12.1

Lecture 12 QED finale

Amplitudes with external photons

Incoming ϵ^μ

Outgoing $\epsilon^{\mu *}$

Lorentz gauge $\partial_\mu A^\mu = 0 \Rightarrow \boxed{\epsilon \cdot k = 0}$

Still free to shift $(\epsilon^\mu \to \epsilon^\mu + (\text{constant}) \cdot k^\mu)$

and get same answer — "Ward identity"

— consequence of gauge symmetry.

We discussed Coulomb gauge ϵ's before. $\epsilon \begin{pmatrix} \epsilon^0 = 0 \\ \vec{\epsilon} \cdot \vec{k} = 0 \end{pmatrix}$

E.g. $\epsilon^\mu_{(x)} = (0, \vec{\epsilon}_x) = (0, 1, 0, 0)$

$\epsilon^\mu_{(y)} = (0, \vec{\epsilon}_y) = (0, 0, 1, 0)$ for $\vec{k} = |\vec{k}|\hat{z}$

$k = |k|(1\,0\,0\,1)$

$$\Rightarrow \sum_{\substack{\text{pol's} \\ \lambda}} \epsilon^\mu_\lambda \epsilon^{*\nu}_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \delta^{ij} - \hat{k}^i \hat{k}^j$$

~~But if we let $\epsilon^\mu_\lambda \to \epsilon^\mu_\lambda + \frac{k^\mu}{|k|}$~~

But this is messy, not covariant.

Let $\mathcal{M}(\epsilon^\mu) = \epsilon^\mu \mathcal{M}_\mu = (\epsilon^\mu + (\text{const.}) k^\mu) \mathcal{M}_\mu$

$\Rightarrow k^\mu \mathcal{M}_\mu = 0$. Here, $k^\mu \mathcal{M}_\mu = |\vec{k}| (\mathcal{M}^0 - \mathcal{M}^3)$

$\Rightarrow \mathcal{M}^0 = \mathcal{M}^3$

$$\Rightarrow \sum_{\substack{\text{pol's} \\ \lambda}} \epsilon^\mu_\lambda \epsilon^{*\nu}_\lambda \mathcal{M}^\mu \mathcal{M}^{\nu *} = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2$$

$$= -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2$$

$$= -g_{\mu\nu} \mathcal{M}^\mu \mathcal{M}^{*\nu}$$

Thus $\boxed{\sum_{\substack{\text{pol's} \\ \lambda}} \epsilon^\mu_\lambda \epsilon^{*\nu}_\lambda = -g_{\mu\nu}}$ can always be used for the photon polarisation sums.

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Example: $\boxed{e^+e^- \to \gamma\gamma}$ $\quad m_e \to 0$

$$\mathcal{M} = \frac{e^2}{(p_1-p_3)^2}\bar{v}(2)\not\epsilon_4(\not p_1-\not p_3)\not\epsilon_3 u(1)$$

$$+ \frac{e^2}{(p_1-p_4)^2}\bar{v}(2)\not\epsilon_3(\not p_1-\not p_4)\not\epsilon_4 u(1)$$

First check Ward identity: result should vanish. ③ Let $\underline{\epsilon_3 = p_3}$

$$\mathcal{M} \to \frac{e^2}{-2p_1\cdot p_3}\bar{v}(2)\not\epsilon_4(\not p_1-\underbrace{\not p_3)\not p_3}_{0} u(1)$$

$$+ \frac{e^2}{-2p_2\cdot p_3}\bar{v}(2)\underbrace{\not p_3(-\not p_2+\not p_3}_{0})\not\epsilon_4 u(1)$$

$$= (-)\frac{e^2}{-2p_1\cdot p_3}\bar{v}(2)\not\epsilon_4 u(1)(2p_1\cdot p_3) + \frac{e^2}{-2p_2\cdot p_3}\bar{v}(2)\not\epsilon_4 u(1)(-2p_2\cdot p_3)$$

$= 0$ ✓

Casimir's trick + ϵ pol. sum $\Rightarrow$

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left\{\frac{1}{(2p_1\cdot p_3)^2}\mathrm{tr}\left[\not p_2\gamma^{\mu_4}(\not p_1-\not p_3)\gamma^{\mu_3}\not p_1\gamma_{\mu_3}(\not p_1-\not p_3)\gamma_{\mu_4}\right]\right.$$

$$+ \frac{1}{(2p_4\cdot p_3)(2p_1\cdot p_3)}\mathrm{tr}\left[\not p_2\gamma^{\mu_4}(\not p_1-\not p_3)\gamma^{\mu_3}\not p_1\gamma_{\mu_4}(\not p_1-\not p_4)\gamma_{\mu_3}\right]$$

$$\left. + (3 \leftrightarrow 4)\right\}$$

First trace is $4\,\mathrm{tr}\left[\not p_2(\not p_1-\not p_3)\not p_1(\not p_1-\not p_3)\right] = 8p_1\cdot p_3\,\mathrm{tr}[\not p_2\not p_3] - 4\,\mathrm{tr}[\not p_2\not p_1\not p_3\not p_3]$ $\quad p_3^2=0$

$= 32\,p_1\cdot p_3\,p_2\cdot p_3 = \underline{8tu}$

Second trace is zero!

$$\Rightarrow \boxed{\langle|\mathcal{M}|^2\rangle = 2e^4\left[\frac{t}{u}+\frac{u}{t}\right]}$$

$R = +$
$L = -$

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Second example: $\boxed{e^+e^- \to \gamma\gamma}$

- A clean test of QED at very high energies. Look for new $e^+e^-\gamma\gamma$ interactions.

2 helicity amplitudes: $e^+_L e^-_R \to \gamma_R \gamma_R$

0 in lowest-order QED for $m_e = 0$ it turns out!

$e^+_L e^-_R \to \gamma_L \gamma_R$

$$\boxed{\mathcal{M}(e^+_L e^-_R \to \gamma_L \gamma_R) = 2e^2\sqrt{\frac{u}{t}}}$$

$J_z = 1 \to J_z = \pm 2$

$\Rightarrow$ more helicity suppression one way than other. ✓

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} \propto \frac{u}{t} + \frac{t}{u} = \frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}}$$

- Compares very well (also in magnitude) with data from the LEP detector L3 at $\sqrt{s} = 183$ GeV, 189 GeV

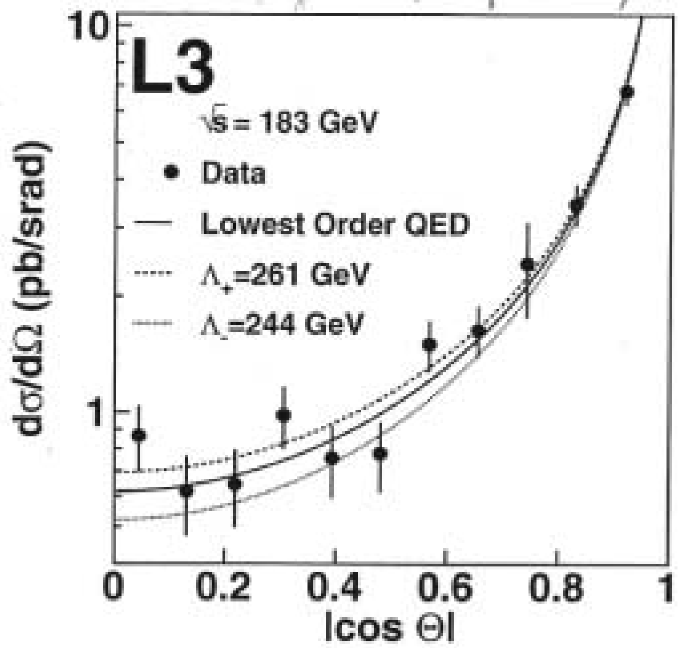
see hep-ex/0002036.

- Can put limits of $\Lambda_\pm \gtrsim 300$ GeV on new interactions producing

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{QED}\left(1 + \frac{s^2}{\alpha}\frac{1}{\Lambda^4}\sin^2\theta\right)$$

L3 Collab. (LEP) hep-ex/0002036 12.4

Phys. Lett. B475, 198 (2000)



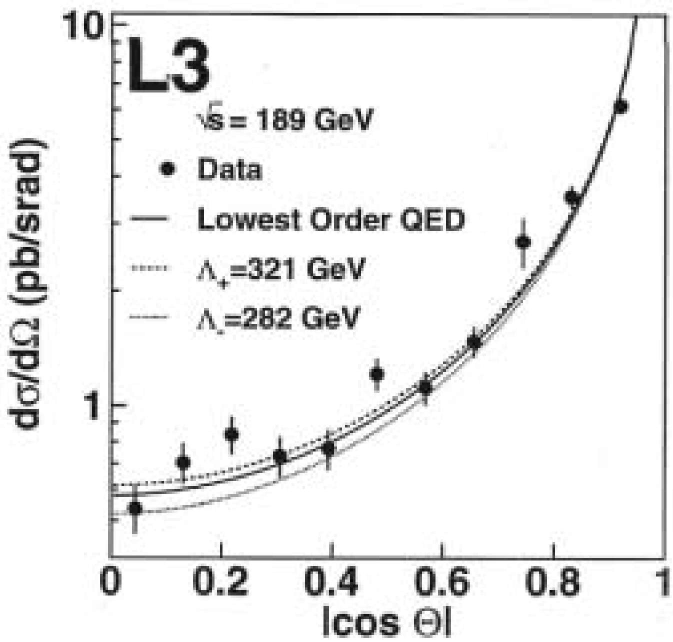


Figure 4: Differential cross section as a function of $\cos\theta$ for the process $e^+e^- \to \gamma\gamma(\gamma)$. The points show the measurements corrected for efficiency and additional photons. The solid line corresponds to the lowest order QED prediction. The dashed and dotted lines represent the limits obtained for deviations from QED, taking into account all the L3 data at centre-of-mass energies up to that presented in the corresponding plot.

QED divergences & renormalization

Consider 1-loop correction to $e\mu$ scattering, in particular

"vacuum polarization" due to virtual e^+e^- pairs

Neglecting m_e,

$$\mathcal{M} = +\frac{ie^2}{q^4}\,\bar u_3 \gamma_\mu u_1 \underbrace{\left[e^2\int \frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{Tr}[\gamma^\mu \not k \gamma^\nu (\not k - \not q)]}{k^2(k-q)^2}\right]}_{\equiv I^{\mu\nu}(q)} \bar u_4 \gamma_\nu u_2$$

· Note that

$$q_\mu I^{\mu\nu}(q) = e^2\int \frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{Tr}[\overset{\not k - (\not k - \not q)}{\not q}\,\not k \gamma^\nu (\not k - \not q)]}{k^2(k-q)^2}$$

$$= 4e^2\int \frac{d^4k}{(2\pi)^4}\left[\frac{(k-q)^\nu}{(k-q)^2} - \frac{k^\nu}{k^2}\right] = 0$$

at least, it *should* be if regulator is OK

Similarly,

$$\boxed{q_\nu I^{\mu\nu}(q) = 0}$$

$$\therefore\; I^{\mu\nu}(q) = (g^{\mu\nu} q^2 - q^\mu q^\nu)\, I(q^2)$$

only this term contributes to $\mathcal{M}$

this term is easier to work out

Feynman's trick to combine denominators:

$$\frac{1}{k^2}\frac{1}{(k-q)^2} = \frac{1}{q^2 - 2q\cdot k}\left[\frac{1}{k^2} - \frac{1}{(k-q)^2}\right] = \int_0^1 dx\,\frac{1}{[k^2 - 2x\,k\cdot q + xq^2]^2}$$

$$= \int_0^1 dx\,\frac{1}{(\ell^2 + x(1-x)q^2)^2} \quad \text{where } \begin{cases} \ell = k - xq \\ k = \ell + xq \end{cases}$$

(12.6) (4)

So,

$$I^{\mu\nu} = e^2 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{\mathrm{Tr}\left[\gamma^\mu (\not{l} + x\not{q}) \gamma^\nu (\not{l} - (1-x)\not{q})\right]}{\left[l^2 + x(1-x)q^2\right]^2}$$

$$\frac{1}{4}\mathrm{Tr}[\ldots] = (l^\mu + x q^\mu)(l^\nu - (1-x)q^\nu) + \mu \leftrightarrow \nu$$
$$- g^{\mu\nu}(l^2 + (2x-1)\, l \cdot q - x(1-x)q^2)$$

$$= 2 l^\mu l^\nu - g^{\mu\nu} l^2 - 2x(1-x) q^\mu q^\nu + g^{\mu\nu} x(1-x) q^2$$
$$+ [\text{linear in } l^\nu, l^\mu]$$

integrate to ϕ due to $l^\mu \leftrightarrow -l^\mu$ symmetry

$$\int \frac{d^4 l}{(2\pi)^4} \frac{l^\mu l^\nu}{[l^2 + x(1-x)q^2]^2} = g^{\mu\nu} \cdot f(l^2)$$

We just want the $q^\mu q^\nu$ terms

"UV cutoff"

$$\Rightarrow I(q^2) = e^2 \cdot 2 \cdot 4 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{x(1-x)}{(l^2 + x(1-x)q^2)^2}$$

(vol. of 4-sphere $= 2\pi^2$)

Now

$$\int^{M} \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 + x(1-x)q^2)^2} \approx \frac{\int d\Omega_4}{(2\pi)^4} \int_0^M \frac{dl\, l^3}{l^4}$$

$$\approx \frac{2\pi^2}{(2\pi)^4} \cdot \frac{1}{2} \int^{M^2} \frac{dl^2}{l^2} = \frac{1}{16\pi^2}(\ln M^2 + \text{finite})$$

$$\text{So, } I(q^2) = \frac{e^2}{2\pi^2} \ln M^2 \underbrace{\int_0^1 dx\, x(1-x)}_{\frac{1}{2} - \frac{1}{3} = \frac{1}{6}} + \text{finite}$$

$$I(q^2) = \frac{e^2}{12\pi^2} \ln M^2 + \text{finite}$$

calling our original $e \to e_0(M)$

So, finally, "the bare charge"

$$\boxed{\mathcal{M} = -e_0^2 \Big[\bar{u}_3 \gamma^\mu u_1 \frac{g_{\mu\nu}}{q^2} \Big\{ 1 - \frac{e_0^2}{12\pi^2} \Big[\ln\Big(\frac{M^2}{m_e^2}\Big) - f\Big(\frac{-q^2}{m_e^2}\Big) \Big] \Big\} \times \bar{u}_4 \gamma^\mu u_1}$$

- To remove the divergence, we define the renormalized charge

$$\boxed{e_R = e_0(M) \sqrt{1 - \frac{e_0^2(M)}{12\pi^2} \ln\Big(\frac{M^2}{m_e^2}\Big)}}$$ such that e_R is finite as $M \to \infty$

Then

$$\boxed{\mathcal{M}_{(ren)} = -e_R^2 \, \bar{u}_3 \gamma^\mu u_1 \frac{g_{\mu\nu}}{q^2} \Big\{ 1 + \frac{e_R^2}{12\pi^2} f\Big(-\frac{q^2}{m_e^2}\Big) \Big\} \bar{u}_4 \gamma^\nu u_2}$$

$\mathcal{M}_{(ren)}$ is finite as $M \to \infty$

For $q^2 \gg m_e^2$, $f\Big(-\frac{q^2}{m_e^2}\Big) \simeq \ln\Big(\frac{q^2}{m_e^2}\Big)$

(to cancel m_e dependence in $\Big[\ln\Big(\frac{M^2}{m_e^2}\Big) - f\Big(-\frac{q^2}{m_e^2}\Big)\Big]$)

- We can absorb this dependence in $\mathcal{M}$ too, in terms of a running coupling:

$$e_R(q^2) = e_R(0) \sqrt{1 + \frac{e_R(0)^2}{12\pi^2} f\Big(-\frac{q^2}{m_e^2}\Big)}$$

or

$$\boxed{\begin{aligned} \alpha(q^2) &= \alpha(0) \Big[1 + \frac{\alpha(0)}{3\pi} f\Big(-\frac{q^2}{m_e^2}\Big) \Big] \\ &\simeq \alpha(0) \Big[1 + \frac{\alpha(0)}{3\pi} \ln\Big(\frac{q^2}{m_e^2}\Big) \Big] \end{aligned}}$$ $|q| \gg m_e$

[α increases logarithmically as q increases]