Solution for HW3 :

$\boxed{1}$ a) we are supposed to find relations between traces of generators of vector and spinor representations of Lorentz group:

$$\mathrm{Tr}[\mathcal{J}_1^{\rho\sigma} \mathcal{J}_1^{\nu\lambda}] = (\mathcal{J}_1)^{\rho\sigma}{}_{\alpha\beta} (\mathcal{J}_1^{\nu\lambda})^{\beta\alpha}$$

$$= -(\delta^\rho_\alpha \delta^\sigma_\beta - \delta^\sigma_\alpha \delta^\rho_\beta)(\eta^{\nu\beta}\eta^{\lambda\alpha} - \eta^{\nu\alpha}\eta^{\lambda\beta})$$

$$= 2(\eta^{\nu\rho}\eta^{\lambda\sigma} - \eta^{\nu\sigma}\eta^{\lambda\rho})$$

on the other hand for spinor rep., we have:

$$\mathrm{Tr}[\mathcal{J}_{1/2}^{\rho\sigma} \mathcal{J}_{1/2}^{\nu\lambda}] = -\frac{1}{16}\mathrm{Tr}\left[[\gamma^\rho,\gamma^\sigma][\gamma^\nu,\gamma^\lambda]\right],$$

$$\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma[\gamma^\nu,\gamma^\lambda]\right] = \mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\lambda - \gamma^\rho\gamma^\sigma\gamma^\lambda\gamma^\nu\right]$$

$$= 4(\eta^{\rho\sigma}\eta^{\nu\lambda} - \eta^{\rho\nu}\eta^{\sigma\lambda} + \eta^{\rho\lambda}\eta^{\sigma\nu})$$

$$-4(\eta^{\rho\sigma}\eta^{\nu\lambda} - \eta^{\rho\lambda}\eta^{\sigma\nu} + \eta^{\rho\nu}\eta^{\sigma\lambda}) = 8(\eta^{\rho\lambda}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\lambda})$$

Now, exchange position ρ,σ in above $\Rightarrow$

$$\mathrm{Tr}\left[[\gamma^\rho,\gamma^\sigma][\gamma^\nu,\gamma^\lambda]\right] = 16(\eta^{\rho\lambda}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\lambda})$$

Therefore,

$$\underline{\mathrm{Tr}[\mathcal{J}_1^{\rho\sigma} \mathcal{J}_1^{\nu\lambda}] = 2\,\mathrm{Tr}[\mathcal{J}_{1/2}^{\rho\sigma} \mathcal{J}_{1/2}^{\nu\lambda}]}.$$

b) This part is similar to (a) but its more tedious.

First: $\mathrm{Tr}[\mathcal{J}_1^{\rho\sigma} \mathcal{J}_1^{\nu\lambda} \mathcal{J}_1^{\alpha\beta}] = -i(\eta^{\nu\rho}\delta^\sigma_\delta - \eta^{\nu\sigma}\delta^\rho_\delta)(\eta^{\delta\nu}\delta^\lambda_\epsilon - \eta^{\delta\lambda}\delta^\nu_\epsilon)$

$$\times(\eta^{\epsilon\alpha}\delta^\beta_\gamma - \eta^{\epsilon\beta}\delta^\alpha_\gamma)$$

after some algebra:

$$\mathrm{Tr}[J_1^{\rho\sigma} J_1^{\nu\lambda} J_1^{\alpha\beta}] = -i\,(\eta^{\beta\rho}\eta^{\sigma\nu}\eta^{\lambda\alpha} - \eta^{\alpha\rho}\eta^{\sigma\nu}\eta^{\lambda\beta} - \eta^{\beta\sigma}\eta^{\rho\nu}\eta^{\lambda\alpha} - \eta^{\beta\rho}\eta^{\sigma\lambda}\eta^{\nu\alpha}$$
$$+\eta^{\beta\sigma}\eta^{\rho\lambda}\eta^{\nu\alpha} + \eta^{\alpha\sigma}\eta^{\rho\nu}\eta^{\lambda\beta} + \eta^{\alpha\rho}\eta^{\sigma\lambda}\eta^{\nu\beta} - \eta^{\alpha\sigma}\eta^{\rho\lambda}\eta^{\nu\beta}).$$

Now, using anticommutation relation of γ-matrices, Tr of any even number of them can be computed easily:

$$\mathrm{Tr}[\gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\lambda\gamma^\alpha\gamma^\beta] = 4\,[\eta^{\rho\sigma}(\eta^{\nu\lambda}\eta^{\alpha\beta} - \eta^{\nu\alpha}\eta^{\lambda\beta} + \eta^{\nu\beta}\eta^{\lambda\alpha})$$
$$-\eta^{\rho\nu}(\eta^{\sigma\lambda}\eta^{\alpha\beta} - \eta^{\sigma\alpha}\eta^{\lambda\beta} + \eta^{\alpha\beta}\eta^{\lambda\alpha})$$
$$+\eta^{\rho\lambda}(\eta^{\sigma\nu}\eta^{\alpha\beta} - \eta^{\sigma\alpha}\eta^{\nu\beta} + \eta^{\sigma\beta}\eta^{\nu\alpha})$$
$$-\eta^{\rho\alpha}(\eta^{\sigma\nu}\eta^{\lambda\beta} - \eta^{\sigma\lambda}\eta^{\nu\beta} + \eta^{\sigma\beta}\eta^{\nu\lambda})$$
$$+\eta^{\rho\beta}(\eta^{\sigma\nu}\eta^{\lambda\alpha} - \eta^{\sigma\lambda}\eta^{\nu\alpha} + \eta^{\sigma\alpha}\eta^{\nu\lambda})]$$

If we anti-symmetrize with respect to each pair, then we find (after doing the algebra):
$$\mathrm{Tr}\,[[\gamma^\rho,\gamma^\sigma][\gamma^\nu,\gamma^\lambda][\gamma^\alpha,\gamma^\beta]] = 32\,\mathrm{Tr}[J_1^{\rho\sigma}J_1^{\nu\lambda}J_1^{\alpha\beta}]$$

$$\Longrightarrow \quad \underline{\mathrm{Tr}[J_{1/2}^{\rho\sigma}J_{1/2}^{\nu\lambda}J_{1/2}^{\alpha\beta}] = \tfrac{1}{2}\mathrm{Tr}[J_1^{\rho\sigma}J_1^{\nu\lambda}J_1^{\alpha\beta}]}.$$

c) First note that the effective action of $N=4$ theory is:

$$i\,\Gamma^{N=4}[A] = -\tfrac{1}{2}\log\det\Delta_{G,1} + \log\det\Delta_{G,1/2} - \tfrac{1}{2}\log\det\hat{\Delta}_{G,0},$$ (A is the background gauge field)

and $$\log\det\Delta_{G,j} = \mathrm{Tr}\log(-D^2 + F^a_{\rho\sigma}t^a_G J^{\rho\sigma}_j)$$
$$= \mathrm{Tr}\log(-\partial^2 + \Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}),$$

where:

$$\Delta^{(1)} := i\left(\partial\cdot A^a t^a + A^a t^a\cdot\partial\right) \quad , \quad \Delta^{(2)} := A^a t^a\cdot A^b t^b \quad ,$$

$$\Delta^{(J)} := F^b_{\rho\sigma}\, J_J^{\rho\sigma}\, t^b_G$$

now, if we start expanding the effective action, we find :

$$\log\det\Delta_{G,J} = \log\det(-\partial^2) + \mathrm{Tr}\left((-\partial^2)^{-1}\left(\Delta^{(1)}+\Delta^{(2)}+\Delta^{(J)}\right) + \text{higher orders}\right)$$

$$\Rightarrow i\Gamma^{\mathcal{N}=4}[A] = \mathrm{Tr}\Big[\left(-\tfrac{1}{2}(-\partial^2)^{-1}\left(\Delta^{(1)}+\Delta^{(2)}+\Delta^{J=1}\right)+\cdots\right) + (-\partial^2)^{-1}\left(\Delta^{(1)}+\Delta^{(2)}+\Delta^{J=\frac{1}{2}}+\cdots\right)$$

$$-\tfrac{1}{2}(-\partial^2)^{-1}\left(\Delta^{(1)}+\Delta^{(2)}\right)+\cdots\Big] ,$$

Now look at terms with the same power of background field. For quadratic terms we simply find that they are cancelled against eachother. For

$\frac{1}{4}\mathrm{Tr}\left((-\partial^2)^{-1}\Delta^{J=1}(-\partial^2)^{-1}\Delta^{J=1}\right)$, we know that it is proportional to $\mathrm{Tr}[J_1 J_1]$ and because of $\mathrm{Tr}[J_1J_1] = \frac{1}{2}\mathrm{Tr}[J_{\frac12}J_{\frac12}]$, it is cancelled by :

$-\frac{1}{2}\mathrm{Tr}\left((-\partial^2)^{-1}\Delta^{J=\frac12}(-\partial^2)^{-1}\Delta^{J=\frac12}\right)$. Therefore, for $m=2$ all terms are cancelled. Now, lets examine the case $m=3$. Again terms which are independent of $\Delta^{(J)}$ are cancelled because the are in common for vectors and spinors. Terms which involve $\Delta^{(J)}$ are also cancelled using $\mathrm{Tr}[J_1J_1J_1] = 2\,\mathrm{Tr}[J_{\frac12}J_{\frac12}J_{\frac12}]$ as the previous case. So no term for $m=3$.

Note:

For $m=4$, ~~[illegible]~~ a similar identity between four generators of vector and spinor

~~[illegible]~~ reps can <u>NOT</u> be constructed if we use the commutation relation of two J's to reduce it to the cubic case, because the last term $[JJJJ]$ has the same sign and it will be cancelled by the orginal term. But the point is we do not need a relation like that. Note that for ~~any~~ $m=4$ (as other cases) the external gluons should couple to on J and since Δ has no p defendence, we find that four point function is proportional to <u>scalar box diagram</u> which is of course finite in $d=4$.

Therefore, schematically, we get the following result for ~~the~~ m-point functions

$$\underbrace{\langle AA\cdots A\rangle}_{m\text{-times}} \sim \begin{cases} \int d^4p \,\dfrac{(p^\mu)^{m-4}}{(p^2)^m}, & m \geq 4 \\ \\ 0, & m=1,2,3 \end{cases}$$

d) The theory is obviously finite. ~~and~~ Specially it is clear that even for $m=4$, there is no factor of p^μ in the numerator, which leads to a finite result.

$\boxed{2}$ a) Let us assume that $\bar{\lambda} = \bar{\lambda}(\lambda) = \lambda + a\lambda^2 + b\lambda^3 + c\lambda^4 + \cdots$

$$\bar{\beta}(\bar{\lambda}) = M\frac{\partial\bar{\lambda}}{\partial M} = M\frac{\partial\bar{\lambda}}{\partial\lambda}\frac{\partial\lambda}{\partial M} = \left(\frac{\partial\bar{\lambda}}{\partial\lambda}\right)M\frac{\partial\lambda}{\partial M} = \frac{\partial\bar{\lambda}}{\partial\lambda}\beta(\lambda)$$

$$\Rightarrow \bar{\beta}(\bar{\lambda}) = \frac{\partial\bar{\lambda}}{\partial\lambda}\beta(\lambda)$$

Now let's assume: $\bar{\beta}(\bar{\lambda}) = \bar{A}\bar{\lambda} + \bar{B}\bar{\lambda}^2 + \cdots$, $\beta(\lambda) = A\lambda + B\lambda^2 + \cdots$

$$\Rightarrow \bar{A}(\lambda + a\lambda^2 + b\lambda^3 + \cdots) + \bar{B}(\lambda + a\lambda^2 + b\lambda^3 + \cdots)^2 + \cdots = (1 + 2a\lambda + 3b\lambda^2)(A\lambda + B\lambda^2 \cdots)$$

Then the relations between coefficients can be extracted from the above relation.

For example, for the leading term: λ and λ^2 term:

$$\underline{\bar{A} = A}, \quad a\bar{A} + \bar{B} = B + 2aA \Rightarrow aA + \bar{B} = B + 2aA \Rightarrow \underline{\bar{B} = B + aA}.$$

and etc.

b) Note that: $\bar{\beta}(\bar{\lambda}) = \frac{\partial\bar{\lambda}}{\partial\lambda}\beta(\lambda)$, if $\lambda = \lambda_F$ is a fixed point, i.e. $\beta(\lambda_F) = 0$

$$\Rightarrow \bar{\beta}(\bar{\lambda}_F) = \left.\frac{\partial\bar{\lambda}}{\partial\lambda}\right|_{\lambda_F}\cancelto{0}{\beta(\lambda_F)} = 0 \Rightarrow \bar{\beta}(\bar{\lambda}_F) = 0$$

$$\Rightarrow \bar{\lambda}_F = \bar{\lambda}_F(\lambda_F) \text{ is a fixed point.}$$

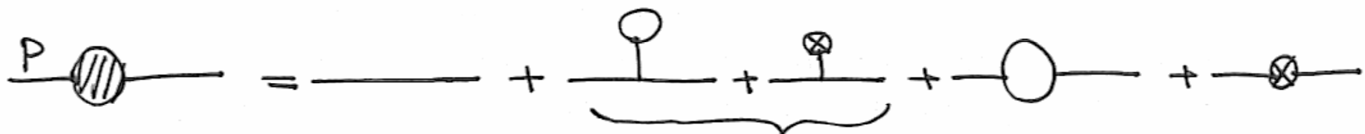
$\boxed{3}$ a) In $d=6$, the dimension of scalar field is $[\phi]=2$. Therefore the interaction term $\lambda\phi^3$ has the following dimension: $[\lambda\phi^3]=[\lambda]+3[\phi]=[\lambda]+6$.

On the other hand, we know $[\lambda\phi^3]=6 \Rightarrow [\lambda]=0$. Since the coupling is dimensionless, the theory is renormalizable.

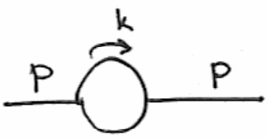
b) The one-loop β-function in ϕ^3-theory is given by:

$$\beta(\lambda) = M\frac{\partial}{\partial M}\left(-\delta_\lambda^{(1)} + \frac{3}{2}\lambda\,\delta_Z^{(1)}\right).$$

First, let us compute $\delta_Z^{(1)}$.



Cancelled by tadpole condition.

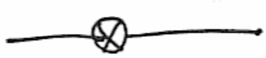


$$= \frac{1}{2}\int \frac{d^6k}{(2\pi)^6}\,\frac{i}{k^2}\,\frac{i}{(p-k)^2}(-i\lambda)^2 = \frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^6k}{(2\pi)^6}\,\frac{1}{(k^2-2xk\cdot p+xp^2)^2}$$

$$= \frac{\lambda^2}{2}\int_0^1 dx \left(\frac{1}{-x(1-x)p^2}\right)^{2-d/2} \frac{i}{(4\pi)^{d/2}}\,\frac{\Gamma(2-d/2)}{\Gamma(2)}$$

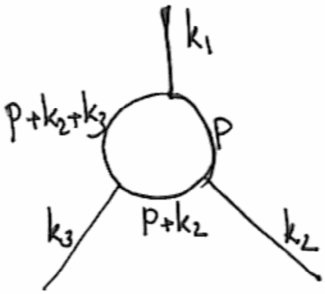
$$= \frac{\lambda^2}{2}\,\frac{i}{(4\pi)^3}\int_0^1 dx\, \Gamma(2-d/2)\,(-x(1-x)p^2)$$

$$= -\frac{\lambda^2}{2}\,\frac{i}{(4\pi)^3}\,\frac{p^2}{6}\,\Gamma(2-d/2)$$

Note: $i p^2 \delta_Z^{(1)} =$ 

$$\Rightarrow \delta_Z^{(1)} = \frac{\lambda^2}{12(4\pi)^3}\,\frac{\Gamma(2-d/2)}{(-M^2)^{3-d/2}} = \frac{\lambda^2}{12(4\pi)^3}\log(-M^2)+\cdots$$

Now, lets compute $\delta_\lambda^{(1)}$:



$$= (-i\lambda)^3\int\frac{d^6p}{(2\pi)^6}\,\frac{i}{p^2}\,\frac{i}{(p+k_2)^2}\,\frac{i}{(p+k_2+k_3)^2}$$

$$= \lambda^3\int dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^6\ell}{(2\pi)^6}\,\frac{2}{(\ell^2-\Delta)^3}$$

$$= \lambda^3\int dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{\Gamma(3-d/2)}{\Delta^{3-d/2}}\left(\frac{-i}{(4\pi)^3}\right)$$

$$\Rightarrow \delta_\lambda^{(1)} = -\lambda^3\int dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{1}{(4\pi)^3}\,\frac{\Gamma(\epsilon/2)}{(-M^2)^{-\epsilon/2}}$$

$$= \frac{\lambda^3}{2(4\pi)^3}\log(-M^2)+\cdots$$

Now, we can easily compute the one-loop β-function:

$$\beta(\lambda) = -\frac{\lambda^3}{(4\pi)^3}+\frac{3}{2}\lambda\,\frac{\lambda^2}{6(4\pi)^3} = -\frac{3\lambda^3}{4(4\pi)^3}.$$

As is clear, $\beta(\lambda)<0$ and the theory is <u>asymptotically free</u>