Lecture 13 Quantum Chromodynamics (QCD)

(← Chapter 9 of Griffiths) mostly (some Ch. 8)

quarks & gluons — the constituents of hadrons — actually interact through a gauge theory, similar in many ways to QED, electrons & photons.

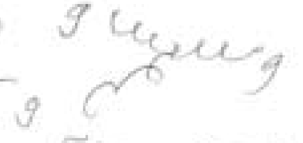
The biggest difference is that the gluons themselves are <u>charged</u> under the new "charge", so they can self-interact.

I.e. besides

[analog of

in QED]

there are also



and

[no QED analog]

new contribution to vacuum polarization has <u>opposite</u> <u>sign</u>

$$\Rightarrow \alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi}(11N_c - 2N_f)\ln(q^2/\mu^2)}$$

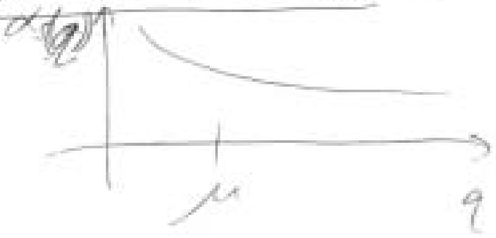
$|q| >> \mu$

N_c: # colors = 3

N_f: # of light quarks

QED formula is same with
$\alpha_s \to \alpha$ $f \to 2$
$n \to 0$ $\mu \to m_e$.

Opposite sign is critical:



- Coupling blows up (gets very strong) as $q \to 0$

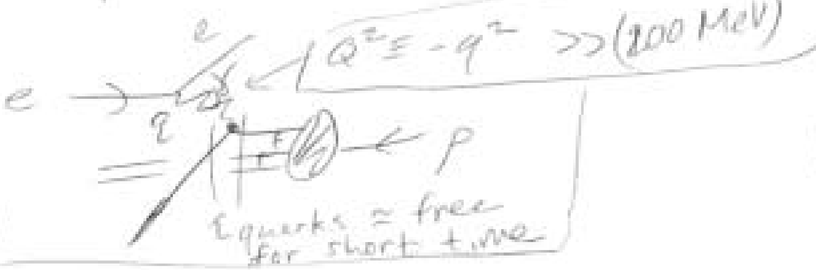
⇒ quarks, gluons "stuck in molasses", confined into hadrons.

- Simplicity of QCD is only apparent at high energies.

E.g. $e^+e^- \to q\bar{q}(g)$
$\to$ 2 or 3 jets of particles

or

$ep \to eX$



Also applies to electroweak theory...

• The QCD Lagrangian & Nonabelian Gauge Sym.

Let ψ be an N_c dimensional vector $i = 1, 2, \ldots, N_c$

N_c = "number of colors"

Instead of $\psi(x) \to \underbrace{e^{-ie\lambda(x)}}_{\text{(simple phase rotation)}} \underbrace{\psi(x)}_{\text{(complex number)}}$

↑ preserves magnitude: $\bar\psi\psi \to \bar\psi\psi$

We can now consider

$$\psi_i(x) \longrightarrow U_i^j(x)\psi_j(x)$$

$$\bar\psi\psi = \bar\psi^i\psi_i \to \bar\psi U^\dagger U \psi = \bar\psi\psi$$

if $U^\dagger U = \mathbb{1}$ ($UU^\dagger = 1$ too)

U is an $N \times N$ unitary matrix

• Actually, we can separate out overall phase rotations,

$e^{-ie\lambda(x)} \cdot \mathbb{1}$ "QED"

and let $\det U = 1$

⇒ special unitary group on N elements

"SU(N)"

• Can we again make $\bar\psi \gamma^\mu D_\mu \psi$ locally invariant, for

$$D_\mu = \partial_\mu \cdot \mathbb{1} - ig A_\mu \quad ?$$

(QCD coupling) ↗ g; A_μ: $N \times N$ matrix

13.4

Yes: we let $\psi \to U(x)\psi(x)$

$$\bar{\psi} \to \bar{\psi} U^\dagger$$

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger$$

Then

$$\partial_\mu \psi - ig A_\mu \psi$$

$$\to (\partial_\mu U)\psi + U\partial_\mu\psi - ig\, U A_\mu \cancel{U^\dagger} U\psi$$

$$+ U(\partial_\mu U^\dagger) U \psi$$

$$= U\Big[\partial_\mu\psi - igA_\mu\psi + \underbrace{\Big(U^\dagger\partial_\mu U + \overset{(\partial_\mu U^\dagger)U}{[illegible]}\Big)}_{\partial_\mu(U^\dagger U) = \partial_\mu \mathbb{1} = 0}\psi\Big]$$

$$= U(\partial_\mu\psi - igA_\mu\psi)$$

$$\Rightarrow \bar{\psi}\not{D}\psi \to \bar{\psi}U^\dagger U\not{D}\psi = \bar{\psi}\not{D}\psi \quad ✓$$

What about the pure-gluon terms?

D_μ transforms "nicely", $(D_\mu \to U D_\mu U^\dagger)$ so we should define the field strength by

$$\underline{F_{\mu\nu}} \equiv \frac{-1}{ig}[D_\mu, D_\nu] = -\frac{1}{ig}[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu]$$

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \underbrace{- ig[A_\mu, A_\nu]}_{\text{new term — quadratic}}}$$

$$F_{\mu\nu} \to U F_{\mu\nu} U^\dagger$$

$$\Rightarrow \boxed{\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})} \to \mathrm{tr}\left[U F_{\mu\nu} U^\dagger U F^{\mu\nu} U\right] = \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

$$\Rightarrow \underbrace{\boxed{\mathcal{L}_{QCD} = -\frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \sum_i \bar{\psi}_i\left(i\not{D}-m\right)\psi_i}}_{\text{is gauge invariant}}$$

ψ is an N-vector of spinors

- For $\underset{SU(3)}{\underline{QCD}}$, $N=3$ "color" $\Rightarrow$ $\underset{\uparrow}{\psi} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_{red} \\ u_{blue} \\ u_{green} \end{pmatrix}$

up quark, for example

each u_i is itself a Dirac spinor

- For the electroweak interactions, $N=2$ "flavor"

$SU(2)_L$
$\uparrow$
only left-handed spinors interact

$P_L = \frac{1}{2}(1-\gamma_5)$

$\Rightarrow \psi = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$ or $\begin{pmatrix} c_L \\ d_L \end{pmatrix}$ or $\begin{pmatrix} t_L \\ b_L \end{pmatrix}$ or $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ ...

- But what kind of object is A_μ?

It is not a unitary matrix, but ~~takes values~~ ~~as~~ a linear combination of Hermitian matrices "generators" of $SU(N)$

Let $U = e^{i\overset{\uparrow}{\alpha_a} T^a}$ $\qquad U^\dagger = e^{-i\alpha_a T^a} = U^{-1}$ ✓

real $\qquad$ $\boxed{\text{Hermitian, traceless}}$ $(T^{a\dagger} = T^a)$

Then A_μ shifts under a gauge transformation by

$$U\partial_\mu U^\dagger = e^{+i\alpha_a T^a}(-i\alpha_a T^a)(e^{-i\alpha_a T^a})$$

commute

$$= -i\alpha_a T^a$$

$\therefore$ Write $A_\mu = A_\mu^a \underset{\uparrow}{T^a} = A_\mu^a \cdot \frac{\lambda^a}{2} \leftarrow$ for $SU(3)$, Gell-Mann matrices

generators of Lie algebra

of $N \times N$ Hermitian matrices : $\left(\begin{pmatrix} r_1 & c_{12} & & \\ c^* & r_2 & & \cdot \\ & \cdot & \ddots & \\ & & & r_N \end{pmatrix}\right) = N^2$

tracelessness $\Rightarrow$ $\underline{\underline{N^2 - 1}}$

$SU(3) \Rightarrow 3^2 - 1 = 8$ matrices.

$$\lambda^i = \left(\begin{array}{c|c} \sigma^i & 0 \\ \hline 0 & 0 \end{array}\right) \qquad i = 1, 2, 3$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- Normalized to $\mathrm{tr}(\lambda^a \lambda^b) = 2\delta^{ab}$

"Nonabelian" means $U_1 U_2 \neq U_2 U_1$ in general.

$U_i = e^{i\alpha_i T_i}$ $\Rightarrow$ $[T_i, T_j] \neq 0$ in general.

"Lie algebra" defined by $\boxed{[T^a, T^b] = i f^{abc} T^c}$

$\left([\lambda^a, \lambda^b] = 2 i f^{abc} \lambda^c \right)$

$$\mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) \supseteq (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 \quad \text{Maxwell-like}$$

$$\left.\begin{array}{l} + f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{\mu b}_{\rho} A^{\nu c}_{\rho} \\ + f \cdot f \text{ (mess)} \end{array}\right\} \text{gluon self interactions}$$

Feynman rules for QCD

[3-vector carrying color index] (column)

Quark { incoming  $u^{(s)}(p) \cdot c$

outgoing  $\bar{u}^{(s)}(p)$ ~~$\otimes$~~ $c^{\dagger}$ ← row vector)

Antiquark { incoming  $\bar{v}^{(s)}(p)\, c^{\dagger}$

outgoing  $v^{(s)}(p)\, c$

Gluon { incoming  $\epsilon_{\mu}(p)\, a^{\alpha}$ ← (3×3 matrix)

outgoing  $\epsilon_{\mu}^{*}(p)\, a^{\alpha *}$

propagators:

quark  $\dfrac{i(\not{q}+m)}{q^{2}-m^{2}}$ ← (color identity)

gluon  $\dfrac{-i g_{\mu\nu}\, \delta^{\alpha\beta}}{q^{2}}$

vertices:

$-\frac{ig}{2}\lambda^{\alpha}\gamma^{\mu}$

$$-g f^{\alpha\beta\gamma}\left[g_{\mu\nu}(k_1-k_2)_{\lambda} + g_{\nu\lambda}(k_2-k_3)_{\mu} + g_{\lambda\mu}(k_3-k_1)_{\nu}\right]$$

[see Griffiths]

- Now, in addition to "Dirac algebra", we have "color algebra" to perform.

Quark-antiquark scattering, and potential

Like $e\bar{\mu}$ scattering, except for color factors

$\boxed{u+\bar{d}\to u+\bar{d}}$

again only 1 Feynman graph.

$$\mathcal{M} = -\frac{g^2}{4}\frac{1}{q^2}\,\bar{u}(3)\gamma^\mu u(1)\,\bar{v}(2)\gamma_\mu v(4)\,(c_3^\dagger\lambda^\alpha c_1)(c_2^\dagger\lambda^\alpha c_4)$$

- What force does this represent?

In QED, we have a Coulomb potential,

$$\boxed{V_{QED}(r) = ~~[illegible]~~ + q_1 q_2\frac{\alpha}{r}}$$

e ↓ g   ⊕ add f

Here we get $\boxed{V_{q\bar{q}} = -f\frac{\alpha_s}{r}}$

where $\boxed{f = \frac{1}{4}(c_3^\dagger\lambda^\alpha c_1)(c_2^\dagger\lambda^\alpha c_4)}$

What values can this have?

- What color states can $u+\bar{d}$ be in?

9 possibilities $r\,b\,g \otimes \begin{pmatrix}\bar{r}\\ \bar{b}\\ \bar{g}\end{pmatrix}$

$\begin{pmatrix}r\\ b\\ g\end{pmatrix}\to U\begin{pmatrix}r\\ b\\ g\end{pmatrix}$

- But 1 is special: $\frac{1}{\sqrt{3}}(r\bar{r}+b\bar{b}+g\bar{g})$ ← color singlet
  - doesn't depend on basis.

In general, $c_j^\dagger c_i \to c_\ell U^\dagger_{\ell j} U_{ik} c_k$

But $c_i^\dagger c_i \to c_\ell^\dagger U^\dagger_{\ell i} U_{ik} c_k$

$= c_\ell^\dagger (U^\dagger U)_{\ell k} c_k$

$= c_k^\dagger c_k$ ✓

- The other $8^{\text{(octet)}}$ are all equivalent

E.g. do $r\bar{b}\to r\bar{b}$

$$\Rightarrow f = \frac{1}{4}\left[(1\,0\,0)\lambda^\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[(0\,1\,0)\lambda^\alpha\begin{pmatrix}0\\1\\0\end{pmatrix}\right]$$

$$= \frac{1}{4}\lambda^\alpha_{11}\lambda^\alpha_{22} \qquad \alpha = 3,8$$

$$= \frac{1}{4}\left[1(-1) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\right] \Rightarrow \boxed{f_{\text{octet}} = -\frac{1}{6}}$$

$$\frac{1}{\sqrt3}(r\bar r + b\bar b + g\bar g) \rightarrow \frac{1}{\sqrt3}(r\bar r + b\bar b + g\bar g)$$

$$f_{singlet} = \frac14\left(\frac{1}{\sqrt3}\right)^2 \sum_{ij=1}^{3} \lambda^\alpha_{ij}\lambda^\alpha_{ji}$$

$$= \frac{1}{12}\,Tr(\lambda^\alpha\lambda^\alpha) = \frac{1}{12}\cdot 2\,\underbrace{\delta^{\alpha\beta}\delta^{\alpha\beta}}_{\delta^\alpha_\alpha = 8} = \frac{1}{12}\cdot 2\cdot 8$$

$$\boxed{f_{singlet} = +\frac43}$$

∴ Singlet channel is <u>attractive</u> $\boxed{V \propto (-)\frac{(f)}{r}}$

⇒ $q + \bar q$ would <u>like</u> to bind as color singlet. E.g. $\boxed{\text{mesons}}$ like $\pi^+ = \frac{1}{\sqrt3}(u_r\bar d_r + u_b\bar d_b + u_g\bar d_g)$ $= \frac{1}{\sqrt3}\sum_{j=1}^{3} u_j\bar d_j$

(Except we can't study this quantitatively without resorting to computer simulations..)

δ_{jk} color coupling

<u>Similar analysis for $q + q$ scattering:</u>

$$3 \times 3 = 6 + \bar 3$$

(symmetric) (antisymmetric)

sign convention as above flipped from Griffiths

$f_{sextet} = +\frac13$

$f_{triplet} = -\frac23$

Shows that $\bar 3$ channel is more attractive. But not yet color singlet.

To do that, combine with <u>another 3</u>:

$$(q \times q) \times q = (3\times3)\times 3 = \underbrace{(6\times3)}_{10 + 8 \cancel{+1}} + \underbrace{(\bar3\times3)}_{8 + 1}$$

more attractive

most attractive

$$q_i q_j \rightarrow \sum_{k=1}^{3}\epsilon_{ijk}\,\text{"}\bar q_k\text{"}$$

⇒ Baryon, e.g. proton, is given by $\boxed{p = \frac{1}{\sqrt6}\sum_{i,j,k=1}^{3} u_i u_j d_k}$

($n = \frac{1}{\sqrt6}\sum u_i d_j d_k$)

(This is only an heuristic explanation of why color singlets only!)

The parton model

(First real evidence for QCD)

- Even though quarks always seem to be bound into hadrons, if we do short distance experiments they can look free.

"Deep Inelastic Scattering"

$\boxed{e^- + p \to e^- + X}$ ↑ (any hadrons)

Interaction happens so fast the quark does not have time to exchange gluons with its neighbors. →

$-(q_e - q_e')^2 \equiv Q^2$

$Q^2 \gg (1\,\text{GeV})^2$

$$\langle |M|^2 \rangle = \frac{e^4}{(q^2)^2} \underbrace{L^{\mu\nu}_{(\text{electron})}} \underbrace{K_{\mu\nu}(x)}$$

trace of electron line just like in $e\mu \to e\mu$

don't know too well in general

$$|p_3|^2 d(E')$$

$$\boxed{d\sigma = \frac{e^4 L^{\mu\nu}}{4 q^4 \underset{ME}{p_1 \cdot p_2}} \frac{d^3 p_3}{(2\pi)^3 \underset{2(E')}{2E_3}} 4\pi M W_{\mu\nu}}$$

$$\boxed{\frac{d\sigma}{dE' d\Omega_e} = \left(\frac{\alpha}{q^2}\right)^2 \frac{E'}{E} L^{\mu\nu} W_{\mu\nu}}$$

with
$$W_{\mu\nu} = \frac{1}{4\pi M} \sum_X \int \cdots \int K_{\mu\nu}(x) \left(\frac{d^3 p_4}{(2\pi)^3 2E_4}\right) \cdots \left(\frac{d^3 p_n}{(2\pi)^3 2E_n}\right)$$

symmetric $W_{\mu\nu}$ depends on p^α and q^β. Also, gauge invariance $\Rightarrow q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$

$$\Rightarrow W^{\mu\nu} = \frac{F_1}{M}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)$$

$$+ \left(\frac{F_2}{M q\cdot p}\right) \left[p^\mu - \left(\frac{q\cdot p}{q^2}\right) q^\mu\right]\left[p^\nu - \left(\frac{q\cdot p}{q^2}\right) q^\nu\right]$$

where

$$F_1 = F_1(q^2, q\cdot p) \qquad (q^2 = M_X^2 \text{ is fixed})$$

Plug $W^{\mu\nu}$ above into formula for $\frac{d\sigma}{dE'}$

using $L^{\mu\nu} = 4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu - g^{\mu\nu} p_1\cdot p_3\right)$

$$\Rightarrow \boxed{\frac{d\sigma}{dE'\,d\Omega} = \left(\frac{\alpha}{2E\sin^2\theta/2}\right)^2 \left[2\frac{F_1}{M}\sin^2\theta/2 + \overset{F_2\left(\frac{M}{Mq\cdot p}\right)}{Q_2}\cos^2\theta/2\right]}$$

"structure functions".

$$\boxed{\begin{aligned} q^2 &= -4EE'\sin^2\theta/2 \\ q\cdot p &= M(E-E') \end{aligned}}$$

Bjorken: Don't use E', θ to plot data, or $q^2, q\cdot p$, but

$x, (-q^2 = Q^2)$ where $x = \frac{-q^2}{2q\cdot p}$

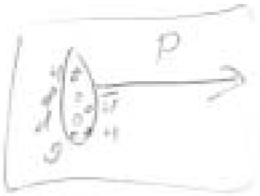
Note that elastic scattering [$e\mu \to e\mu$, or $ep \to ep$ at low energies] is recovered by setting $p'^2 = (p+q)^2 = q^2 + 2q\cdot p + M^2 = M^2 \Rightarrow \boxed{x = 1}$

And

$$F_1(q^2, x) = \frac{1}{2}\delta(x-1)$$

$$F_2(q^2, x) = \delta(x-1)$$

· Suppose now we treat the proton as an incoherent beam of parallel quarks [spin $\frac{1}{2}$, like proton], where ith quark has charge Q_i, and probability $f_i(z_i)$ of ~~being~~ having momentum $\boxed{p_i = z_i p.}$



(We neglect M^2 here $p_i^2 = 0$ OK)

$0 < z_i < 1 \equiv$ quark momentum fraction

The quark scattering rate goes like Q_i^2.

Since $x = -\frac{q^2}{2q\cdot p}$ and $p \to z_i p$ now,

$x_i = \frac{x}{z_i}$ is the variable that now enters

$$F_1^{(1)}(x) = \int_0^1 \frac{1}{2}\delta(x_i - 1)\cdot Q_i^2 \left(\frac{1}{z_i}\right) f_i(z_i)\, dz_i$$

← [fudge factor to account for steps I glossed over]

$$\Rightarrow \boxed{F_1^{(1)}(x) = \frac{1}{2}\sum_i Q_i^2 f_i(x)}$$

Similarly, $F_2^{(1)}(x) = \int_0^1 \delta(x_i - 1)\, Q_i^2 f_i(z)\, dz_i$

$$\Rightarrow \boxed{F_2(x) = x\sum_i Q_i^2 f_i(x)}$$

In proton,
$F_2(x) = x\left\{\left(\frac{2}{3}\right)^2 u(x) + \left(\frac{1}{3}\right)^2 d(x)\right\}$

Note $\boxed{F_2(x) = 2x F_1(x)}$ (spin $\frac{1}{2}$ quarks — Callan-Gross relation)

· Thus we can measure quark distributions in the proton by deep inelastic scattering.

They are not actually independent of Q^2
as you increase resolving power,
you can "see" virtual $q\bar{q}$ pairs, gluons, etc.

$\Rightarrow$ $u(x,Q^2)$ $d(x,Q^2)$, $s(x,Q^2)$, ..., $g(x,Q^2)$
<u>evolve</u> logarithmically with Q^2
(like running coupling though technically more complicated since also x-dependent).

There are constraints from momentum conservation,

$$\int_0^1 dx\, x\,[u+\bar{u}+d+\bar{d}+s+\bar{s}+\dots+g] = 1$$

and flavor conservation:

$$\int_0^1 dx\,(u-\bar{u}) = 2$$
$$\int_0^1 dx\,(d-\bar{d}) = 1$$
$$\int_0^1 dx\,(s-\bar{s}) = 0$$