Physics 332, Spring 2008, Problem Set 3
— due Friday, May 9, 3pm, Masoud Soroush's mailbox in Varian

Problem 1 [4 pts]: In class we considered the one-loop effective action for a theory with a nonabelian gauge boson, 4 Majorana fermions and 6 real scalars, all transforming in the adjoint representation of the gauge group. (This theory is known as $\mathcal{N} = 4$ super-Yang-Mills theory.) We showed that

$$i\Gamma^{\mathcal{N}=4}[A] = -\frac{1}{2}\ln\det\Delta_{G,1} + \ln\det\Delta_{G,1/2} - \frac{1}{2}\ln\det\hat{\Delta}_{G,0}\,, \qquad (1)$$

where the operator

$$\Delta_{G,J} = -D^2 + (F^b_{\rho\sigma}t^b_G)\mathcal{J}^{\rho\sigma}_J \qquad (2)$$

involves the Yang-Mills field strength in the adjoint representation G. The "hat" over $\Delta_{G,0}$ corresponds to redefining a scalar to have 4 states, to match the number for a Dirac fermion ($J = 1/2$) and for a vector boson ($J = 1$). The generators of infinitesimal Lorentz transformations in the spin J representation are $\mathcal{J}^{\rho\sigma}_J$. Explicitly, for $J = 1/2$ and $J = 1$,

$$\begin{aligned} \mathcal{J}^{\rho\sigma}_{1/2} &= \frac{i}{4}[\gamma^\rho\,,\,\gamma^\sigma]\,, \\ (\mathcal{J}^{\rho\sigma}_1)_{\alpha\beta} &= i\,[\delta^\rho_\alpha\,\delta^\sigma_\beta - \delta^\sigma_\alpha\,\delta^\rho_\beta]\,. \end{aligned} \qquad (3)$$

(a) Show that

$$\mathrm{Tr}[\mathcal{J}^{\rho\sigma}_1\mathcal{J}^{\nu\lambda}_1] = 2\,\mathrm{Tr}[\mathcal{J}^{\rho\sigma}_{1/2}\mathcal{J}^{\nu\lambda}_{1/2}]\,, \qquad (4)$$

where the traces are over the vector and Dirac representations, respectively. (Note that the trace of a pair of Lie algebra generators is only nonzero if the two generators are the same, $\mathrm{Tr}[T^aT^b] \propto \delta^{ab}$, so you can pick the indices $\rho, \sigma, \nu, \lambda$ accordingly, to simplify things.)

(b) Similarly, show that

$$\mathrm{Tr}[\mathcal{J}^{\rho\sigma}_1\mathcal{J}^{\nu\lambda}_1\mathcal{J}^{\alpha\beta}_1] = 2\,\mathrm{Tr}[\mathcal{J}^{\rho\sigma}_{1/2}\mathcal{J}^{\nu\lambda}_{1/2}\mathcal{J}^{\alpha\beta}_{1/2}]\,, \qquad (5)$$

(c) Use eqs. (1), (4) and (5) to argue that the leading loop-momentum behavior of the one-loop effective action for m-external gluons in $\mathcal{N} = 4$ super-Yang-Mills theory is $(p^\mu)^{m-4}$, where p^μ is the loop momentum. What is the first value of m for which the result is nonzero?

(d) Use this result to show that the amplitudes with external gluons in this theory have no ultraviolet divergences at one loop. (In fact all amplitudes in this theory are ultraviolet-finite, to all loop orders.)

Problem 2 [3 pts]: Consider a general quantum field theory in which λ is the coupling constant. Suppose that one makes a general redefinition of the coupling constant by replacing

$$\bar{\lambda} = \bar{\lambda}(\lambda) \ . \tag{6}$$

In order to make the perturbation theory have the same structure, we require that for small λ

$$\bar{\lambda}(\lambda) = \lambda + \mathcal{O}(\lambda^2) \ . \tag{7}$$

1. How is the new β-function $\bar{\beta}(\bar{\lambda})$ related to the old β-function $\beta(\lambda)$? **Hint**: Expand both β-functions in terms of the coupling constants and find the relation between the coefficients of the expansions.

2. Show that if $\lambda = \lambda_F$ is a fixed point of the theory, then this fixed point is realized at $\bar{\lambda} = \bar{\lambda}_F$ in terms of the new coupling constant. Note that λ_F and $\bar{\lambda}_F$ are related via eq. (6).

Problem 3 [3 pts]: Consider ϕ^3 theory in $d = 6$.

1. Based on power counting, show that this theory is renormalizable.

2. Compute the one loop β-function of this theory. Show that the theory is asymptotically free.