LECTURE 10 QED (CONT.) 10.1

Parity: Claim: $\psi \to \psi' = \gamma^0 \psi$ $\quad \bar{\psi} = \psi^\dagger \gamma^0 \to \psi^\dagger (\gamma^0)^2 = \bar{\psi}\gamma^0$

· Check it on $\bar{\psi}\gamma^\mu\psi$:

transforms to $\hookrightarrow \bar{\psi}\underbrace{\gamma^0\gamma^\mu\gamma^0}\psi$

$\gamma^0 \quad \mu = 0$

$-\gamma^i \quad \mu = i$

Like $\begin{pmatrix} E \to E \\ \vec{p} \to -\vec{p} \end{pmatrix}$ ✓

· How does $\bar{\psi}\psi$ transform under P?

$\bar{\psi}\psi \to \bar{\psi}\gamma^0\gamma^0\psi = +\bar{\psi}\psi \Rightarrow$ true scalar

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is an important matrix because it anti-commutes with all γ^μ: $\{\gamma^\mu, \gamma^5\} = 0$

$\therefore [\sigma^{\mu\nu}, \gamma^5] = i\gamma^\mu\gamma^\nu\gamma^5 - i\gamma^5\gamma^\mu\gamma^\nu$
$= -i\gamma^\mu\gamma^5\gamma^\nu + i\gamma^\mu\gamma^5\gamma^\nu = 0$

$[\sigma^{\mu\nu}, \gamma^5] = 0$ means spinors can be split into 2 sets, $\psi_\pm = \underbrace{\tfrac{1}{2}(1 \pm \gamma^5)}\psi$

$\sigma^{\mu\nu}$ (Lorentz gen's)

chirality ($\approx$ helicity) projection operators

Also it means that

$\bar{\psi}\gamma^5\psi$ is ~~another~~ a (pseudo)-scalar

because P: $\bar{\psi}\gamma^5\psi \to \bar{\psi}\gamma^0\gamma^5\gamma^0\psi = -\bar{\psi}\gamma^5\psi$

$\psi_0^* \psi_j$ contains 16 independant components
(4×4)

These break up into combinations with nice Lorentz transformations as

(γ's inserted)				
0	$\bar\psi \psi$	scalar	1	component
1	$\bar\psi \gamma^\mu \psi$	vector	4	"
2	$\bar\psi \sigma^{\mu\nu} \psi$	antisym. tensor	6	"
3	$\bar\psi \gamma^\mu \gamma^5 \psi$	pseudovector	4	"
4	$\bar\psi \gamma^5 \psi$	pseudoscalar	1	"

E&M revisited: Recall Maxwell's eqn's

$$\vec\nabla \cdot \vec E = 4\pi\rho \qquad \vec\nabla \cdot \vec B = 0$$

$$\vec\nabla \times \vec E + \frac{\partial \vec B}{\partial t} = 0 \qquad \vec\nabla \times \vec B - \frac{\partial \vec E}{\partial t} = 4\pi \vec J$$

and our packaging $F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

LHS of inhomogeneous Maxwell's eq's

$$\ni \quad \partial_\mu F^{\mu 0} = +\partial_i F^{i0} = -\partial_i F^{0i} = -\nabla \cdot E = -4\pi J^0$$

where $J^\mu = (\rho, \vec J)$

$\partial_i F^{xi} = -\partial_y B_z + \partial_z B_y = (\nabla \times B)_x$

And $\ni \quad \partial_\mu F^{\mu j} = \partial_0 F^{0j} + \partial_i F^{ij} = -(\vec\nabla \times \vec B)_j + \frac{\partial}{\partial t} E_j = -4\pi J^j$

$\therefore \; \partial_\mu F^{\mu\nu} = -4\pi J^\nu$ is relativistic statement of inhomogeneous eqn's

• Note that

$$\underset{\text{sym}}{\partial_\mu \partial_\nu} \underset{\text{antisym.}}{F^{\mu\nu}} = 0 \Rightarrow \boxed{\partial_\nu J^\nu = 0}$$

"continuity eqn"

J^ν must ~~be~~ a "conserved current"

$$Q = \int d^3x \, \rho = \int d^3x \, J^0$$

$$\frac{dQ}{dt} = \int d^3x \, \partial_0 J^0 = -\int d^3x \, \underbrace{\frac{\partial}{\partial x^i} J^i}_{\text{total derivative}} = 0$$

$$\partial_\nu J^\nu = 0 \Leftrightarrow \frac{d}{dt} Q = 0$$

(electric charge conservation)

• Homogeneous eqn's are solved by introducing a vector potential:

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$0 = \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) \Rightarrow \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

Equivalent to

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

where $A^\mu = (V, \vec{A})$

• In terms of A^μ, Maxwell's eqn's read

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = (-) 4\pi J^\nu$$

• We can derive this equation, and the Dirac eqn. ~~[illegible]~~ in an EM field, from the QED Lagrangian:

$$\mathcal{L}_{QED}(A^\mu, \bar\psi, \psi) = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\gamma^\mu \partial_\mu - m)\psi - e\bar\psi\gamma^\mu\psi A_\mu$$

$$= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\gamma^\mu D_\mu - m)\psi$$

$$\left[\begin{array}{l} D_\mu \equiv \partial_\mu + ieA_\mu \\ \text{"gauge covariant derivative"} \end{array}\right]$$

A^μ eqn. of motion:

$$0 = \frac{\partial}{\partial x^\mu}\frac{\delta \mathcal{L}}{\delta(\partial_\mu A^\nu)} - \frac{\delta \mathcal{L}}{\delta A^\nu}$$

$$= \partial_\mu \left[\left(-\frac{1}{4}\right)(2)(2) F^{\mu\nu}\right] - e\bar\psi\gamma^\nu\psi$$

varying each F is equivalent

$\partial^\mu A^\nu - \partial^\nu A^\mu$

equivalent after $\mu \leftrightarrow \nu$

$$\Leftrightarrow \boxed{\partial_\mu F^{\mu\nu} = -J^\nu \quad \text{with} \quad J^\nu = e\bar\psi\gamma^\nu\psi}$$

$\bar\psi$ treated indep. of ψ. [complex field]

$\bar\psi$ eqn. of motion: $\quad 0 = \frac{\delta\mathcal{L}}{\delta\bar\psi} = \boxed{(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0}$

$\boxed{\text{Gauge invariance}}$ local symmetry

Lagrangian is invariant under a
$A_\mu \to A_\mu + \partial_\mu\lambda$ (Any $\lambda(x)$)

$$\Rightarrow F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu(\partial_\nu\lambda) - \partial_\nu(\partial_\mu\lambda) = F_{\mu\nu}$$

With electron:

$$\psi(x) \to e^{-ie\lambda}\psi(x)$$

$$\partial_\mu\psi(x) \to e^{-ie\lambda}\partial_\mu\psi(x) - ie(\partial_\mu\lambda)\psi(x)$$

$$\Rightarrow (\partial_\mu + ieA_\mu)\psi(x) \to e^{-ie\lambda}(\partial_\mu + ieA_\mu)\psi(x) + ie\cancel{(\partial_\mu\lambda - \partial_\mu\lambda)\psi(x)}$$

• So gauge-covariant ~~[illegible]~~ derivative makes $\mathcal{L}_{\psi}$ gauge invariant:

$$\bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \rightarrow \bar{\Psi}\cancel{e^{+i\lambda}} \cdot \cancel{e^{-i\lambda}}(i\gamma^\mu D_\mu - m)\Psi \quad \checkmark$$

• Using $A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \lambda$,
we can arrange that $\boxed{\partial_\mu A^\mu = 0}$ (Lorentz gauge)

$$\Rightarrow \boxed{(\partial_\nu \partial^\nu) A^\mu = (-)4\pi J^\mu} \qquad (\Box \equiv \partial^2 \equiv \partial_\mu \partial^\mu)$$

Equivalent to KG eqn. for $J^\mu = 0$

• Gauge is still not completely fixed, since λ with $\Box\lambda = 0$ leaves $\partial_\mu A^\mu = 0$.

• Plane wave solutions to free photon wave eqn:

$$\boxed{A^\mu = \underset{\text{norm.}}{a}\, \epsilon^\mu(p)\, e^{-ip\cdot x}} \quad \text{with } p^2 = 0$$

$\epsilon^\mu(p)$: polarization vector

$\epsilon^\mu \rightarrow \epsilon^\mu + p^\mu$ should give equivalent results

Lorentz gauge: $\Rightarrow (\epsilon \cdot p = 0)$

~~[illegible]~~ often choose $\epsilon^0 = 0$

$$\Rightarrow \vec{\epsilon} \cdot \vec{p} = 0$$

E.g. for $\vec{p}$ in $+\hat{z}$ direction, can use

$\vec{\epsilon}_x = (1, 0, 0)$ (x, y, z) — $\hat{x}$ plane polarized

$\vec{\epsilon}_y = (0\ 1\ 0)$ — $\hat{y}$ plane polarized

$\vec{\epsilon}^{+} = \frac{1}{\sqrt{2}}(1\ i\ 0)$ — RH circularly polarized "(+) helicity"

$\vec{\epsilon}^{-} = \frac{1}{\sqrt{2}}(1\ -i\ 0)$ — LH " " "(−) helicity"

helicity ≡ spin angular momentum along direction of motion

10.6

Feynman rules for QED

(1) Notation (as before).

Label in + out momenta $p_1, p_2, \dots, p_n$

Also spins of $e^\pm$, γ $s_1, s_2, \dots, s_n$

- Arrows on fermion lines preserve "direction of flow", given external arrow assignments. Arrows on (internal) photon lines $\Leftrightarrow$ arbitrary choice

(2) External lines: Factors:

Electrons: Incoming u
Outgoing $\bar{u}$

Positrons: Incoming $\bar{v}$
Outgoing v

Photons: Incoming ε^μ
Outgoing $\varepsilon^{\mu *}$

(3) Vertex factors: ($|q_e| = 1$) $\rightarrow$ All $\boxed{ie\gamma^\mu}$

Griffith writes $e\sqrt{4\pi} = \sqrt{4\pi\alpha}$ Gaussian

I write $e = \sqrt{4\pi\alpha}$ Heaviside Lorentz

(4) Propagators: $e^\pm$: $\frac{i}{\not{q} - m} = \frac{i(\not{q} + m)}{q^2 - m^2}$

γ: $\frac{-i g_{\mu\nu}}{q^2}$

(5) EM cons. (as before) $(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$ (all outgoing)

(6) Integrate over Internal Momenta $\int \frac{d^4 q}{(2\pi)^4}$

(7) Cancel Overall $(2\pi)^4 \delta^4(p_1 + p_2 + \dots - p_n)$ $\Longrightarrow$ $-i\mathcal{M}$.

(8) (Fermi statistics) Include a relative $(-)$ sign between any 2 diagrams that differ only in the interchange of 2 incoming e^- (or 2 e^+) or 2 outgoing e^- (or 2 e^+)

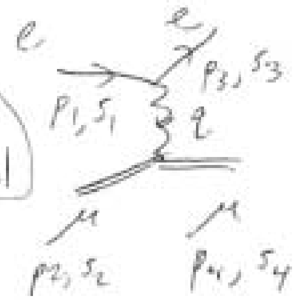
(9) Include an overall (-1) for each closed fermion loop

e.g. 

Example 7.1: $e^-\mu^- \to e^-\mu^-$ scattering

a.k.a. ("Mott scattering" $M \gg m$; "Rutherford scattering" $v \ll c$)

Only 1 diagram at tree-level



Write fermion spinor strings backward along arrow from $\bar{u} \to u$

$$\Rightarrow (2\pi)^4 \int [\bar{u}^{(s_3)}(p_3)\, ie\gamma^\mu u^{(s_1)}(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}^{(s_4)}(p_4)\, ie\gamma^\nu u^{(s_2)}(p_2)] \times \delta^{(4)}(p_1 - p_3 - q)\, \delta^{(4)}(p_2 + q - p_4)\, d^4 q$$

$$\Rightarrow \boxed{\mathcal{M} = -\frac{e^2}{(p_1 - p_3)^2} [\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)] [\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)]}$$