Physics 152 Lecture 2b

Relativistic Kinematics

- First, a word about <u>units</u>
- In particular physics, it is traditional to set speed of light $\boxed{c = 1}$ and reduced Planck's constant $\boxed{\hbar \equiv \frac{h}{2\pi} = 1}$.
- This is the <u>same</u> as converting <u>times</u> and <u>distances</u> into <u>energies</u>, for which the standard unit is $\boxed{1\ \text{GeV} = 10^9\ \text{electron volts}}$.

(Convenient because the rest energy of the proton is $\boxed{m_p c^2 = 0.938\ \text{GeV}}$.)

- <u>Times → distances</u> using $\boxed{c \simeq 3 \times 10^{10}\ \text{cm s}^{-1} = 1\ \text{foot ns}^{-1} = 30\ \text{cm ns}^{-1}}$

Example:
- Several unstable particles (B's, D's, ...) have <u>lifetimes</u> $\boxed{\tau \approx 1\ \text{ps}}$

$\Rightarrow \boxed{c\tau \approx 0.3\ \text{mm}}$ ← (very approximate flight distance before decay, if $v \approx c$)

- <u>Microscopic distances</u>:
  - Atomic physicists often use $\boxed{1\,\text{Å} = 10^{-10}\ \text{m} = 10^{-8}\ \text{cm}}$ $\approx r_{atom}$
  - Nuclear/particle physicists use $\boxed{1\ \text{fermi} = 1\ \text{fm} \equiv 1\ \text{femtometer} = 10^{-15}\ \text{m} = 10^{-13}\ \text{cm} \approx r_{proton}}$

- <u>Distances → Energies</u> using

$$\boxed{1 = \hbar c = 197\ \text{MeV fm}\ \ (\text{nuclear}) = 0.197\ \text{GeV fm}\ \ (\text{particle})}$$

$$\boxed{[L] = [T] = \frac{1}{[E]}}$$

- <u>"Cross sections"</u> ≡ area ⟺ probability of interaction
  <u>measured by nuclear physicists</u> in "barns" $1\ \text{b} = 10^{-24}\ \text{cm}^2$.
  Typical particle physics <u>cross sections</u>: $1\ \text{nb} = 10^{-33}\ \text{cm}^2$, $1\ \text{pb} = 10^{-36}\ \text{cm}^2$, $1\ \text{fb} = 10^{-39}$

$\boxed{1 = (\hbar c)^2 = 3.89 \times 10^8\ \text{GeV}^2\ \text{pb}}$ ← Use to convert $1/\text{GeV}^2 \to \text{pb}$

Special Relativity

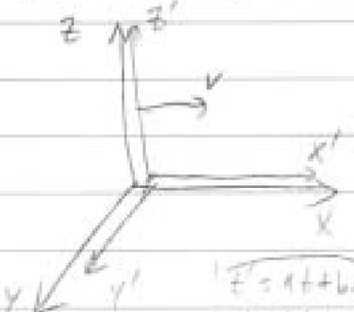
· Laws of physics apply in any reference frame, provided that we make appropriate transformations of coordinates (and other quantities).

· Speed of light = constant, c in all frames.
(Michelson-Morley experiment)

"Galilean transformations" $\begin{cases} t' = t \\ x' = x - vt \end{cases}$ ← 2 frames with rel. velocity v in x direction

no good
– leads to wrong velocity addition formula, $u = u' + v$
· For $u = c \Rightarrow u' = c + v \neq c$

· Need to generalize to $t' = at + bx$
$x' = dt + ex$



$\begin{cases} y' = y \\ z' = z \end{cases}$ to avoid paradox! does ball get through cylinder or not?

Light in $\pm x$ direction
$t = x \Rightarrow t' = x'$
$\Rightarrow a + b = d + e$
$t = -x \Rightarrow t' = -x'$
$\Rightarrow -a + b = d - e$
$\Rightarrow d = b,\ e = a$

$t' = at + bx$
$x' = bt + ax$

$t = h$

$t' = ah$
$x' = bh$

$t = 1 = \frac{\sqrt{t'^2 + h^2}}{t'} = \frac{\sqrt{b^2+1}}{a} \Rightarrow \sqrt{b^2+1} = a$

$\beta \equiv \frac{v}{c}$ often

Satisfied by $a = \gamma \equiv \frac{1}{\sqrt{1 - v^2}}$, $b = -v\gamma$

$\sqrt{1 + v^2\gamma^2} = \sqrt{1 + \frac{v^2}{1 - v^2}} = \frac{1}{\sqrt{1 - v^2}} = \gamma$ ✓

$$\Rightarrow \boxed{\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(-vt + x) \\ y' &= y \\ z' &= z \end{aligned}}$$

⇐ [reduces to Galilean for $v \ll c$]

$\boxed{\text{Inverse transformation} \Leftrightarrow v \leftrightarrow -v}$

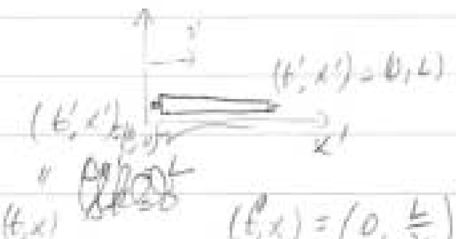
Consequences:

(1) $\Delta t' = \gamma(\Delta t - v\Delta x)$

so $\Delta t = 0$ can $\Rightarrow$ $\Delta t' \neq 0$ "relativity of simultaneity"

(2) Lorentz contraction

moving stick,
length L in rest frame
looks like length L/γ.



using $x' = \gamma(x - vt)$ with $x' = L$, $t = 0$

$\Rightarrow$ Protons & nuclei "pancake" at high energies.

(3) Time dilation.

$(t',x') = (0,0) \rightarrow (t',x') = (\tau, 0)$

$= (t,x)$

$t = \gamma(t' + vx')$

$\Rightarrow \boxed{\Delta t = \gamma\tau}$

Particle lifetimes are longer by γ.

Average flight length is $\Delta x = \beta\Delta t = \boxed{\beta\gamma\tau}$

can be $\gg c\tau$ for $\beta \simeq 1$

(e.g. B mesons with $c\tau \simeq 0.3$mm can fly several mm before decaying.)

Velocity addition formula becomes $\boxed{u = \frac{u'+v}{1+uv}}$

$u' = c = 1 \Rightarrow \boxed{u = \frac{1+v}{1+v} = 1}$ ✓

$\boxed{\text{4-vectors}}$

$x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z$

x^μ

- Write Lorentz transformation as

$$(*)\quad \boxed{x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu}$$

implicit $\sum_{\nu=0}^{3}$

where for v in x-direction,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Anything transforming like x^μ in $(*)$ is called a ~~co~~ contravariant 4-vector.

- Recall proper time interval $(\Delta\tau)^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$

$= \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$

(time elapsed on clock travelling along from $(0,0,0,0) \to (\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3)$)

where $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is Minkowski metric

Used to raise/lower indices; e.g. $x_\mu \equiv \eta_{\mu\nu} x^\nu$

called a covariant 4-vector

$\eta_{\mu\nu}$ is an invariant tensor,

$$\boxed{\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}}$$

$$\text{"="} \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & +\beta\gamma \\ -\beta\gamma & -\gamma \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- True for all Λ, in fact it defines the group of Lorentz transformations: $SO(1,3)$

(real matrices) preserves $(+\,-\,-\,-)$

4-vectors can be combined into Lorentz invariant quantities (Lorentz scalars) using $\eta_{\mu\nu}$ (or $\eta^{\mu\nu} = \eta_{\mu\nu}^{-1} = \eta_{\mu\nu}$)

$$a \cdot b \equiv a^\mu b_\mu = \eta_{\mu\nu} a^\mu b^\nu$$

$$\xrightarrow{\text{Lorentz}} \eta_{\mu\nu} \Lambda^\mu{}_\alpha a^\alpha \Lambda^\nu{}_\beta b^\beta$$

$$= \eta_{\alpha\beta} a^\alpha b^\beta = a \cdot b$$

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

$$a^2 = (a^0)^2 - \vec{a}^2 \begin{cases} > 0 & \text{timelike} \\ = 0 & \text{lightlike} \\ < 0 & \text{spacelike} \end{cases}$$

. Tensors transform with more indices

$$T^{\mu\nu} \to \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}, \ldots$$

Energy & momentum of particles

. Ordinary velocity $\vec{v} \equiv \frac{d\vec{x}}{dt} = \frac{d\vec{x}}{dx^0}$ does not have simple Lorentz transformation properties because of x^0 in denominator.

Better to put proper time interval $d\tau$ in denominator. $dt = \gamma\, d\tau$ (time dilation)

$\Rightarrow \eta^i \equiv \frac{dx^i}{d\tau} = \gamma \frac{dx^i}{dt}$ will transform like 3 of the 4-components of a 4-vector.

The ~~4th~~ 0^{th} component is

$$\eta^0 \equiv \frac{dx^0}{d\tau} = \frac{dt}{d\tau} = \gamma.$$

$\Rightarrow$ 4-velocity $\eta^\mu \equiv \gamma(1, v_x, v_y, v_z)$

Lorentz invariant magnitude of 4-velocity: $\eta^\mu \eta_\mu = \gamma^2(1 - v^2) = 1$!

Momentum of a particle

- Because $\vec{v} = \frac{d\vec{x}}{dt}$ does not have simple transformation properties, neither does $\vec{p} = m\vec{v}$.
- Better to use $\vec{p} = m\vec{\eta} = m\frac{d\vec{x}}{d\tau} = m\gamma\vec{v}$.

Agrees with old $\vec{p}$ for $v \ll 1$.

- The full 4-vector is (*) $\boxed{p^\mu = m\eta^\mu = m\gamma(1, v_x, v_y, v_z)}$
- The time component $\boxed{p^0 \equiv E}$ ← total energy of particle: rest energy ($E_{rest} = mc^2$) + kinetic energy

$= m\gamma$

$$\boxed{p^\mu = (E, p_x, p_y, p_z)}$$

$p^\mu p_\mu$ is Lorentz invariant. Evaluate in particle rest frame $\Rightarrow p^\mu = (m, 0, 0, 0)$ ← (rest energy only)

or directly from (*).

$\Rightarrow$ $\boxed{p^2 = p^\mu p_\mu = m^2}$ ← "on-shell condition" (mass)

- Thus $\boxed{E^2 - \vec{p}^2 = m^2}$

$\boxed{\text{Can also expand for small } v}$ $\Rightarrow E = m\gamma = \frac{m}{\sqrt{1-v^2}} = m + \frac{1}{2}mv^2 + \frac{3}{8}mv^4 + \ldots$

rest energy; nonrelativistic kinetic energy; relativistic correction

- As $v \to c$, m becomes inconvenient and irrelevant. $\gamma \to \infty$

$\vec{p} = m\vec{v}\gamma \gg m$

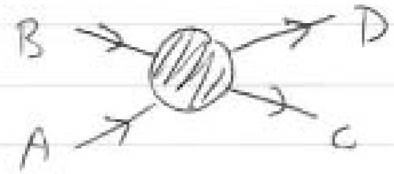
$E = m\gamma \gg m$.

$E^2 - \vec{p}^2 \simeq 0$, or $\boxed{E \simeq |\vec{p}|}$ ← (massless (γ) or high energy particle)

- Difference between $E =$ 2 eV and 3 eV photon is not the mass, but (via Quantum mechanics) the frequency, $\underline{E = h\nu}$, or wavelength, $\underline{E = hc/\lambda}$

Collisions

$A + B \to C + D$ ~~[illegible]~~ "$2 \to 2$" (most frequent)



"free particles at infinity" $\Rightarrow$ p_i = constants at ∞
$i = A, B, C, D$

- Collision takes place over a very short time/distance interval $\Rightarrow$ ignore all external forces & fields.
- Then energy-momentum conservation

$$\Rightarrow \quad \boxed{p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu} \qquad (1)$$

0th component: $E_A + E_B = E_C + E_D$ (2)

total energy is conserved.

- But this includes rest energy, which can change if $m_A + m_B \neq m_C + m_D$
- So total kinetic energy is not conserved.

An extreme example: $e^+e^- \to \tau^+\tau^-$ at threshold.

with $E_{e^+} = E_{e^-} = 1.8$ GeV $\qquad m_{\tau^\pm} = 1.777$ GeV

Eq. (2) + symmetry $\Rightarrow E_{\tau^+} = E_{\tau^-} = 1.8\,\text{GeV} = \gamma_\tau \cdot 1.777\,\text{GeV}$

$\Rightarrow \gamma_\tau = 1.01 = \frac{1}{\sqrt{1-v_\tau^2}} \Rightarrow v_\tau = \sqrt{1-\frac{1}{1.01^2}} = 0.16c$

τ's are nonrelativistic,

$(K.E.)_\tau = \frac{1}{2} m_\tau v_\tau^2 = 22$ MeV $\ll m_\tau$

while
$(K.E.)_e \simeq E_e = 1.8$ GeV

- How many kinematic quantities characterize the "process" or "reaction" $A + B \to C + D$?

- Not counting $m_A, \ldots, m_D$ which are attributes of the particles, not the process.

- Start with $4 \times 4 = 16$ quantities,

  $p_A^\mu \quad p_B^\mu \quad p_C^\mu \quad p_D^\mu$

- But $p_i^2 = m_i^2$ removes 4 $\quad (i = A, B, C, D)$

  $p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu$ removes 4 more

  $16 - 4 - 4 = \underline{8}$.

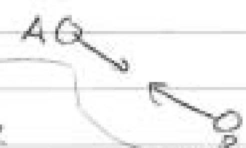
- Still not done. There are $\underline{3}$ boosts (along x, y, z axes) and $\underline{3}$ rotations (around x, y, z axes) which only change the frame, not the essential attributes of the reaction.

  $8 - 3 - 3 = \boxed{2} \leftarrow$ true number of kinematic quantities

- What are they?

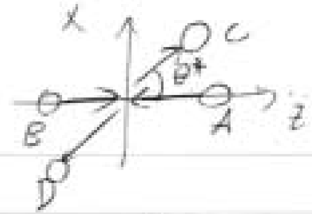
- Let's use boost invariance to go to center of mass frame

  $\vec{p}_{TOT} = \vec{p}_A + \vec{p}_B = 0$
  $(= \vec{p}_C + \vec{p}_D)$



- Then use rotation invariance to line $\vec{p}_A$ & $\vec{p}_B$ up (the "beamline") along z direction. $\quad A \circ\!\longrightarrow \quad \longleftarrow\!\circ B$

- Rotate around z axis until C & D are in (say) xz plane



$\boxed{\theta^* \equiv \theta_{CM} \text{ is center of mass scattering angle.}}$

The other quantity is just the
$\boxed{\text{center-of-mass energy, } E_{cm}}$

- Let's count a second way (Mandelstam): construct Lorentz invariants $(p_i \pm p_j)^2$. (problem 3.22 of Griffiths)
- How many are there?

  $i=j$ gives 0 or $(2p_i)^2 = 4m_i^2$ (we don't count this)
- So take $i \neq j$.

  Since $(p_i+p_j)^2 + (p_i-p_j)^2 = 2(p_i^2+p_j^2) = 2(m_i^2+m_j^2)$, we can take $+$ or $-$ but not both.

Mandelstam chose

$$\boxed{\begin{aligned} s &\equiv (p_A+p_B)^2 = (p_C+p_D)^2 \\ t &\equiv (p_A-p_C)^2 = (p_B-p_D)^2 \\ u &\equiv (p_A-p_D)^2 = (p_B-p_C)^2 \end{aligned}}$$

with one constraint eqn:

$$\begin{aligned} 0 &= (p_A+p_B-p_C-p_D)^2 \\ &= \left(\sum_{i=1}^{4} m_i^2\right) + 2(p_A\cdot p_B + p_C\cdot p_D) - 2(p_A\cdot p_C + p_B\cdot p_D) - 2(p_A\cdot p_D + p_B\cdot p_C) \\ &= \left(\sum_{i=1}^{4} m_i^2\right) + 2s - \sum_{i=1}^{4} m_i^2 + 2t - \sum_{i=1}^{4} m_i^2 + 2u - \sum_{i=1}^{4} m_i^2 \end{aligned}$$

$$\Rightarrow \boxed{s+t+u = \sum_{i=1}^{4} m_i^2}$$

Thus u (say) can be eliminated.

$\Rightarrow$ 2 variables: $\boxed{s, t}$

Now, in CM frame $s = (\underbrace{E_A + E_B}_{E_{cm}}, \vec{0})^2$

$\Rightarrow s = E_{cM}^2$

t must be a function of E_{cM} and θ_{cM}

$t = f(E_{cM}, \theta_{cM})$

- In the case where all particles are massless, this relation is particularly simple:

$E = \frac{E_{cM}}{2}$

$p_A = E(1, 0, 0, 1)$
$p_B = E(1, 0, 0, -1)$
$p_C = E(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
$p_D = E(1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta)$
$(t \quad x \quad y \quad z)$

$\Rightarrow t = (p_A - p_C)^2 = -2p_A \cdot p_C = -2E^2(1-\cos\theta)$
$= -2\left(\frac{E_{cM}}{2}\right)^2(1-\cos\theta)$

$\Rightarrow t = -\frac{1}{2}E_{cM}^2(1-\cos\theta) = -\frac{1}{2}s(1-\cos\theta)$

$u = -\frac{1}{2}s(1+\cos\theta)$

26.11

2 Body Decay is Simpler

$\boxed{\text{Ex. 3.3 of Griffiths, } \pi^+ \to \mu^+ \nu_\mu}$ (π at rest)

$m_{\nu_\mu} \simeq 0$. What is velocity of muon? (Also momentum)

Use 4-vectors, $p_{\nu}^2 = 0$

$\Rightarrow$ rewrite $p_\pi = p_\mu + p_\nu$ as $p_\nu = p_\pi - p_\mu$, square it

$$\Rightarrow \quad 0 = p_\nu^2 = (p_\pi - p_\mu)^2 = m_\pi^2 - 2 p_\pi \cdot p_\mu + m_\mu^2$$

$$p_\pi = (m_\pi, \vec{0}) \qquad \Rightarrow \quad p_\pi \cdot p_\mu = m_\pi E_\mu$$
$$p_\mu = (E_\mu, \vec{O})$$

$$\Rightarrow \quad \boxed{E_\mu = \frac{m_\pi^2 + m_\mu^2}{2 m_\pi}}$$

$$p_\mu = \sqrt{E_\mu^2 - m_\mu^2} = \sqrt{\frac{(m_\pi^2 + m_\mu^2)^2 - 4 m_\pi^2 m_\mu^2}{4 m_\pi^2}}$$

$$\Rightarrow \quad \boxed{p_\mu = \frac{m_\pi^2 - m_\mu^2}{2 m_\pi}}$$ $\leftarrow$ (measure with $\vec{B}$ field in detector)

$$v_\mu = p_\mu / E_\mu = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}$$

$m_\pi = 140$ MeV
$m_\mu = 106$ MeV

$\Downarrow$

$\boxed{v_\mu = 0.271}$ $\leftarrow$ (semi-relativistic)

- Note that v_μ, p_μ, E_μ are all fixed by the particle masses in a 2-body decay. "Mono-energetic muon".
- In a 3-body decay, like $^{60}Co \to {}^{60}Ni + e^- + \bar{\nu}_e$ this is not true: electron can have a spectrum of different energies. Key to Pauli's proposal of the neutrino in 1930.

Use of CM frame

Ex. 3.4 of Griffiths

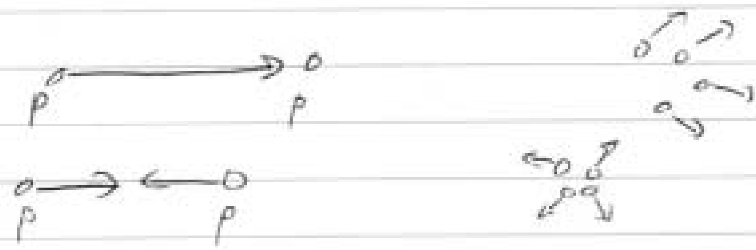
"Bevatron at Berkeley built to produce antiprotons.

$$\text{(beam)} \to p + \underset{\text{rest}}{p} \to p + p + p + \bar{p}$$

What is minimum beam energy E^{min} required?

Lab frame

CM frame



- Minimum beam energy $\Longleftrightarrow$ minimum CM energy

$\Longleftrightarrow$ all particles at rest in CM (no wasted K.E.)

$\Rightarrow E_{cm}^{min} = 4m_p$

Now we compute $s = E_{cm}^2$ before collision, in lab frame.

$$s = \left((E,\vec{p}) + (m,\vec{0})\right)^2 = (E+m,\vec{p})^2 = (E+m)^2 - p^2$$

$$= \cancel{E^2} + 2m_p E + m_p^2 - (\cancel{E^2} - m_p^2) = 2m_p(E+m_p)$$

So, we have $2m_p(E^{min} + m_p) = 16 m_p^2$

$$\Rightarrow \boxed{E^{min} = 7m_p}$$

- For $E \gg m$, $s \approx 2E_{beam}\, m_{target}$ for fixed target expt.

$\overset{\longrightarrow}{E}$ m

- In colliding beams, ~~$s = ((E,\vec{p}) + (E,-\vec{p}))^2 = 4E^2$~~

$\boxed{s = 4E_{beam}^2} \gg 2E_{beam}\, m_{target}$ $\quad E \longrightarrow \longleftarrow E$

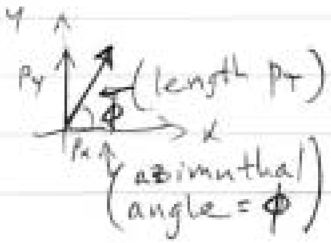
e.g. Tevatron 1 TeV + 1 TeV $\Longleftrightarrow$ 2000 TeV fixed target !!!

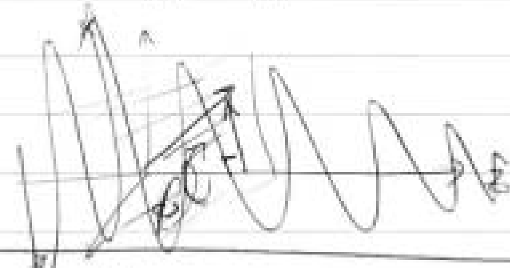
Remark:

Transverse momenta with respect to z are boost-invariant for a boost along z:

$$E' = \gamma(E - \beta p_z)$$
$$p_z' = \gamma(-\beta E + p_z)$$
$$p_x' = p_x$$
$$p_y' = p_y$$

$$\boxed{p_T \equiv \sqrt{p_x^2 + p_y^2}}$$





- A better longitudinal variable is sometimes the rapidity

$$y = \frac{1}{2} \ln\left(\frac{E + p_z}{E - p_z}\right)$$

under a boost

$$y \to y' = \frac{1}{2} \ln\left(\frac{E' + p_z'}{E' - p_z'}\right) = \frac{1}{2} \ln\left(\frac{\gamma(1-\beta)(E + p_z)}{\gamma(1+\beta)(E - p_z)}\right)$$

$$\boxed{y' = y + \frac{1}{2} \ln\left(\frac{1-\beta}{1+\beta}\right)}$$

(p_T, η, ϕ) Used often in hadron colliders, where you don't know the CM frame of the quark collisions