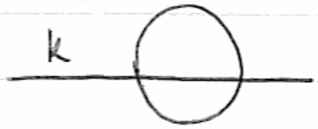
Solution for HW4.

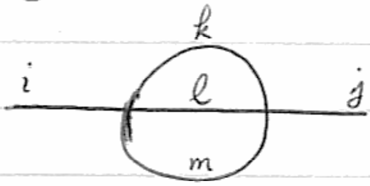
13.2 :

From the result of (10.3), we can extract the leading correction to

δ_{Z_ϕ} :  $= -ik^2 \dfrac{\lambda^2}{12(4\pi)^4}\left(-\dfrac{1}{\epsilon} + \log k^2 + \cdots\right)$

In $\overline{MS}$, the divergent part gives the counterterm: $\delta_{Z_\phi} = \dfrac{\lambda^2}{12(4\pi)^4}\log M^2$.

Therefore, $\gamma = \frac{1}{2} M \dfrac{\partial \delta_Z}{\partial M} = \dfrac{\lambda^2}{12(4\pi)^4} + \mathcal{O}(\lambda^3)$.

Similarly in $O(N)$-model, the leading correction to the field strength renormalization factor comes from :  $=$ (index structure) $\times$ (loop integral as ϕ^4-theory).

(index structure) $= (-2i\lambda)^2 \left(\delta^{ik}\delta^{lm} + \delta^{il}\delta^{km} + \delta^{im}\delta^{kl}\right)$

$\times\left(\delta^{jk}\delta^{lm} + \delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl}\right) = -12\lambda^2(N+2)\,\delta^{ij}$

So, $\gamma(\lambda) = (N+2)\dfrac{\lambda^2}{(4\pi)^4} + \mathcal{O}(\lambda^3)$.

In $O(N)$-model, the Wilson-Fisher fixed point occurs at $\lambda_* = \dfrac{8\pi^2}{N+8}\epsilon$

and $\eta = 2\gamma(\lambda_*) = \dfrac{N+2}{2(N+8)^2}\epsilon^2 +$ higher order terms.

Note that the coupling λ of $O(N)$ model is 6 times larger than the coupling of ϕ^4 and this explains why we don't get the same answer if we put $N=1$ in $O(N)$-model.

2: Coleman_Weinberg problem:

a) The global $U(1)$_symmetry is spontaneously broken when $m^2 = -\mu^2 < 0$.

Setting $\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}(\sigma(x) + i\pi(x))$, and rewrite the bare Lagrangian in terms of (σ, π) scalars, we find:

$$\mathcal{L} = \left(\mu^2 \phi_0^2 - \frac{\lambda}{6}\phi_0^4\right) + \frac{1}{2}(\partial\sigma)\cdot(\partial\sigma) - \frac{1}{2}(\lambda\phi_0^2 - \mu^2)\sigma^2 + \frac{1}{2}\partial\pi\cdot\partial\pi$$

$$-\frac{1}{2}\left(\frac{\lambda}{3}\phi_0^2 - \mu^2\right)\pi^2 + \frac{1}{4}(F_{\mu\nu})^2 + e^2\phi_0^2 A_\mu A^\mu + \sqrt{2}\mu^2\phi_0\sigma + \sqrt{2}e^2\phi_0\sigma A_\mu A^\mu$$

$$-\frac{\sqrt{2}}{3}\lambda\sigma\phi_0^3 - \frac{\sqrt{2}}{6}\lambda\phi_0\sigma^3 - \frac{\lambda}{24}\sigma^4 - \frac{\sqrt{2}}{6}\lambda\phi_0\sigma\pi^2 - \frac{\lambda}{12}\sigma^2\pi^2 - \frac{\lambda}{24}\pi^4$$

$$+\frac{1}{2}e^2 A_\mu A^\mu(\sigma^2 + \pi^2) + \sqrt{2}e\phi_0 A_\mu\partial^\mu\pi + eA_\mu\partial^\mu\pi\sigma - e\pi A_\mu\partial^\mu\sigma,$$

Now for scalar Lagrangian $V(\phi) = -\mu^2\phi^2 + \frac{\lambda}{6}\phi^4$, we find the critical value of $\phi = \phi_0$:
$$\left.\frac{\partial V}{\partial\phi}\right|_{\phi_0} = 0 \Rightarrow \underline{\phi_0^2 = \frac{3\mu^2}{\lambda}}.$$

If we plug back this value in the above Lagrangian, we find that π is massless and all linear terms in $\mathcal{L}$ drop. Also, we find a mass for the gauge field: $\underline{m_A = \sqrt{2}e\phi_0}$.

b) First note that all linear graphs in σ are cancelled by tadpole condition.

Then, the quadratic part of the action is given by:

$$S = \int d^4x\left(-\frac{1}{4}(F_{\mu\nu})^2 + e^2\phi_{cl}^2 A_\mu A^\mu + \frac{1}{2}\partial\sigma\cdot\partial\sigma - \frac{1}{2}(\lambda\phi_{cl}^2 - \mu^2)\sigma^2 + \frac{1}{2}\partial\pi\cdot\partial\pi\right.$$

$$-\frac{1}{2}\left(\frac{\lambda}{3}\phi_0^2-\mu^2\right)\pi^2+\mu^2\phi_{cl}^2-\frac{\lambda}{6}\phi_{cl}^4\Big).$$

The one-loop effective action is then given by:

$$\Gamma[\phi_{cl}] = \frac{i}{2}\log\det\left(\frac{\delta^2\mathcal{L}}{\delta A^\mu\delta A_\mu}\right)+\frac{i}{2}\log\det\left(\frac{\delta^2\mathcal{L}}{\delta\sigma\,\delta\sigma}\right)+\frac{i}{2}\log\det\left(\frac{\delta^2\mathcal{L}}{\delta\pi\,\delta\pi}\right)$$

First, lets consider

$$\int \mathcal{D}A\,\mathcal{D}\sigma\,\mathcal{D}\pi\, e^{i\int d^4x\left(-\frac{1}{4}(F_{\mu\nu})^2+e^2\phi_{cl}^2A_\mu A^\mu\right)}$$

$$=\int \mathcal{D}A\,\mathcal{D}\sigma\,\mathcal{D}\pi\, e^{\frac{i}{2}\int d^4x\, A_\mu\left(\partial^2g^{\mu\nu}-\cancel{\partial^\mu\partial^\nu}+2e^2\phi_{cl}^2g^{\mu\nu}\right)A_\nu+\cdots}$$

$\to$ in ~~Landau~~ Lorentz gauge.

In Landau gauge, we have $k^\mu A_\mu=0$ and this leads to three independent degrees of freedom for A. Then, integration over A will give us:

$$\int \mathcal{D}\sigma\,\mathcal{D}\pi\, e^{i\int d^4x\left(\frac{1}{2}\partial\sigma\cdot\partial\sigma+\frac{1}{2}\partial\pi\cdot\partial\pi+\cdots\right)}\left(\det\left(-\partial^2-2e^2\phi_{cl}^2\right)^{-\frac{1}{2}}\right)^3$$

$$=\int \mathcal{D}\sigma\,\mathcal{D}\pi\, e^{i\int d^4x\left(\frac{1}{2}\partial\sigma\cdot\partial\sigma+\frac{1}{2}\partial\pi\cdot\partial\pi-\frac{1}{2}(\lambda\phi_{cl}^2-\mu^2)\sigma^2-\frac{1}{2}\left(\frac{\lambda}{3}\phi_{cl}^2-\mu^2\right)\pi^2\right)-\frac{3}{2}\mathrm{Tr}\log(\partial^2+2e^2\phi_{cl}^2)}$$

$$=e^{i\Gamma[\phi_{cl}]},$$

On the other hand, from calculation in the book we know

$$\mathrm{Tr}(\partial^2+m^2)=-i(VT)\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}(m^2)^{d/2}\quad\text{and}\quad V_{eff}(\phi_{cl})=-\frac{1}{VT}\Gamma[\phi_{cl}].$$

$$\Rightarrow V_{eff}(\phi_{cl})=-\mu^2\phi_{cl}^2+\frac{\lambda}{6}\phi_{cl}^4-\frac{1}{2}\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}\Big[(\lambda\phi_{cl}^2-\mu^2)^{d/2}$$

$$+\left(\frac{\lambda}{3}\phi_{cl}^2-\mu^2\right)^{d/2}+3\left(2e^2\phi_{cl}^2\right)^{d/2}\Big]+\delta_\mu\phi_{cl}^2+\frac{\delta_\lambda}{6}\phi_{cl}^4,$$

So, as $d \to 4$ if we set $\delta\mu = -\frac{1}{2}\frac{\Gamma(2-d/2)}{(4\pi)^2}\left(\frac{4\lambda}{3}\mu^2\right)+\cdots$ and similarly

$\delta_\lambda = \frac{2}{3}\frac{\Gamma(2-d/2)}{(4\pi)^2}\left(\frac{10}{9}\lambda^2+12e^4\right)+\cdots$, V_{eff} will be finite. Therefore,

with $\overline{MS}$, we have:

$$V_{eff}(\phi_{cl}) = -\mu^2\phi_{cl}^2+\frac{\lambda}{6}\phi_{cl}^4+\frac{1}{4(4\pi)^2}\left[(\lambda\phi_{cl}^2-\mu^2)^2\left(\log\left(\frac{\lambda\phi_{cl}^2-\mu^2}{M^2}\right)-3/2\right)\right.$$

$$\left.+\left(\frac{\lambda}{3}\phi_{cl}^2-\mu^2\right)^2\left(\log\left(\frac{\lambda/3\,\phi_{cl}^2-\mu^2}{M^2}\right)-3/2\right)+3(2e^2\phi_{cl}^2)^2\left(\log\left(\frac{2e^2\phi_{cl}^2}{M^2}\right)-3/2\right)\right].$$

c) In the limit $\underline{\mu^2 \to 0}$ and $\underline{\lambda \sim (e^2)^2}$, we find:

$$V_{eff}(\phi_{cl}) = \phi_{cl}^4\left(\frac{\lambda}{6}+\frac{3}{16\pi^2}e^4\left(\log\frac{2e^2\phi_{cl}^2}{M^2}-3/2\right)\right).$$

Now, if we minimize V_{eff}, we find the critical value of ϕ_{cl}^* as:

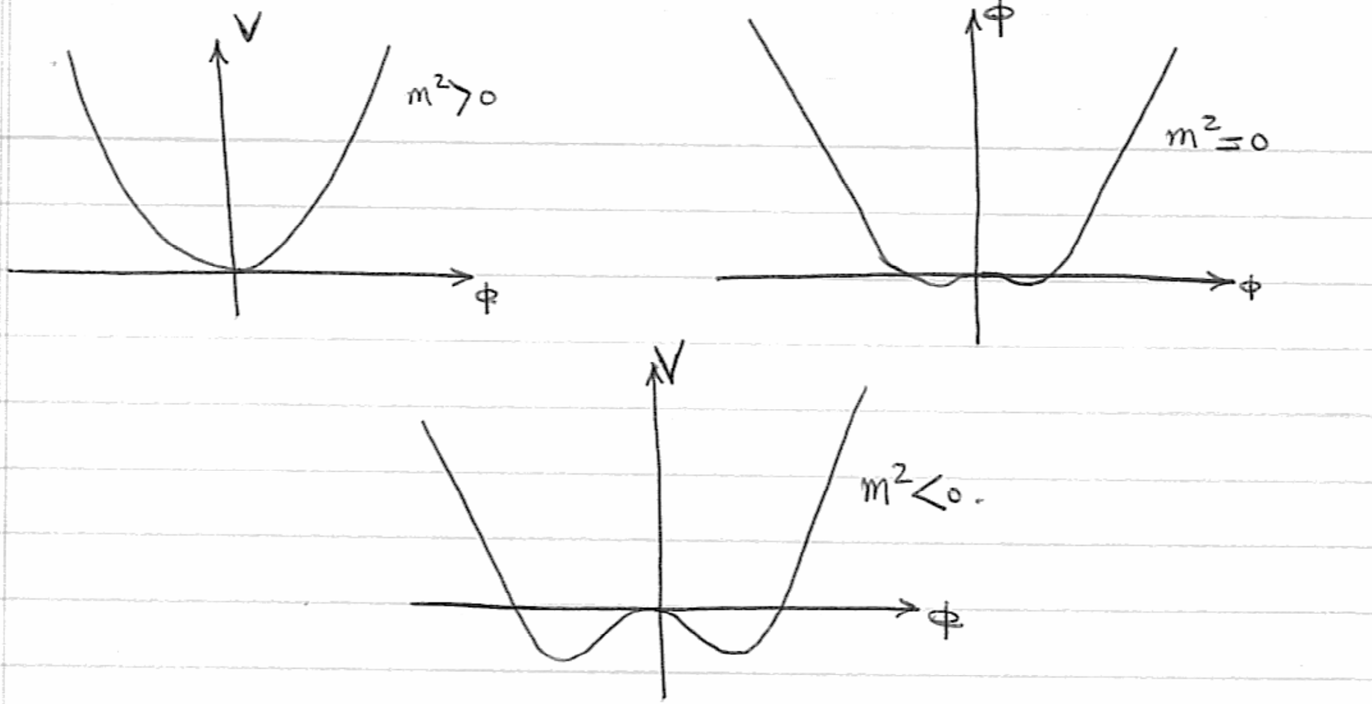
$$\left.\frac{\partial V_{eff}}{\partial\phi_{cl}}\right|_{\phi_{cl}^*}=0 \Longrightarrow 1-\frac{8\pi^2\lambda}{9e^4}=\log\frac{2e^2\phi_{cl}^{*2}}{M^2} \quad \text{or} \quad \phi_{cl}^*=0.$$

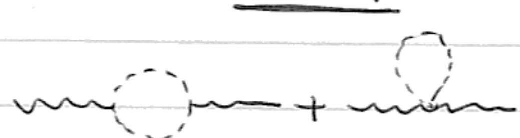
$V_{eff}(\phi_{cl}^*=0)=0.$ (large logarithms) $=0$.

$V_{eff}(\phi_{cl}^*\text{ (the finite one)}) = \phi_{cl}^4\left(\frac{-3}{32\pi^2}e^4\right)<0 \Rightarrow$ true minimum.

Therefore even at the limit $\mu^2\to 0$, the theory has spontaneously broken symmetry by quantum effects. Since there is no large logarithms involved, the perturbation is expected to be valid.

d) We consider 3 different cases: $m^2>0$, $m^2=0$, $m^2<0$.



e) In general, if a theory has two coupling constants, then β-functions are functions of the two couplings. However, at <u>one-loop order</u>, the only diagrams contribute to $\beta(e^2)$ are  as the usual scalar QED (also note that by Ward identity $Z_{1e} = Z_{2e}$). Theref

we have an identical β-function for e^2 at <u>one-loop</u>:

$$\beta(e) = \frac{e^3}{16\pi^2}\left(\frac{1}{3}\right) = \frac{e^3}{48\pi^2}.$$

On the other hand, the Callen-Symanzik equation states:

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \beta(e)\frac{\partial}{\partial e} - \gamma\phi_0\frac{\partial}{\partial \phi_0}\right)V_{eff} = 0.$$

We already found the effective potential, so if we compute γ, we can solve the above equation for $\beta(\lambda)$.

To compute γ, we first need to find δ_{Z_ϕ}. At one-loop:

in Landau gauge

$$= (-ie)^2\int\frac{d^dk}{(2\pi)^d}\,\frac{(2p-k)^\mu(2p-k)^\nu\left(g_{\mu\nu}-k_\mu k_\nu/k^2\right)}{k^2\,(p-k)^2}$$

$$= -e^2\int\frac{d^dk}{(2\pi)^d}\,\frac{4p^2k^2-4(p\cdot k)^2}{(k^2)^2(p-k)^2} = -e^2\int\frac{d^dk}{(2\pi)^d}\,dx\,dy\,dz\,\frac{4p^2k^2-4(p\cdot k)^2}{\left(xk^2+yk^2+z(p-k)^2\right)^3}$$

$$= -e^2\int\frac{d^d\ell}{(2\pi)^d}\,dz\,\frac{3\ell^2p^2}{(\ell^2-\Delta)^3}, \quad \ell=k-zp, \quad \Delta=-z(1-z)p^2$$

$$= -\frac{3i\,p^2e^2}{(4\pi)^2}\int dz\,\Gamma(2-d/2)\,\frac{1}{\Delta^{2-d/2}}$$

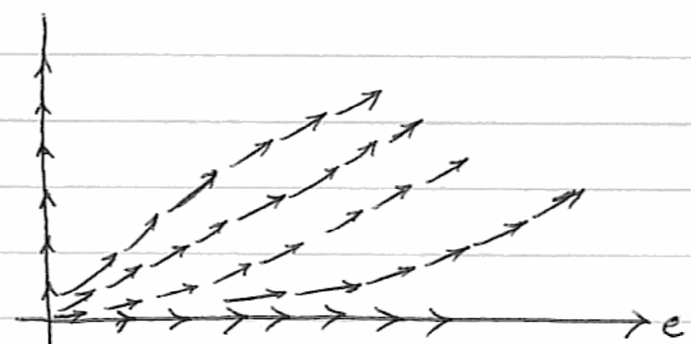
In $\overline{MS}$ $\Longrightarrow$ $\delta_{Z_\phi} = -\frac{3e^2}{(4\pi)^2}\log M^2 + \cdots$

Now, we can plug this back into Callen-Symanzik equation. Note that to the lowest order, $\beta(e)\sim e^3$ and $\frac{\partial V_{eff}}{\partial e}\sim e^3$, so we drop $\beta(e)\frac{\partial}{\partial e}V_{eff}$ in compare with $\lambda^2, \lambda e^2, e^4$ terms. Then:

$$\beta(\lambda) = 6\left(-\frac{3e^2}{(4\pi)^2}\cdot\frac{2}{3}\lambda + \frac{1}{2}\,\frac{1}{(4\pi)^2}\left(\frac{10}{9}\lambda^2+12e^4\right)\right)$$

$$\Longrightarrow \quad \beta(\lambda) = \frac{1}{24\pi^2}\left(5\lambda^2-18\lambda e^2+54e^4\right).$$

The flow:



f) By dimensional analysis $V_{eff} = \phi_{cl}^4 \, v\left(\phi_{cl}/M, \lambda, e^2\right)$ and the Callan–Symanzik equation becomes:

$$\left(\phi_{cl}\frac{\partial}{\partial \phi_{cl}} - \frac{\beta(\lambda)}{1+\gamma}\frac{\partial}{\partial \lambda} - \frac{\beta(e^2)}{1+\gamma}\frac{\partial}{\partial e^2} + \frac{4\gamma}{1+\gamma}\right) v = 0.$$

The solution of this equation should agree with part c. The RG-improved effective potential is:

$$V_{eff}(\phi_{cl}) = \phi_{cl}^4\left[\frac{\bar\lambda}{6} + \frac{1}{4(4\pi)^2}\left(\bar\lambda^2(\log\bar\lambda - 3/2) + \frac{\bar\lambda^2}{9}\left(\log\frac{\bar\lambda}{3} - \frac{3}{2}\right) + 3(2\bar e^2)^2 (\log 2\bar e^2 - 3/2)\right)\right]\exp\left(-\int_M^{\phi_{cl}} d\log\phi_{cl}\,\frac{4\gamma(\bar e^2)}{1+\gamma(\bar e^2)}\right).$$

To the lowest order in e^2,

$$\beta(e^2) = \frac{(e^2)^2}{24\pi^2}, \quad \beta(\lambda) \simeq \frac{9}{4}\frac{(e^2)^2}{\pi^2}$$

$$\Rightarrow \bar e^2 \simeq e^2 + \frac{e^4}{24\pi^2}t, \quad \bar\lambda \simeq \lambda + \frac{9}{4}\frac{e^4}{\pi^2}t, \text{ where } t = \log\frac{\phi_{cl}}{M}.$$

Also, $V_{eff} \simeq \phi_{cl}^4\left(\frac{\lambda}{6} + \frac{3}{8\pi^2}e^4 t + \frac{3}{16\pi^2}e^4(\log 2e^2 - 3/2)\right)$.

Now, let us minimize V_{eff}:

$$\frac{\partial V_{eff}}{\partial \phi_{cl}} = 0 \Rightarrow \phi_{cl} = 0 \text{ or } \left(\frac{2\lambda}{3} + \frac{3}{2\pi^2}e^4 t + \frac{3}{8\pi^2}e^4 + \frac{3}{4\pi^2}(e^4)(\log 2e^2 - 3/2)\right)$$

$$\Longrightarrow \underline{\phi_{cl}^* = \frac{M}{(2e^2)^{1/2}} e^{1/2} \cdot e^{-\frac{4\pi^2}{9}\frac{\lambda}{e^4}}} \text{ which agrees with part c.}$$

Then $m_\sigma^2 = \bar\lambda\phi_0^2 \simeq \frac{9e^2M^2}{8\pi^2}(2.71)\left(\frac{1}{2} - \frac{1}{4}\log 2e^2\right)$

and $m_\sigma^2/m_A^2 \approx \frac{9}{8\pi^2}e^2\left(\frac{1}{2} - \frac{1}{4}\log 2e^2\right)$.