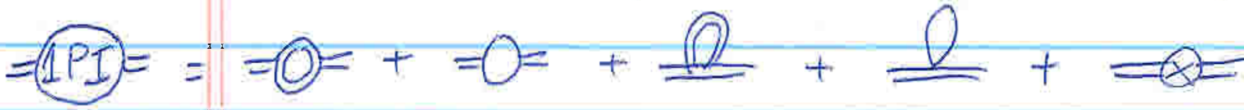
$\boxed{\text{Solution (HW2)}}$

(1a) For the mass correction to the sigma field, we should compute (eq. (11.34) of PS):



where  $= -i(2v^2\delta_\lambda) - i(\delta_\mu + v^2\delta_\lambda) + ip^2\delta_Z$

- The renormalization condition  $= 0$ means that these contributions cancel:

- Also, PS show that  $+ (-i(\delta_\mu + v^2\delta_\lambda))$ $= 0$ (including finite terms)

- So,

 $- i(2v^2\delta_\lambda) + ip^2\delta_Z$

- δ_λ is divergent; it will cancel a pole in  and just leave a $\ln M^2$, using $\overline{MS}$ scheme for λ.
- δ_Z is finite, but we <u>need</u> <u>it</u> because at $p^2 = 2\mu^2$, it gives a nonvanishing mass shift.
- Let $I_1 =$ , $I_2 =$ 
- δ_Z is determined by

$$0 = \frac{d}{dp^2}\left(=\!(1PI)\!=\right)\Big|_{p^2=2\mu^2} = \frac{d}{dp^2}(I_1 + I_2)\Big|_{p^2=2\mu^2} + i\delta_Z$$

$$\Rightarrow \delta_Z = i\frac{d}{dp^2}(I_1 + I_2)\Big|_{p^2=2\mu^2}$$

(1a) (CONT.)

· From eqs. (10.26), (10.27) of PS,

$= \text{(1PI)} = \; = -iM(p^2)$, with full propagator

$$= \text{(///)} = \; = \frac{i}{p^2 - m^2 - M(p^2)} \simeq \frac{i}{p^2 - m^2 - M(2\mu^2)}$$

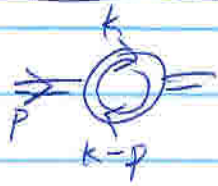
for $p^2 \simeq 2\mu^2$, the tree-level mass.

· Thus the mass^2 shift is

$$\boxed{\begin{aligned} \delta m_\sigma^2 &= M(2\mu^2) = i\left(= \text{(1PI)} =\right)\Big|_{p^2 = 2\mu^2} \\ &= i\left\{ I_1 + I_2 - 2\mu^2 \frac{d}{dp^2}(I_1 + I_2)\right\}\Big|_{p^2 = 2\mu^2} \end{aligned}}$$

(from finite δ_Z)

· Now $I_1 =$



$$= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} (-6i\lambda v)^2 \frac{i}{k^2 - 2\mu^2} \frac{i}{(k-p)^2 - 2\mu^2}$$

$$= 18\lambda^2 v^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - 2k\cdot p x + x p^2 - 2\mu^2)^2}$$

$$= 18 i \lambda^2 v^2 \int_0^1 dx \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2}, \quad \Delta = -x(1-x)p^2 + 2\mu^2$$

$$= \frac{18 i \lambda^2 v^2}{(4\pi)^{d/2}} \Gamma(2 - d/2) \int_0^1 dx\, \Delta^{-(2-d/2)}$$

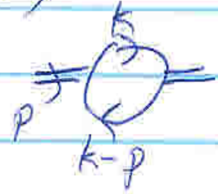
$$= \frac{18 i \lambda^2 v^2}{(4\pi)^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \gamma - \ln 4\pi - \ln \Delta\right)$$

After $\overline{\text{MS}}$ renormalization,

$$\boxed{I_1 = -\frac{18 i \lambda^2 v^2}{(4\pi)^2} \int_0^1 dx \ln\left(\frac{-x(1-x)p^2 + 2\mu^2}{M^2}\right)}$$

(1a) (CONT.)

Similarly, $I_2 =$



$$= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} (-2i\lambda v)^2 \delta^{ij}\delta^{kl} \frac{i\delta^{jk}}{k^2}\,\frac{i\delta^{il}}{(k-p)^2}$$

$$= \frac{(N-1)}{3^2} * I_1(p^2)\Big|_{\mu^2\to 0}$$

After $\overline{MS}$ renormalization,

$$\Rightarrow \boxed{I_2 = -\frac{2i(N-1)\lambda^2 v^2}{(4\pi)^2}\int_0^1 dx \ln\left(\frac{-x(1-x)p^2}{M^2}\right)}$$

• So, $$\delta m_\sigma^2 = \frac{(\lambda v)^2}{8\pi^2}\Bigg\{ 9\int_0^1 dx \left[\ln\left(\frac{-x(1-x)p^2+2\mu^2}{M^2}\right) - 2\mu^2\frac{d}{dp^2}\ln\left(\frac{-x(1-x)p^2+2\mu^2}{M^2}\right)\right]$$
$$+ (N-1)\int_0^1 dx\left[\ln\left(\frac{-x(1-x)p^2}{M^2}\right) - 2\mu^2\frac{d}{dp^2}\ln\left(\frac{-x(1-x)p^2}{M^2}\right)\right]\Bigg\}\Bigg|_{p^2=2\mu^2}$$

• Take real part here, imaginary part contributes to decay width of σ

$$\delta m_\sigma^2 = \frac{(\lambda v)^2}{8\pi^2}\Bigg\{ 9\int_0^1 dx\left[\ln\left(\frac{2\mu^2}{M^2}\right) + \ln(1-x(1-x)) - \frac{\cancel{2\mu^2}}{\cancel{2\mu^2}}\,\frac{-x(1-x)}{1-x(1-x)}\right]$$
$$+ (N-1)\int_0^1 dx\left[\ln\left(\frac{2\mu^2}{M^2}\right) + \ln(x(1-x)) - \frac{\cancel{2\mu^2}}{\cancel{2\mu^2}}\cdot 1\right]$$

$$= \frac{(\lambda v)^2}{8\pi^2}\Bigg\{ 9\left[\ln\left(\frac{2\mu^2}{M^2}\right) + \left(-2+\frac{\pi}{\sqrt{3}}\right) + \left(-1+\frac{2\pi}{3\sqrt{3}}\right)\right]$$
$$+ (N-1)\left[\ln\left(\frac{2\mu^2}{M^2}\right) + (-2) + (-1)\right]\Bigg\}$$

(1a) (CONT.)

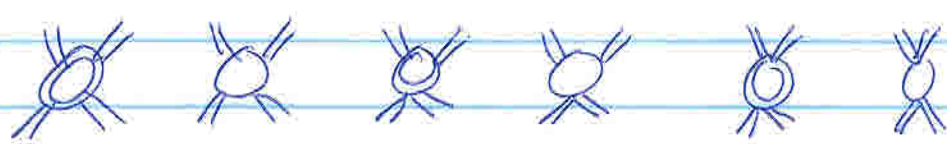
$$\Rightarrow \boxed{\delta M_\sigma^2 = \frac{(\lambda v)^2}{8\pi^2}\left\{(N+8)\left[\ln\left(\frac{2\mu^2}{M^2}\right)-3\right] + 5\sqrt{3}\,\pi\right\} \quad \text{with} \quad \mu^2 = \lambda v^2, \quad \lambda \equiv \lambda^{\overline{MS}}}$$

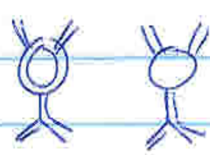
(1b) As noted above, the finite part of δ_Z is required, because it enters —(1PI)— as $ip^2\delta_Z \to i2\mu^2\delta_Z \neq 0$, near the tree-level ~~mass~~ pole position, $p^2 = 2\mu^2$.

(1c) Using on-shell renormalization of λ would have required computation of the finite parts of all the one-loop contributions to the four-σ scattering amplitude, to determine δ_λ from

 $= -6i\lambda$ at $s = 4m^2$, $t = u = 0$

$\Rightarrow$ You would have had to compute finite parts of graphs like

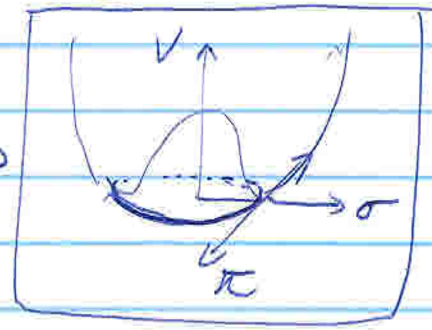


 and several more (!)

(2)

It is true that if we move from one vacuum to a nearby vacuum, it shouldn't cost any energy. However the vacuum "manifold" is curved, while (π^k, σ) are Cartesian coordinates, and the π^k don't follow the direction of zero energy past leading order.

For example, for $N=2$ the flat direction is a circle S^1 and the π field is tangent to the circle.



Once we leave the flat direction, we will need to invest energy.