Lecture 11 QED (CONT.)

11.1

2 ways to proceed now

(1) For unpolarized cross-sections, especially when $m_i^2 \simeq s, t, u$, it is often most efficient to not compute $\mathcal{M}$ explicitly, but go directly to $\left(\frac{1}{\prod_i s_i}\right)\sum_{\text{spins}} |\mathcal{M}|^2 \equiv \langle |\mathcal{M}|^2 \rangle$

• For this we'll use Casimir's trick:
From the fermion line

we have $\mathcal{M} \ni \bar{u}(a)\Gamma_1 u(b)$

some strings of γ matrices

$$\mathcal{M}^* \ni [\bar{u}(a)\Gamma_2 u(b)]^*$$

[might be another diagram in "interference"]

Typically encounter in $\langle|\mathcal{M}|^2\rangle$:

$\Rightarrow$ $\boxed{G = [\bar{u}(a)\Gamma_1 u(b)]\,[\bar{u}(a)\Gamma_2 u(b)]^\dagger}$

1×1 matrix $\rightarrow$ Hermitian conjugate it

$\Rightarrow u^\dagger(b)\Gamma_2^\dagger \gamma^{0\dagger} u(a)$

$= \bar{u}(b)\bar{\Gamma}_2 u(a)$ $\quad$ $\bar{\Gamma}_2 \equiv \gamma^0 \Gamma_2^\dagger \gamma^0$

• Now $\sum_{s_b} G = \bar{u}(a)\Gamma_1 \left\{\sum_{s_b} u(b)\bar{u}(b)\right\} \bar{\Gamma}_2 u(a)$

Here we need the completeness relation:

(e^-) $\quad \sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = \not{p} + m$

Also: (e^+) $\quad \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = \not{p} - m$

and for photons:

(γ) $\quad \sum_{s=1,2} (\epsilon_{(s)})_i (\epsilon^*_{(s)})_j = \delta_{ij} - \hat{p}_i \hat{p}_j$ (Coulomb gauge)

$$\underbrace{V_i M_{ij} W_j = M_{ij} W_j V_i}$$

So $\displaystyle\sum_{b \text{ spins}} G = \bar{u}(a)\Gamma_1(\not p_b + m)\bar{\Gamma}_2 u(a)$

$$\sum_{a,b \text{ spins}} G = \mathrm{Tr}\Big[\Gamma_1(\not p_b + m)\bar{\Gamma}_2 \sum_{a \text{ spins}} u(a)\bar{u}(a)\Big]$$

$$= \mathrm{Tr}\left[\Gamma_1(\not p_b + m)\bar{\Gamma}_2(\not p_a + m)\right]$$

Note
$\bar{\gamma}_\mu = \gamma^0 \gamma_\mu^\dagger \gamma^0 = \gamma_\mu$

Thus for e-μ scattering, apply trick twice (e, μ)

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4}\frac{e^4}{((p_1 - p_3)^2)^2} \mathrm{Tr}\left[\gamma^\mu(\not p_1 + m)\gamma^\nu(\not p_3 + m)\right] \times \mathrm{Tr}\left[\gamma_\mu(\not p_2 + M)\gamma_\nu(\not p_4 + M)\right]$$

initial spins averaged $(\frac{1}{2})^2$

Trace rules: $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

$\mathrm{Tr}(\alpha A) = \alpha\,\mathrm{Tr}(A)$ $\quad \alpha = $ number

$g_{\mu\nu} g^{\mu\nu} = 4$ $\quad \mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ (cyclic)

$\gamma^\mu \gamma^\nu = 2g^{\mu\nu} - \gamma^\nu \gamma^\mu$ [move γ's around]

$\Leftrightarrow \not a \not b = 2a\cdot b - \not b \not a$

$\gamma_\mu \gamma^\mu = 4$

$\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu \quad \longleftrightarrow \quad \gamma_\mu \not a \gamma^\mu = -2\not a$

$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda} \quad \longleftrightarrow \quad \gamma_\mu \not a \not b \gamma^\mu = 4a\cdot b$

Trace theorems $\quad \mathrm{Tr}[\text{odd } \# \ \gamma\text{'s}] = 0$

$\mathrm{Tr}(1) = 4$

$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$

And with γ^5: If even $\#$ γ^5's put them adjacent, use $(\gamma^5)^2 = 1$

If odd $\#$, use

$\mathrm{Tr}(\gamma^5) = 0$

$\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu) = 0$

$\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}$

$\epsilon^{\mu\nu\lambda\sigma}$: -1 even perms of 0123, $+1$ odd perms

(11.3) (Mbb)

Apply trace rules to $e\mu$ scattering

$$\mathrm{Tr}\left[\gamma^\mu(\not p_1+m)\gamma^\nu(\not p_3+m)\right] = \mathrm{Tr}\left[\gamma^\mu \not p_1 \gamma^\nu \not p_3\right] + m^2\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]$$
$$= 4\left(p_1^\mu p_3^\nu - g^{\mu\nu} p_1\cdot p_3 + p_3^\mu p_1^\nu + m^2 g^{\mu\nu}\right)$$

$$\Rightarrow \langle|M|^2\rangle = \frac{4e^4}{(p_1-p_3)^2)^2}\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2 - p_1\cdot p_3)\right]\left[p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + g_{\mu\nu}(M^2 - p_2\cdot p_4)\right]$$

$$\boxed{\langle|M|^2\rangle = \frac{8e^4}{((p_1-p_3)^2)^2}\left[p_1\cdot p_2\; p_3\cdot p_4 + p_1\cdot p_4\; p_2\cdot p_3 - M^2(p_1\cdot p_3) - m^2 p_2\cdot p_4 + 2m^2M^2\right]}$$

$((-4+4)\; p_1\cdot p_3\; p_2\cdot p_4 \to 0)$

skip in class

Ex: $e\mu \to e\mu$ (or $ep\to ep$) with $E_e \ll M$.

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi M)^2}\langle|M|^2\rangle$$

$$\langle|M|^2\rangle = \left(\frac{e^2 M}{\vec p^{\,2}\sin^2\frac{\theta}{2}}\right)^2 \left[E^2 - \frac{m^2}{2} - \vec p^{\,2}\sin^2\frac{\theta}{2} - \frac{m^2}{2} + m^2\right]$$

$4\pi\alpha$

$m^2+\vec p^{\,2}$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{2\vec p^{\,2}\sin^2\frac{\theta}{2}}\right)^2\left[m^2+\vec p^{\,2}\cos^2\frac{\theta}{2}\right]}$$

Mott formula

$p_1 = (E, \vec p_1)$

$p_2 = (M, \vec 0)$

$p_3 = (E, \vec p_3)$

$p_4 = (M, \vec 0)$ (approx)

$$(p_1-p_3)^2 = -(\vec p_1 - \vec p_3)^2 = -2\vec p^{\,2}(1-\cos\theta) = -4\vec p^{\,2}\sin^2\frac{\theta}{2}$$

$$p_1\cdot p_3 = \frac{1}{2}\left[p_1^2 + p_3^2 - (p_1-p_3)^2\right] = m_e^2 + 2\vec p^{\,2}\sin^2\frac{\theta}{2}$$

$$p_1\cdot p_2\; p_3\cdot p_4 = p_1\cdot p_4\; p_2\cdot p_3 = ME^2$$

$p_2\cdot p_4 = M^2$

$\boxed{\begin{array}{l} m, M \to 0 \\ HE \text{ limit} \end{array}}$

$p_1 \cdot p_2 = p_3 \cdot p_4 = s/2$

$p_1 \cdot p_4 = p_2 \cdot p_3 = -t/2$

$(p_1 - p_3)^2 = u$

$$\langle |M|^2 \rangle = 2e^4 \left[\left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2 \right]$$

$16\pi^2\alpha^2$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2 s} 2e^4 \left[\left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2 \right]$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \frac{s^2 + t^2}{u^2} \longleftarrow \begin{array}{l} t = -s/2(1+\cos\theta) \\ u = -s/2(1-\cos\theta) \end{array}$$

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(2) <u>2nd approach</u>: With longitudinal polarization, and/or high energy limit, compute <u>helicity amplitudes</u>

$\sigma \cdot \hat{p} = \pm 1$ <u>Helicity states</u> <u>for $m=0$</u>

$$u^\pm = N \begin{pmatrix} u_A \\ \pm u_A \end{pmatrix}$$

$$u_B = \frac{p \cdot \sigma}{E} u_A = \pm u_A$$

$$N = \sqrt{\frac{1}{2|\vec{p}|(|\vec{p}| \pm p_z)}} \qquad u_A = \begin{pmatrix} p_z \pm |\vec{p}| \\ p_x + ip_y \end{pmatrix}$$

$$\sigma \cdot \vec{p}\, u_A = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} p_z \pm |\vec{p}| \\ p_x + ip_y \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{p_z^2 + 0 + p_x^2 + p_y^2}^{|p|^2} \pm p_z |\vec{p}| \\ \pm |\vec{p}| (p_x + ip_y) \end{pmatrix} = \pm |\vec{p}|\, u_A$$

- Also <u>chirality</u> (γ^5) eigenstates (for $m=0$)

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \gamma^5 u_\pm = \pm u_\pm$$

$$\boxed{\text{helicity} = \text{chirality for } m=0 \;\begin{array}{l|l} LH & \vec{\sigma}\cdot\hat{p} = +1 \Leftrightarrow \frac{1}{2}(1+\gamma_5)u = u \\ RH & \phantom{\vec{\sigma}\cdot\hat{p}} = -1 \Leftrightarrow \frac{1}{2}(1-\gamma_5)u = u \end{array}}$$

11.5 11/4/16

QED interaction preserves chirality

$\{\gamma^\mu, \gamma^5\} = 0$

$$\Rightarrow \quad \bar{u}_{3(-)} \gamma^\mu u_{1(+)} = \bar{u}_{3(-)} \gamma^\mu \tfrac{1}{2}(1+\gamma_5) u_{1(+)}$$

$$= u^\dagger_{3(-)} \tfrac{1}{2}(1-\gamma_5^\dagger) \gamma^0 \gamma^\mu \tfrac{1}{2}(1+\gamma_5) u_{1(+)}$$

$$= \bar{u}_{3(-)} \tfrac{1}{2}(1+\gamma_5) \gamma^\mu \tfrac{1}{2}(1+\gamma_5) u_{1(+)}$$

$$= \bar{u}_{3(-)} \gamma^\mu \tfrac{1}{2}(1-\gamma_5) \tfrac{1}{2}(1+\gamma_5) u_{1(+)} = 0$$

∴ Only $\bar{u}_{3(+)} \gamma^\mu u_{1(+)} \bar{u}_{4(+)} \gamma_\mu u_{2(+)}$

and $\bar{u}_{3(+)} \gamma^\mu u_{1(+)} \bar{u}_{4(-)} \gamma_\mu u_{2(-)}$

⊕ amp's with all spins flipped together are nonzero.

these are related by parity:

$P: \hat{p} \to -\hat{p}$

$\vec{s} \to +\vec{s}$

$\Rightarrow \vec{s}\cdot\hat{p} \to -\vec{s}\cdot\hat{p}$

- There are some nifty methods ("helicity techniques") to complete helicity amplitudes directly.
- However, we don't have time to develop this formalism completely.
- Instead we'll compute $|\mathcal{M}|^2$ for helicity states, then take the square root.

• For $\left|\mathcal{M}(e_1^+ \mu_2^\pm \to e_3^+ \mu_4^\pm)\right|^2 \equiv |\mathcal{M}|^2_\pm$

we need

$$\mathrm{Tr}\left[\gamma^\mu \underbrace{\tfrac{1}{2}(1+\gamma_5)\not p_1}_{\text{from } \frac{1}{2}(1+\gamma_5)u_1\bar{u}_1} \gamma^\nu \cancel{\tfrac{1}{2}(1+\gamma_5)} \not p_3\right]$$

(not needed because of helicity conservation)

$$\times\ \mathrm{Tr}\left[\gamma_\mu \tfrac{1}{2}(1\pm\gamma_5)\not p_2 \gamma_\nu \cancel{\tfrac{1}{2}(1\pm\gamma_5)} \not p_4\right]$$

$$= \tfrac{1}{4}\mathrm{Tr}[\gamma^\mu \not p_1 \gamma^\nu \not p_3]\,\mathrm{Tr}[\gamma_\mu \not p_2 \gamma_\nu \not p_4]$$

$$\pm \tfrac{1}{4}\mathrm{Tr}[\gamma^\mu \gamma_5 \not p_1 \gamma^\nu \not p_3]\,\mathrm{Tr}[\gamma_\mu \gamma_5 \not p_2 \gamma_\nu \not p_4]$$

odd terms in γ_5 $\Rightarrow$ linear in $\varepsilon_{\mu\nu\rho\sigma}$. But antisym $\Rightarrow$ $\varepsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma = 0$ $i=1,3,3,4$; $p_4 = -p_3 + p_1 + p_2$

$$= \tfrac{1}{4}\,(\text{unpolarized trace})$$

$$\pm \tfrac{1}{4}(4i)^2 \varepsilon^{\mu\nu}{}_{\alpha\beta}\, p_1^\alpha p_3^\beta\, \varepsilon_{\mu\nu\gamma\delta}\, p_2^\gamma p_4^\delta$$

Use identity $\varepsilon^{\mu\nu}{}_{\alpha\beta}\varepsilon_{\mu\nu\gamma\delta} = -2(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})$

$$= \tfrac{1}{4}\cdot 32\,(p_1\cdot p_2\, p_3\cdot p_4 + p_1\cdot p_4\, p_2\cdot p_3)$$

$$\pm(-4)\cdot(-2)\,(p_1\cdot p_2\, p_3\cdot p_4 - p_1\cdot p_4\, p_2\cdot p_3)$$

(no $\frac{1}{4}$ for spin avg. here)

$$\Rightarrow |\mathcal{M}|^2_+ = \frac{e^4\cdot 16\,(p_1\cdot p_2)^2}{((p_1-p_3)^2)^2} = 4e^4\left(\frac{s}{u}\right)^2$$

$$|\mathcal{M}|^2_- = \frac{e^4}{((p_1-p_3)^2)^2}\cdot 16\,(p_1\cdot p_4)^2 = 4e^4\left(\frac{t}{u}\right)^2$$

Take square root

$$\Rightarrow \boxed{\begin{aligned} \mathcal{M}(e_1^+\mu_2^+ \to e_3^+\mu_4^+) &= 2e^2\,\frac{s}{u}\ (\text{phase}) \\ \mathcal{M}(e_1^+\mu_2^- \to e_3^+\mu_4^-) &= 2e^2\,\frac{t}{u}\ (\text{phase}) \end{aligned}}$$

doesn't usually matter, if no ⊥ pol.

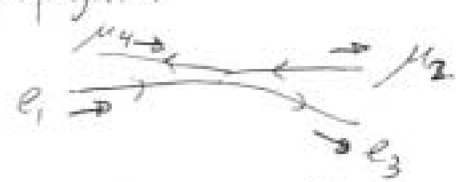
- We can understand the 2nd formula by a helicity argument:

$$t = -\frac{s}{2}(1+\cos\theta)$$
$$u = -\frac{s}{2}(1-\cos\theta)$$

- $u \to 0$ for forward scattering $\theta \to 0$

$\frac{1}{u}$ divergence due to photon propagator

no helicity suppression

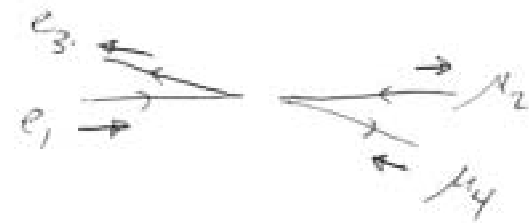


$J_z^{(i)} = +1 \quad \to \quad J_z^{(f)} = +1$

(similarly for other helicity configuration)

- $t \to 0$ for backward scattering $(\theta = \pi)$

Now $J_z^{(i)} = +1 \to J_z^{(f)} = -1$



$\Rightarrow$ amplitude must vanish as $t \to 0$

And for $m=0$, it is a dimensionless function of t/u, with a $1/u$ pole

$\Rightarrow$ Must be proportional to t/u (!) ✓

- Recover unpolarized $\langle |\mathcal{M}|^2 \rangle$ by summing over helicities:

- There are $2^4 = 16$ helicity configurations, but only $2^2 = 4$ are nonzero for $m=0$. And 2 of these are related to the others by P.

$$\Rightarrow \langle |\mathcal{M}|^2 \rangle = \underbrace{\left(\tfrac{1}{2}\right)^2}_{\text{(initial spin avg.)}} \cdot \underbrace{2}_{\text{(parity P)}} \cdot \left[|\mathcal{M}|_+^2 + |\mathcal{M}|_-^2 \right]$$

$$= 2e^4 \left[\left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2 \right]$$

$\boxed{\text{Crossing symmetry}}$ $\boxed{e_1 \mu_2 \to e_3 \mu_4}$

formulas can be re-used for a more <u>experimentally accessible</u> (!) process,

$\boxed{e_2^+ e_1^- \to \mu_3^+ \mu_4^-}$ $\boxed{e_3 \mu_2 \leftrightarrow \bar{e}_2 \bar{\mu}_3}$

- Related by $p_2 \leftrightarrow -p_3$
- Also $\bar{u}(p_3) \longrightarrow \bar{v}(p_2)$
  $u(p_2) \longrightarrow v(p_2)$
- However, this is compensated for in the completeness relation $u\bar{u} \to (\not{p} + m)$ vs. $v\bar{v} \to (\not{p} - m)$

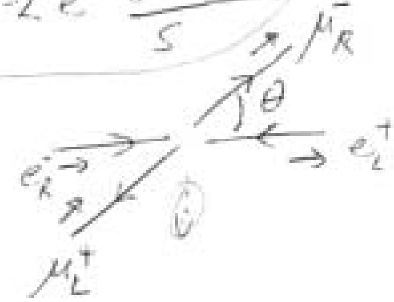
$p_2 \leftrightarrow p_3$

$\Leftrightarrow \; s \leftrightarrow u$

$s = (p_1+p_2)^2$, $\quad u = (p_1-p_3)^2$

- <u>Except</u>: <u>helicity</u> <u>gets</u> <u>flipped</u> in crossing a particle (v ~~helicities~~ spins "backward")

$$\therefore \; \boxed{\mathcal{M}(e_L^+ \underset{(+)}{e_R^-} \to \mu_L^+ \mu_R^-) = -2e^2 \frac{(p_{e^-} - p_{\mu^+})^2}{s} = e^2(1+\cos\theta)}$$



$$\boxed{\mathcal{M}(e_L^+ e_R^- \to \mu_R^+ \mu_L^-) = e^2(1-\cos\theta)}$$

$$\langle |\mathcal{M}|^2 \rangle = \left(\tfrac{1}{2}\right)^2 2 \cdot e^4 \left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right]$$

$$= e^4 \left[1 + \cos^2\theta\right]$$

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi}{(8\pi)^2 s} \underbrace{e^4}_{(4\pi\alpha)^2} \left[1+\cos^2\theta\right] = \frac{\pi\alpha^2}{2s}\left[1+\cos^2\theta\right]$$

$$\sigma = \frac{\pi\alpha^2}{2s} \int_{-1}^{1} dx\, (1+x^2) = \frac{\pi\alpha^2}{2s} \cdot 2 \cdot \left(1+\tfrac{1}{3}\right), \quad \boxed{\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}}$$