\tau\right)$$

$$\boxed{x(\tau) = \underbrace{\left(\frac{|\vec{p}_\perp|}{qB}\right)}_{R_{curv}} \sin\left(\frac{qB}{m}\tau\right)}$$

$$\boxed{R_{curv} = \frac{|\vec{p}_\perp|}{qB}}$$

$$\Rightarrow \boxed{|\vec{p}_\perp| = qB R_{curv}}$$