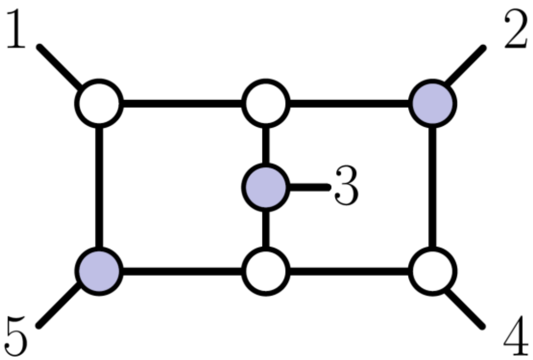
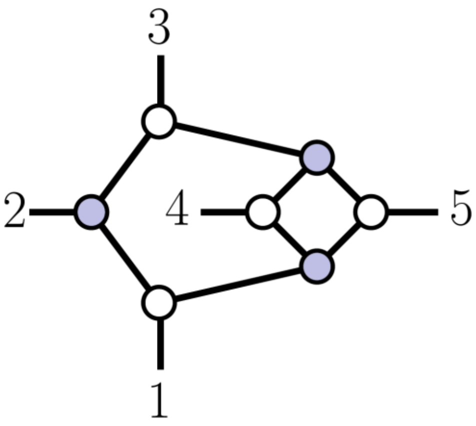
N=4 Meets N=8 at





Lance Dixon (SLAC)

S. Abreu, LD, E. Herrmann, B. Page and M. Zeng, 1812.08941, 1901.08563

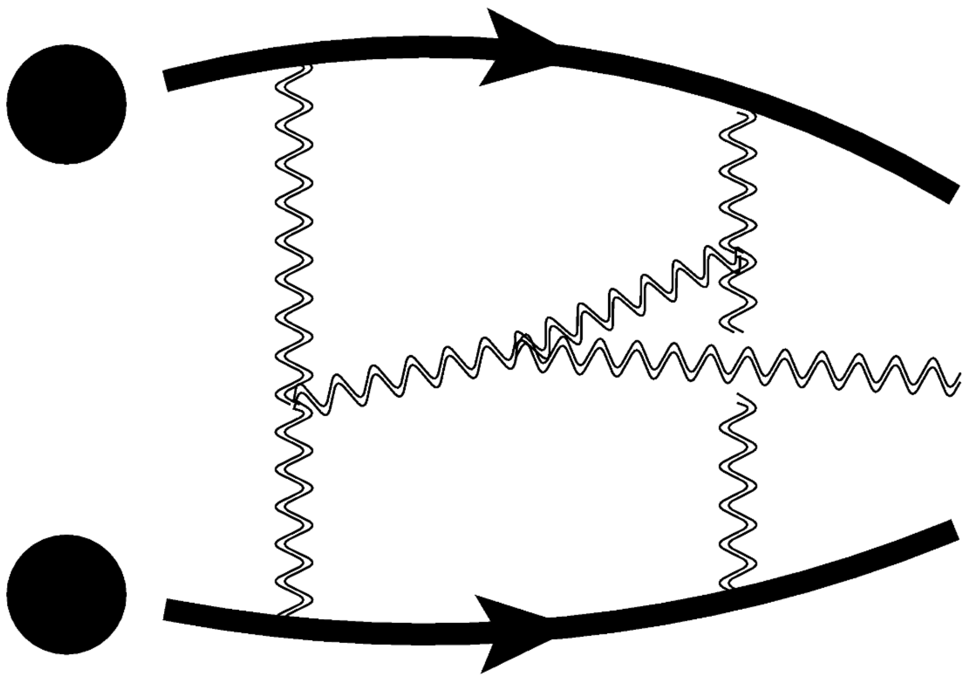
LD, E. Herrmann, K. Yan, H.-X. Zhu, 1912.nnnnn

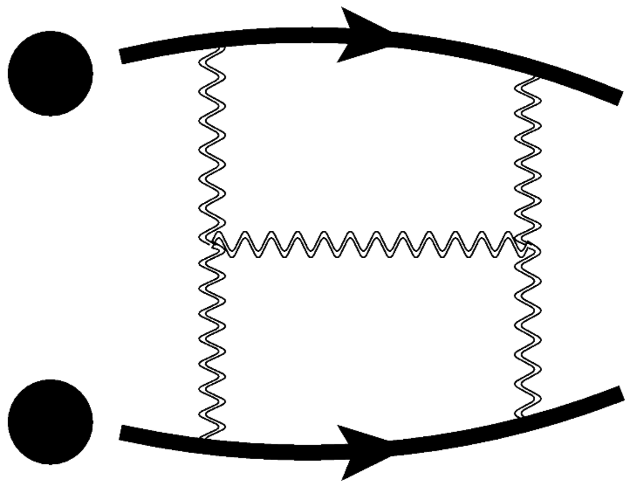
"QCD Meets Gravity"

UCLA, December 13, 2019

SLAC NATIONAL ACCELERATOR LABORATORY

Classical gravity connection





But: $\hbar \neq 0$, $m = 0$, $N_{susy} = 8$,

and symbol level

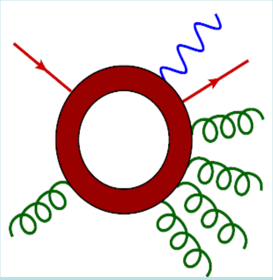
Yet $N = 8$, $m = 0$ still useful for comparison to ACV!

Talk by Julio Parra-Martinez

QCD Loop Amplitude Bottleneck

- NLO: Efficient, "prescriptive" unitarity-based methods for computing one-loop amplitudes at high multiplicity, e.g. the 8-point process $pp \rightarrow W + 5$ jets

Bern, LD, et al., 1304.1253,
BlackHat 1.0



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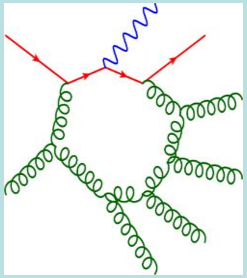
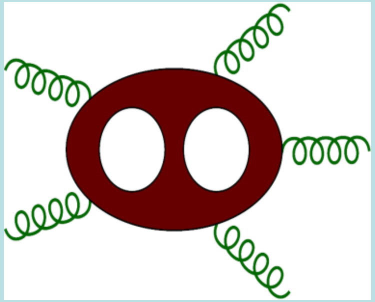


+ 256,264 more diagrams

- NNLO: Two-loop QCD amplitudes unknown beyond 2 → 2 processes, except for recent all massless 2 → 3 cases:



$gg \rightarrow ggg,\ \ qg \rightarrow qgg,\ \ q\bar{q} \rightarrow \gamma\gamma\gamma$ in large N_c (planar) limit

Gehrmann, Henn, Lo Presti, 1511.05409;
Badger, Brønnum-Hansen, Hartanto, Peraro, 1712.02229, 1811.11699;
Abreu, Dormans, Febres Cordero, Ita, Page, Zeng, Sotnikov, 1712.03946, 1812.04586, 1904.00945
Chawdhry, Czakon, Mitov, Poncelet, 1911.00479

Why is two loops so hard?

- Primarily because two-loop integrals are intricate, transcendental, multi-variate functions
- In contrast, at one loop all integrals are reducible to scalar box integrals + simpler

→ combinations of dilogarithms

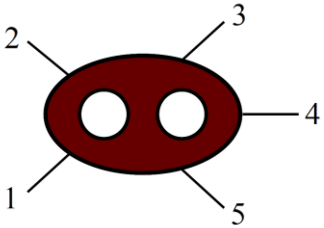
$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

\+ logarithms and rational terms

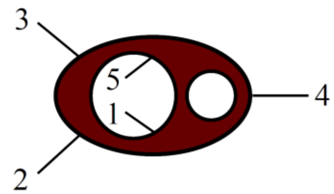
't Hooft, Veltman (1974)

Our favorite toy model(s)

- Explore nonplanar multi-loop, multi-leg amplitudes in N=4 super-Yang-Mills theory (SYM).
  Gauge group $SU(N_c)$, **NOT** in the large N_c (planar) limit
- First two-loop 2 → 3 amplitude with full color dependence – albeit still at level of symbol
- **Spinoff:** same amplitude in N=8 supergravity
- Space of functions encountered here also enters two-loop 5-point amplitudes in full-color QCD.
- Soft limit understood at function level; complicated tripole emission is same in QCD as in N=4 SYM



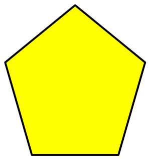
Two-loop color decomposition



Bern, Rozowsky, Yan, hep-ph/9702424

$$\mathcal{A}_5^{(2)} = \left[\frac{N_c g^2 e^{-\epsilon\gamma_E}}{(4\pi)^{2-\epsilon}}\right]^2 \Big\{ \sum_{S_5/D_5} (\mathrm{Tr}[12345] - \mathrm{Tr}[54321])(A^{\mathrm{ST}}[12345] + \frac{A^{\mathrm{SLST}}[12345]}{N_c^2}$$

$$+ \sum_{S_5/(S_3\times Z_2)} \frac{\mathrm{Tr}[15](\mathrm{Tr}[234] - \mathrm{Tr}[432])}{N_c} A^{\mathrm{DT}}[15|234] \Big\}$$



$$\mathrm{Tr}[12345] \equiv \mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}]$$

- Leading color coefficient A^{ST} obeys ABDK/BDS ansatz, Anastasiou, Bern, LD, Kosower, hep-th/0309040, Bern, LD, Smirnov, hep-th/0505205,
- Verified numerically long ago Cachazo, Spradlin, Volovich, hep-th/0602228; Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074
- Given by exponential of one-loop amplitude (need $O(\varepsilon^2)$ terms) Bern, LD, Dunbar, Kosower, hep-th/9611127

Color trace relations

Kleiss, Kuijf (1989); Bern, Kosower, (1991); Del Duca, LD, Maltoni, hep-ph/9910563; Edison, Naculich, 1111.3821; talk by Fei Teng

- Tree-level: $A_n[1,\ldots,n,\ldots]$ given in terms of permutations of $(n$-$2)!$ independent $A_n[1,\ldots,n]$ [Kleiss-Kuijf relations]
- One loop: subleading-color $A^{\rm DT}$ completely determined by permutations of $A^{\rm ST}$
- Both follow from applying Jacobi relations for structure constants f^{abc} to all-adjoint color structures.
- Two loops: Same method → Edison-Naculich relations, which we solve as:

$$A^{\rm SLST}[12345] = 5A^{\rm ST}[13254] + \sum_{\rm cyclic}\left[A^{\rm ST}[12435] - 2A^{\rm ST}[12453] + \frac{1}{2}\left(A^{\rm DT}[12|345] + A^{\rm DT}[13|245]\right)\right]$$

Integrands

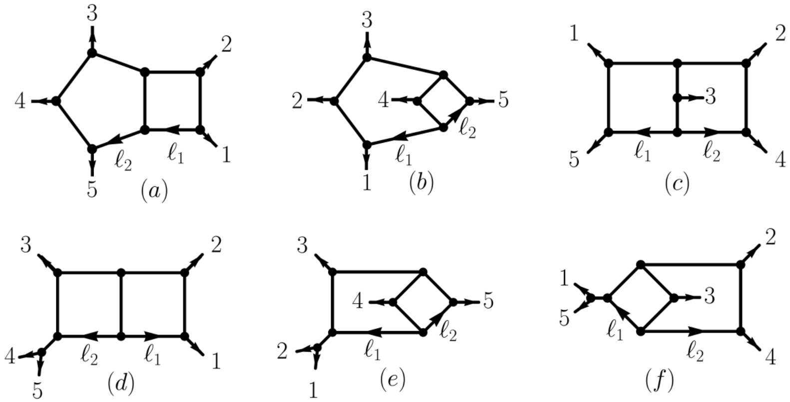
- First obtained Carrasco, Johansson, 1106.4711
in a "BCJ" form Bern, Carrasco, Johansson, 1004.0476
which simultaneously gives the integrand for N=8 supergravity as a "square" of the N=4 SYM integrand.
This integrand is manifestly D-dimensional
- Integrand also given in a four-dimensional form
Bern, Herrmann, Litsey, Stankowicz, Trnka, 1512.08591
which exposes the expected rational prefactors for pure transcendental functions g^{DT} as 6 "KK" independent Parke-Taylor factors,

$$\mathrm{PT}[\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5] \equiv \frac{\delta^8(Q)}{\langle\sigma_1\,\sigma_2\rangle\,\langle\sigma_2\,\sigma_3\rangle\,\langle\sigma_3\,\sigma_4\rangle\,\langle\sigma_4\,\sigma_5\rangle\,\langle\sigma_5\,\sigma_1\rangle}$$

$$A^{\mathrm{DT}}[15|234] = \sum_{\sigma(234)\in S_3} \mathrm{PT}[1\sigma_2\sigma_3\sigma_4 5]\; g^{\mathrm{DT}}_{\sigma_2\sigma_3\sigma_4}$$

← pure

BCJ Integrand



Carrasco, Johansson, 1106.4711

$$\gamma_{12} \equiv \gamma_{12345} \equiv i\frac{[12][23][34][45][51]}{[12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]} = i\frac{[12][23][34][45][51]}{\mathrm{tr}_5}$$

- Linear in loop momentum for N=4 SYM: multiply $N^{(x)}$ by f^{abc} structures
- Quadratic for N=8 SUGRA: square the $N^{(x)}$

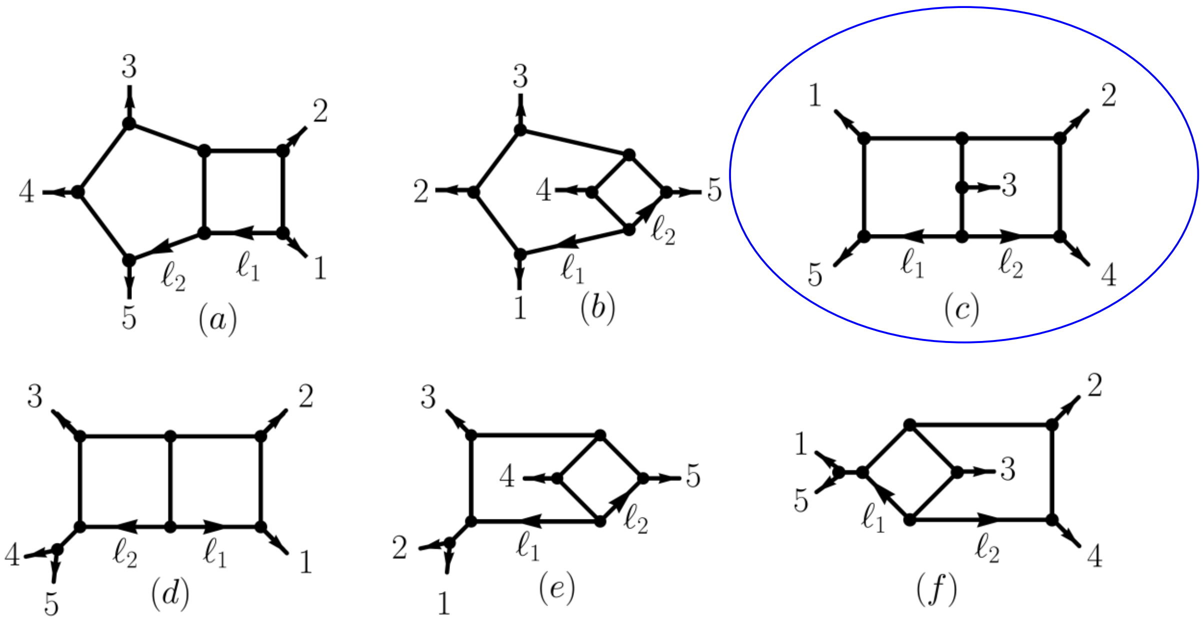
$$N^{(a,b)} = \frac{1}{4}\Big(\gamma_{12}(2s_{45}-s_{12}+\tau_{2\ell_1}-\tau_{1\ell_1})+\gamma_{23}(s_{45}+2s_{12}-\tau_{2\ell_1}+\tau_{3\ell_1}) + 2\gamma_{45}(\tau_{5\ell_1}-\tau_{4\ell_1})+\gamma_{13}(s_{12}+s_{45}-\tau_{1\ell_1}+\tau_{3\ell_1})\Big)\,,$$

$$N^{(c)} = \frac{1}{4}\Big(\gamma_{15}(\tau_{5\ell_1}-\tau_{1\ell_1})+\gamma_{25}(s_{12}-\tau_{2\ell_1}+\tau_{5\ell_1})+\gamma_{12}(s_{34}+\tau_{2\ell_1}-\tau_{1\ell_1}+2[s_{15}+\tau_{1\ell_2}-\tau_{2\ell_2}]) + \gamma_{45}(\tau_{4\ell_2}-\tau_{5\ell_2})-\gamma_{35}(s_{34}-\tau_{3\ell_2}+\tau_{5\ell_2})+\gamma_{34}(s_{12}+\tau_{3\ell_2}-\tau_{4\ell_2}+2[s_{45}+\tau_{4\ell_1}-\tau_{3\ell_1}])\Big)\,,$$

$$N^{(d-f)} = \gamma_{12}s_{45}-\frac{1}{4}\Big(2\gamma_{12}+\gamma_{13}-\gamma_{23}\Big)s_{12}\,,$$

$$s_{ij} = (k_i+k_j)^2 = 2k_i\cdot k_j\,, \quad \tau_{i\ell_j} = 2k_i\cdot\ell_j$$

Integrals



non-planar double pentagon the crux

Most topologies were known previously, e.g.

planar (a) Papadopoulous, Tommasini, Wever, 1511.09404; Gehrmann, Henn, Lo Presti, 1511.05409, 1807.09812;

hexabox (b) Chicherin, Henn, Mitev, 1712.09610

planar (d) Gehrmann, Remiddi, hep-ph/000827

nonplanar (e,f) Gehrmann, Remiddi, hep-ph/0101124

Integrals (cont.)

- Use IBP reduction method of Abreu, Page, Zeng, 1807.11522 building off earlier work based on generalized unitarity and computational algebraic geometry Gluza, Kajda, Kosower, 1009.0472; Ita, 1510.05626; Larsen, Zhang, 1511.01071; Abreu, Febres Cordero, Ita, Page, Zeng, 1712.03946
- Reduction performed numerically, at numerous rational phase space points, over a prime field to avoid enormous intermediate expressions
- Quite sufficient for full analytic reconstruction when structure of the rational function prefactors is so heavily constrained, as in N=4 SYM
- Even works for planar QCD
  Abreu, Dormans, Febres Cordero, Ita, Page, 1812.04586,…
- Our results for the integrals and the amplitude confirmed by
  Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 1812.11057, 1812.11160

Iterated integrals

Chen; Goncharov; Brown

- Generalized polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions f.
- Define by derivatives: $$d\,f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d\ln s_k$$

S = finite set of rational expressions, "symbol letters", and

$$f^{s_k} \equiv \{n-1,1\} \text{ component of a ``coproduct'' } \Delta$$

f^{s_k} are also pure functions, weight n-1

- Iterate: $$d\,f^{s_k} \;\Rightarrow\; f^{s_j,s_k} \equiv \{n-2,1,1\} \text{ component}$$

- Symbol = {1,1,…,1} component (maximally iterated)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example: Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Generalization of classical polylogs:

$$\mathrm{Li}_n(u) \;=\; \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(t) = -\ln(1-t)$$

- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Just two symbol letters: $\mathcal{S} \;=\; \{u, 1-u\}$
- Weight n = length of binary string $\vec{w}$

$$\mathcal{S}[\mathrm{Li}_n(u)] \;=\; -\ (1-u) \otimes \underbrace{u \otimes u \otimes \cdots \otimes u}_{n-1}$$

Symbol alphabet for planar 5-point

5 x 5 + 1 = 26 letters

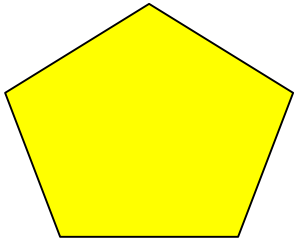
Gehrmann, Henn, Lo Presti, 1511.05409

$$\mathcal{S} = \{s_{i,i+1}, s_{i-1,i} + s_{i,i+1}, s_{i,i+1} - s_{i+2,i+3}, s_{i+3,i+4} - s_{i,i+1} - s_{i+1,i+2}, o_i, \Delta\}$$

$$s_{i,i+1} \equiv (k_i + k_{i+1})^2 \qquad o_1 = \frac{[12]\langle 23\rangle[34]\langle 41\rangle}{\langle 12\rangle[23]\langle 34\rangle[41]}$$

$$\Delta = \mathrm{tr}[\gamma_5 1234] = [12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]$$

Closed under dihedral permutations, $\mathbf{D_5}$, subset of $\mathbf{S_5}$



O_i are odd under parity, $\langle ab\rangle \Leftrightarrow [ab]$

- Most letters seen already in one-mass four-point integrals
- But not O_i or Δ

Symbol alphabet for nonplanar 5-point

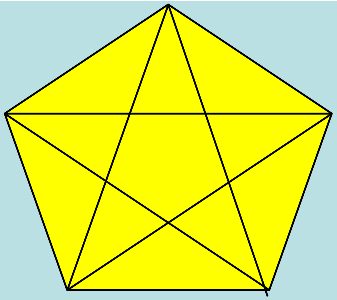
10 + 15 + 5 + 1 = 31 letters

Chicherin, Henn, Mitev, 1712.09610

$$\mathcal{S} = \{s_{i,j},\ s_{i,j} - s_{k,l},\ o_i,\ \Delta\}$$

$$o_1 = \frac{[12]\langle 23\rangle[34]\langle 41\rangle}{\langle 12\rangle[23]\langle 34\rangle[41]} \qquad \Delta = \mathrm{tr}[\gamma_5 1234] = [12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]$$

- Obtained by applying full **S_5** to planar alphabet; only generates 5 new letters
- However, function space is much bigger because branch-cut condition now allows 10 first entries, $s_{i,j} \equiv (k_i + k_j)^2$
- In planar case there were only 5, $s_{i,i+1} \equiv (k_i + k_{i+1})^2$



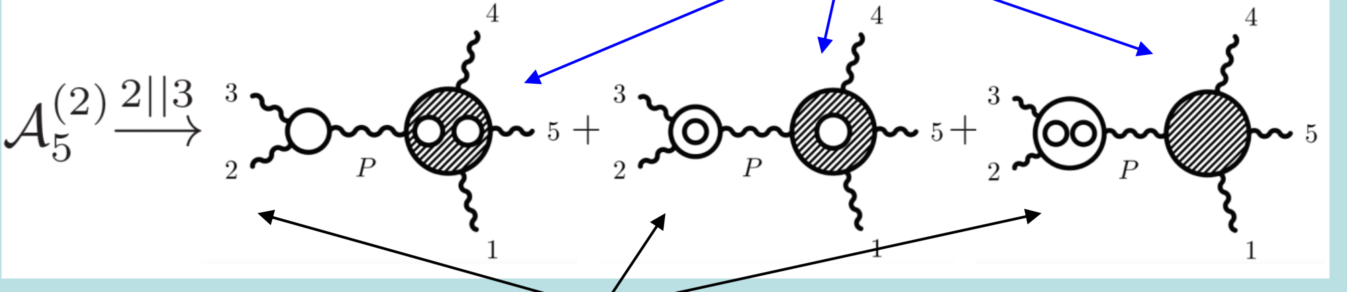
Numerical reduction and assembly

- Given decomposition into 6 PT factors, suffices to perform reduction to master integrals at 6 numerical kinematic points
- Use mod p arithmetic with p a 10-digit prime; reconstruct rational numbers using Wang's algorithm Wang (1981); von Manteuffel, Schabinger, 1406.4513; Peraro, 1608.01902
- Inserting symbols of all master integrals, we obtain symbols of all the pure functions
- Basic result is for g_{234}^{DT} , but also recover M^{BDS}, where $A^{ST}[12345] = \mathrm{PT}[12345]\, M^{BDS}$
- Also computed $A^{SLST}[12345]$, so color algebra could be checked via Edison-Naculich relations

Validation

- Five-point gauge theory amplitudes have stringent set of limiting behaviors as one gluon becomes soft or two partons become collinear.
- E.g. as legs 2 and 3 become collinear:

four-point amplitudes



splitting amplitudes

- We checked the collinear limit, as well as the soft limit, and the IR poles in ε which are predicted by
  Catani, hep-ph/9802439; Bern, LD, Kosower, hep-ph/0404293; Aybat, LD, Sterman, hep-ph/0606254, hep-ph/0607309

Soft gluon emission

LD, E. Herrmann, K. Yan, H.-X. Zhu, 1912.nnnnn

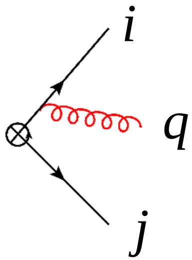
- Compute from Wilson lines → only depends on rescaling invariant combinations of velocities $\beta_m^\mu = p_m^\mu / p_m^0$

$$\boldsymbol{S}^{\pm}(\{\beta_m\}, q) \equiv \sum_{L=0}^{\infty} g_s\, \overline{a}^L\, \boldsymbol{S}^{\pm,(L)}(\{\beta_m\}, q)$$

Catani-Seymour color operator formalism

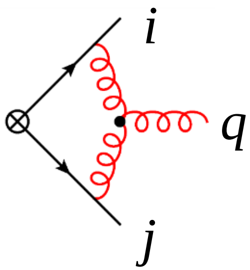
- (+) gluon emission at tree level:

$$\boldsymbol{S}_a^{+,(0)}(\{\beta_m\}) = +\frac{1}{2n} \sum_{i \neq j} (\boldsymbol{T}_i^a - \boldsymbol{T}_j^a) \frac{\langle ij \rangle}{\langle iq \rangle \langle qj \rangle}$$



- At 1 loop, still only dipole emission:

$$\boldsymbol{S}_a^{+,(1)} = \frac{1}{2} C_1(\epsilon) \sum_{i \neq j} f_{a a_i a_j} \boldsymbol{T}_i^{a_i}\, \boldsymbol{T}_j^{a_j}\; V_{ij}^q \frac{\langle ij \rangle}{\langle iq \rangle \langle qj \rangle} \qquad V_{ij}^q = \left[\frac{\mu^2(-s_{ij})}{(-s_{iq})(-s_{qj})} \right]^{\epsilon}$$

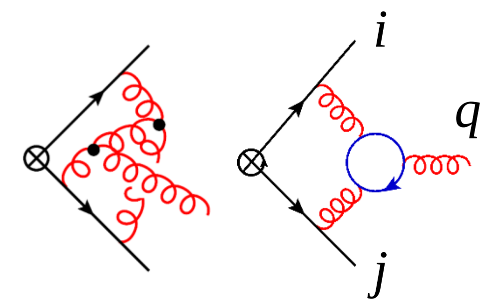


Soft emission at two loops

LD, E. Herrmann, K. Yan, H.-X. Zhu, 1912.nnnnn

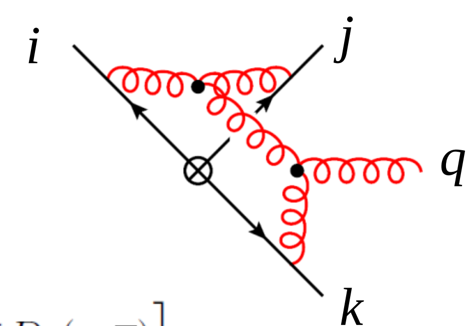
- Have to distinguish dipole terms

$$\boldsymbol{S}^{+,(2)}_{a,ij} = C_2(\epsilon)\, f_{aa_ia_j} \boldsymbol{T}_i^{a_i}\, \boldsymbol{T}_j^{a_j}\ \left(V_{ij}^q\right)^2 \frac{\langle ij\rangle}{\langle iq\rangle\langle qj\rangle}$$



(matter dependent, simple kinematic dependence, but not uniform transcendental)

from tripole terms



$$\boldsymbol{S}^{+,(2)}_{a,\{i,j,k\}} = 2\left(\boldsymbol{S}^{+,(2)}_{a,ikj} + \boldsymbol{S}^{+,(2)}_{a,kji} + \boldsymbol{S}^{+,(2)}_{a,jik}\right)$$

$$= 2\,\boldsymbol{T}_i^{a_i}\boldsymbol{T}_j^{a_j}\boldsymbol{T}_k^{a_k} \left\{ \frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle}\,(V_{ik}^q)^2 \left[f^{aa_jb} f^{ba_ia_k} D_1(z,\bar{z}) + f^{aa_ib} f^{ba_ka_j} D_2(z,\bar{z}) \right] \right.$$

$$\left. +\ \{i \leftrightarrow j\} \right\},$$

Tripole emission

- Subleading color, same in any gauge theory, including QCD, and N=4 SYM
- Hence expect weight 4 transcendentality
- Nontrivial dependence on rescaling invariant ratio,

$$z = z_k^{ij} = \frac{\langle kj\rangle\langle iq\rangle}{\langle ij\rangle\langle kq\rangle} \qquad \bar{z} = \bar{z}_k^{ij} = \frac{[kj][iq]}{[ij][kq]}$$

- D_1 , D_2 are weight 4 SVHPLs F. Brown (2004)

$$D_1(z) = -\frac{1}{\epsilon^2}(\mathcal{L}_1)^2 - \frac{1}{\epsilon}(\mathcal{L}_1)^3 - \frac{7}{12}(\mathcal{L}_1)^4 + 4\mathcal{L}_{1,0,1,0} + 2\mathcal{L}_{1,0,1,1} + 2\mathcal{L}_{1,1,1,0}$$

$$D_2(z) = \frac{1}{\epsilon^2}\mathcal{L}_0\mathcal{L}_1 + \frac{1}{\epsilon}\mathcal{L}_0(\mathcal{L}_1)^2 + \frac{2}{3}\mathcal{L}_0(\mathcal{L}_1)^3 + 6\,\zeta_2\left(\mathcal{L}_{0,1} - \mathcal{L}_{1,0}\right) + 2\big(\mathcal{L}_{0,0,0,1} - \mathcal{L}_{0,0,1,0} + \mathcal{L}_{0,1,0,0} + \mathcal{L}_{0,1,0,1} - \mathcal{L}_{1,0,0,0}\big)\,.$$

- Checks symbol terms, and constrains beyond-symbol terms, in full 2 loop 5 point amplitude

Structure of SYM result for full kinematics

- Symbols are large: M^{BDS} (planar) has "only" 2,365 terms, while g_{234}^{DT} (nonplanar) has 24,653 terms.
- How many functions are there in the full amplitude?
- Take linear span of all 120 permutations of g_{234}^{DT} and M^{BDS}
- At order ε^0, there are 52 weight 4 functions.
  Naively there should be 72 = 12 (planar) + 6 ·10 (nonplanar)
- So there are 20 relations among the permutations, e.g.

$$g[12345] + g[12453] + g[12534] + g[21345] + g[21453] + g[21534]$$
$$- g[12435] - g[12543] - g[12354] - g[21435] - g[21543] - g[21354] = 0$$

- The 20 relations are also obeyed by the lower-weight $1/\varepsilon$ pole terms.
- What do they mean? Do they reflect a nonplanar version of dual conformal invariance or integrated BCJ relations?

Glimpse of 24,653 term symbol

- Simplicity of weight 3 odd space lets us present the odd part of the derivative of the odd part of the basic double trace function:

$$\frac{\partial}{\partial x_i}\left[g_{234}^{\mathrm{DT,odd}}\right]\Big|_{\mathrm{odd}} = \sum_{j=1}^{12} \mathcal{I}_5^{d=6}(\Sigma_j) \sum_\gamma m_{j\gamma} \frac{\partial \log W_\gamma}{\partial x_i}$$

- $\gamma \in \{1, \ldots, 5, 16, \ldots, 20, 31\} = \{s_{ij}, \Delta\}$ only, $\Sigma_j \in S_5/D_5$
- {3,1} coproduct matrix $m_{j\gamma}$:

$$m_{j\gamma} = \begin{pmatrix}
-\frac{17}{4} & -\frac{5}{4} & -6 & -\frac{17}{4} & -\frac{7}{2} & -\frac{17}{4} & -\frac{7}{4} & \frac{1}{2} & -1 & -\frac{17}{4} & 10 \\
\frac{17}{4} & \frac{5}{4} & \frac{5}{4} & \frac{17}{4} & 4 & \frac{17}{4} & \frac{11}{2} & \frac{17}{4} & \frac{1}{2} & \frac{1}{2} & -10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
-\frac{17}{4} & -6 & -\frac{5}{4} & -\frac{17}{4} & -\frac{7}{2} & \frac{1}{2} & -\frac{7}{4} & -\frac{17}{4} & -\frac{17}{4} & -1 & 10 \\
-\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\
\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 & 0 & 0 \\
0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & 0 & 1 & 0 \\
-\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{17}{4} & 6 & 6 & \frac{17}{4} & 9 & -\frac{1}{2} & 4 & -\frac{1}{2} & -\frac{5}{4} & -\frac{5}{4} & -10
\end{pmatrix}$$

rank 8 → only 8 independent linear combinations of final entries appear. Why?

N=8 supergravity

Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 1901.05932;
Abreu, LD, Herrmann, Page, Zeng, 1901.08563

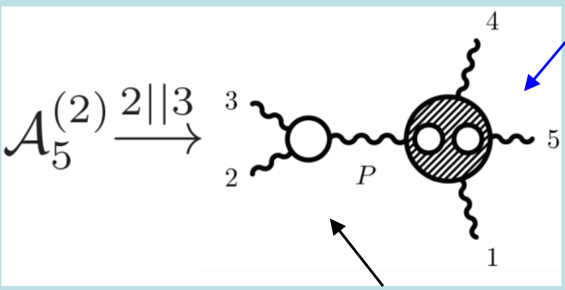
- Same integration methods can be applied to the double-copy N=8 supergravity integrand
  Carrasco, Johansson, 1106.4711
- Loop-momentum numerator is quadratic instead of linear in the loop momentum (QCD would be ~ ninth order)
- Richer set of rational function prefactors
- 40 prefactors can be inferred from four-dimensional leading singularities computed from on-shell diagrams
- 5 more require d-dimensional leading singularities

N=8 supergravity (cont.)

- After reductions for > 45 phase space points, discover 5 additional rational structures (d-dim'l leading sing's)
- Result has uniform transcendentality
- Because there is no color, there are exactly 45 pure function components to the amplitude
- 5 of the 45 are removed by a natural IR subtraction.
- Compare the 45 functions to the 52 for N=4 SYM: They overlap a lot; their span has dimension 62

N=8 Validation

- Five-point gravity amplitudes have stringent set of limiting behavior as a graviton becomes soft
  Weinberg (1965); Berends, Giele, Kuijf (1988);
  Bern, LD, Perelstein, Rozowsky, hep-th/9811140
  or two gravitons become collinear
  Bern, LD, Perelstein, Rozowsky, hep-th/9811140
- E.g. as legs 2 and 3 become collinear:



- Checked collinear limit as well as soft limit, and IR poles in ε which are correctly predicted by Weinberg (1965); Naculich, Nastase, Schnitzer, 0805.2347

Conclusions

- Two-loop five-point nonplanar amplitudes now available at symbol level in maximally supersymmetric theories
- Soft limit known at function level, including intricate tripole terms
- Need to promote symbols → functions, beyond soft limit
- All required master integrals needed for QCD now known at symbol level
- Opens door to full-color 2 → 3 massless QCD amplitudes for e.g. NNLO 3 jet production at hadron colliders
- Can these results give insight into classical gravity too?

Extra Slides

Nonplanar 5-point function space

Chicherin, Henn, Mitev, 1712.09610

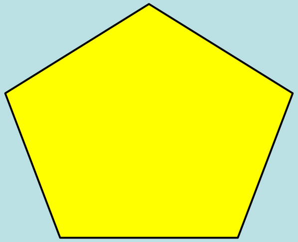
- Also empirical constraint on first 2 entries of the symbol.
- Imposing this condition and integrability, dimension of even | odd part of function space is:

Weight	1	2	3	4
# of integrable symbols for $\mathbb{A}_{\mathrm{P}}$	$5 \mid 0$	$25 \mid 0$	$125 \mid 1$	$635 \mid 16$
after 2nd entry condition	$5 \mid 0$	$20 \mid 0$	$80 \mid 1$	$335 \mid 11$
# of integrable symbols for $\mathbb{A}_{\mathrm{NP}}$	$10 \mid 0$	$100 \mid 9$	$1000 \mid 180$	$9946 \mid 2730$
after 2nd entry condition	$10 \mid 0$	$70 \mid 9$	$505 \mid 111$	$3736 \mid 1191$

- SYM and SUGRA amplitudes both lie in this space

Structure (cont.)

- Take first derivatives, i.e. {3,1} coproducts.
- How many functions are there?
- Weight 3 even: 362 (out of a possible 505).
- But only 40 of them have (two) odd letters. Rest simple.
- Weight 3 odd is even more restricted: only 12 (out of a possible 111)
- They are just the 12 $\mathbf{S_5/D_5}$ permutations of the D=6 one-loop pentagon integral!!



- Weight 2 is not restricted at all; the {2,1,1} coproducts include all 70 even and 9 odd functions obeying the second entry condition.

Comparing function spaces

functions	$\{1,1,1,1\}$	$\{2,1,1\}$	$\{3,1\}$	weight 4
P odd space	0	9	111	1191
no. from $\mathcal{N}=8$	0	9	11	45
no. from $\mathcal{N}=4$	0	9	12	52
no. from both	0	9	12	62
P even space	10	70	505	3736
no. from $\mathcal{N}=8$	10	70	285	40
no. from $\mathcal{N}=4$	10	70	362	52
no. from both	10	70	367	56
P even with odd letters	0	0	45	711
no. from $\mathcal{N}=8$	0	0	40	40
no. from $\mathcal{N}=4$	0	0	40	40
no. from both	0	0	40	44

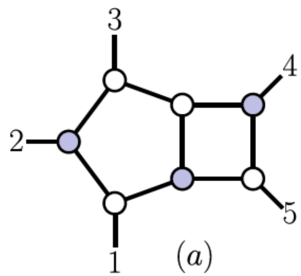
Structure

- Same odd, odd {3,1} coproduct matrix as in N=4 SYM, but now for a component of the N=8 finite remainder:

$$m_{j\alpha_1} = \frac{1}{12}\begin{pmatrix} -3 & -2 & 2 & 2 & -2 & 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & -1 & -3 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 & -2 & 1 & 0 & 5 & 0 & 1 & 0 \\ 3 & 0 & -3 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & -2 & 4 & -3 & 0 & -3 & 0 & 1 & 0 \\ -3 & -1 & 1 & 3 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -3 & 2 & -1 & 3 & -3 & 2 & 0 & 3 & 0 & -3 & 0 \\ 3 & 4 & -2 & 0 & 0 & 1 & 0 & -3 & 0 & -3 & 0 \\ 3 & -1 & 0 & 0 & -1 & 2 & 0 & -5 & 0 & 2 & 0 \\ -3 & 0 & 3 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ -3 & -3 & 3 & -1 & 2 & -3 & 0 & 3 & 0 & 2 & 0 \\ 3 & 2 & -2 & -2 & 2 & -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}$$

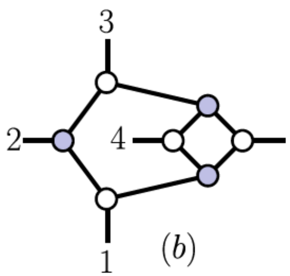
rank 5 → only 5 independent linear combinations of final entries!

N=8 SUGRA 4-dim'l leading singularities



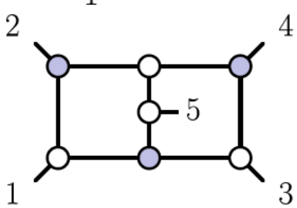
(a)

$$= \frac{[12][23][45]^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle\langle 13\rangle}$$



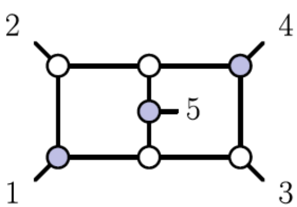
(b)

$$= \frac{[12][23][45]^2}{\langle 12\rangle\langle 23\rangle\langle 14\rangle\langle 34\rangle\langle 35\rangle\langle 51\rangle}$$



(c_1)

$$= \frac{[24][34][12]^2}{\langle 13\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle\langle 51\rangle} + (1 \leftrightarrow 3, 2 \leftrightarrow 4)$$



(c_2)

$$= \frac{[12][34][45][51]}{\langle 12\rangle\langle 13\rangle\langle 24\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle}$$

→

linear span
has dimension 40

d dimensional

$$\frac{s_{12}[12][23][34][45][51]}{\mathrm{tr}_5\,\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

5 more

Shift from "Euclidean" to Minkowski region

$$\mathrm{disc}_{A_1} D_1(z) = 2i\pi\Big\{8\Big[\mathrm{Li}_3(z) + \mathrm{Li}_3\Big(\frac{-z}{1-z}\Big)\Big]$$
$$- \log(1-z)\Big[4\Big(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\Big) + \log^2(1-z) - \log^2(1-\bar{z})\Big]\Big\}$$

$$\mathrm{disc}_{A_1} D_2(z) = 2i\pi\Bigg\{\frac{\log|1-z|^2}{\epsilon^2} - \frac{\log^2|1-z|^2}{\epsilon} + 8\mathrm{Li}_3(z) - 4\mathrm{Li}_3(\bar{z}) + 4\mathrm{Li}_3\Big(\frac{-\bar{z}}{1-\bar{z}}\Big)$$
$$- 2\Big[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\Big]\Big[2\log|z|^2 - \log(1-\bar{z})\Big] + 2\zeta_2 \log\left(\frac{1-z}{1-\bar{z}}\right)$$
$$- \log\left(\frac{1-z}{1-\bar{z}}\right)\log^2|z|^2 + 2\log(1-z)\log(1-\bar{z})\,\log|1-z|^2$$
$$+ \frac{2}{3}\log^3(1-z)\Bigg\}$$
$$- 4\pi^2\left\{2\Big[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\Big] + \log\left(\frac{1-z}{1-\bar{z}}\right)\log|z|^2\right\}.$$

…