Solution for Problem Set 1:

① We want to prove: $\delta_1 = \delta_2$. In d dimensions:

$$\delta_1 = \frac{-e^2}{(4\pi)^{d/2}} \int dz\,(1-z)\left(\frac{\Gamma(\epsilon/2)}{\Delta^{\epsilon/2}}\,\frac{(2-\epsilon)^2}{2} + \frac{\Gamma(1+\epsilon/2)}{\Delta^{1+\epsilon/2}}\left(2(1-4z+z^2)-\epsilon(1-z)^2\right)m^2\right),$$

$$\delta_2 = \frac{-e^2}{(4\pi)^{d/2}} \int dz\, \frac{\Gamma(\epsilon/2)}{\Delta^{\epsilon/2}}\left[(2-\epsilon)z - \epsilon/2\,\left(4-2z-\epsilon(1-z)\right)\frac{2z(1-z)m^2}{\Delta}\right],$$

where $\Delta = (1-z)^2 m^2 + z\mu^2$.

To evaluate δ_1 and δ_2 in $d=4$, we have to send $\epsilon \to 0$. In this limit

we find :

$$\delta_1 - \delta_2 = \frac{-e^2}{(4\pi)^2}\int dz\left[(1-2z)\left(\frac{4}{\epsilon} - 2 - 2\log\Delta - 2\gamma + 2\log 4\pi\right) - \right.$$

$$\left. 2(1-z) + (1-z)\frac{2m^2}{\Delta}(1-4z+z^2) + (1-z)2z(4-2z)\frac{m^2}{\Delta}\right],$$

obviously: $\int_0^1 (1-2z)\,dz = 0$, So

$$\delta_1 - \delta_2 = \frac{-e^2}{(4\pi)^2}\int dz\left[-2(1-z) + (1-z)\,\frac{2(1-z^2)\,m^2}{\Delta} - 2(1-2z)\log\Delta\right],$$

integrating by part the last term, we find:

$$\int_0^1 dz\,(1-2z)\log\Delta = z(1-z)\log\Delta\Big|_0^1 - \int_0^1 dz\, z(1-z)\,\frac{-2(1-z)m^2+\mu^2}{\Delta}$$

$$= \int_0^1 dz\, z(1-z)\,\frac{2(1-z)m^2-\mu^2}{\Delta},$$

using the above result, we find for $\delta_1 - \delta_2$:

$$\delta_1 - \delta_2 = \frac{-e^2}{(4\pi)^2}\int_0^1 dz\,\frac{1-z}{\Delta}\,m^2\left(-4z(1-z) - 2(1-z)^2 + 2(1-z^2)\right) = 0. \checkmark$$

② The bare Lagrangian of the theory is:

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m_0^2 \phi^2 + \bar{\psi}(i\not{\partial} - \cancel{m} M_0)\psi - i g_0 \bar{\psi}\gamma^5 \psi \phi ,$$

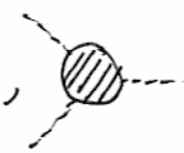
Notice that we have two types of vertices. Therefore, the superficial degree of divergence of the theory is: $D = 4L - 2P_\phi - P_\psi$,

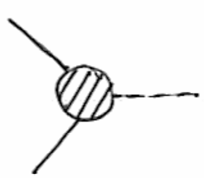
and $L = P_\phi + P_\psi - V + 1$, $V = 2P_\phi + N_\phi = \tfrac{1}{2}(2P_\psi + N_\psi)$.

Substituting L and V for D, we find: $\underline{D = 4 - N_\phi - \tfrac{3}{2} N_\psi}$.

No, we can list all divergent diagrams (1PI):

D=4:  , D=3 

D=2 :  , D=1  , 

D=0 ,  , 

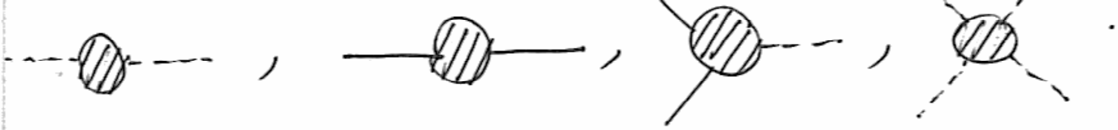
If we require that the scattering amplitudes must respect the parity-symmetry, then some of the above diagram will vanish. For instance,

$$\langle \Omega | \bar{\psi} \gamma^5 \psi | \Omega \rangle = \langle \Omega | P^{-1} P \bar{\psi} P^{-1} \gamma^5 P \psi P^{-1} P | \Omega \rangle$$

$$= \langle \Omega | \bar{\psi} \gamma^0 \gamma^5 \gamma^0 \psi | \Omega \rangle = - \langle \Omega | \bar{\psi} (\gamma^0)^2 \gamma^5 \psi | \Omega \rangle$$

$$= - \langle \Omega | \bar{\psi} \gamma^5 \psi | \Omega \rangle \Longrightarrow \langle \Omega | \bar{\psi} \gamma^5 \psi | \Omega \rangle = 0 .$$

So, the diagrams which are non-vanishing are:



We have now 5 terms in Lagrangian, whereas we have 6 counterterms

(3 for the first diag., 2 for second diag, 1 for third diag, 1 for last diag)

2 by parity ($p \to -p$)

Therefore, we need to add a $\lambda \phi^4$ term to the theory. The bare λ_0 can be set to zero, but radiative corrections will generate it anyway.

Including this ϕ^4 term, the theory will be renormalizable.

Then, the renormalized Lagrangian of the theory can be written as:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_r)^2 - \tfrac{1}{2} m^2 \phi_r^2 + \bar{\psi}_r (i\partial\!\!\!/ - M)\psi_r - ig\,\bar{\psi}_r \gamma^5 \psi_r \phi_r + \tfrac{1}{2}\delta_{Z_\phi}(\partial_\mu \phi_r)^2$$

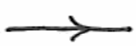
$$- \tfrac{1}{2}\delta_m \phi_r^2 + \delta_{Z_\psi}(\bar{\psi}_r\, i\partial\!\!\!/\, \psi_r) - \delta_M \bar{\psi}_r \psi_r - i\delta_g \bar{\psi}_r \gamma^5 \psi_r \phi_r$$

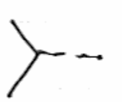
$$- \frac{\lambda}{4!}\phi_r^4 - \frac{\delta_\lambda}{4!}\phi_r^4 .$$

where: $\delta_{Z_\phi} = Z_\phi - 1$, $\delta_{Z_\psi} = Z_\psi - 1$, $\delta_m = m_0^2 Z_\phi - m$, $\delta_M = M_0 Z_\psi - M$

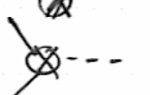
$$\delta_g = g_0 Z_\psi Z_\phi^{\frac{1}{2}} - g, \quad \delta_\lambda = \lambda_0 Z_\phi^2 - \lambda,$$

The Feynman rules are read as:

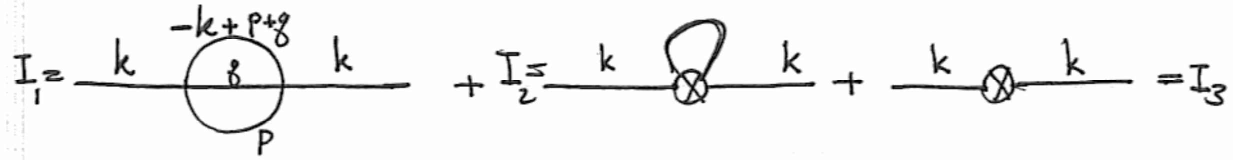
 : $\dfrac{i}{p\!\!\!/ - M}$, : $\dfrac{i}{p^2 - m^2}$,

 : $-ig\gamma^5$,  : $-i\lambda$,

 : $i(p\!\!\!/\, \delta_{Z_\psi} - \delta_M)$,  : $i(p^2 \delta_{Z_\phi} - \delta_m)$,

 : $-i\delta_g \gamma^5$,  : $-i\delta_\lambda$.

③ The two-loop diagrams contributing to the propagator in ϕ^4 theory are:



We want to compute the above diagrams in the limit of vanishing scalar mass.

$$I_1 = (-i\lambda)^2 \int \frac{d^4p\, d^4q}{(2\pi)^8} \frac{-i}{(p^2+i\epsilon)(q^2+i\epsilon)\left((p+q-k)^2+i\epsilon\right)},$$

we use dimensional regularization:

$$I_1 = (-i\lambda)^2 \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+i\epsilon} \int \frac{d^dq}{(2\pi)^d} \frac{(-i)}{(q^2+i\epsilon)\left((p+q-k)^2+i\epsilon\right)},$$

using the usual Feynman parametrization, we have:

$$I_1 = i\lambda^2 \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+i\epsilon} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^2}, \quad \ell = q + x(p-k)$$

and $\Delta = (x-1)x(p-k)^2$.

Taking the integral over momentum ℓ, we have:

$$I_1 = i\lambda^2 \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+i\epsilon} \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{(x(x-1))^{2-d/2}(p-k)^{4-d}}$$

$$= \frac{-\lambda^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{1}{(x(x-1))^{2-d/2}} \int_0^1 dw \int \frac{d^dp}{(2\pi)^d} \frac{w^{1-d/2}}{\left(w(p-k)^2+(1-w)p^2\right)^{3-d/2}}$$

$$\times \frac{\Gamma(3-d/2)}{\Gamma(2-d/2)\,\Gamma(1)} \Gamma(2-d/2)$$

using (10.56) of Peskin.

$$I_1 = \frac{-\lambda^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{1}{(x(x-1))^{2-d/2}} \int_0^1 dw\, w^{1-d/2} \int \frac{d^d\ell'}{(2\pi)^d} \frac{\Gamma(3-d/2)}{(\ell'^2+w(1-w)+wk^2)^{3-d/2}},$$

where $\ell' = p - wk$.

now, taking the l' integral and sending $d \to 4$ $(\epsilon \to 0)$, we find:

$$I_1 = \frac{i\lambda^2}{(4\pi)^4}\int_0^1 dx\left(1-(2-d/2)\ln(x(x-1)) + \mathcal{O}(\epsilon^2)\right)\int_0^1 dw\,\frac{-w(1-w)}{w}k^2\left(\frac{-1}{\epsilon}+\text{finite}\right)$$

$$\times\left(1-\epsilon\ln k^2\right) \qquad \text{, note } (k^2)^{-\epsilon} = e^{\ln(k^2)^{-\epsilon}} = e^{-\epsilon\ln k^2} \simeq 1-\epsilon\ln k^2 + \cdots$$

$$\Rightarrow I_1 = \frac{i\lambda^2}{(4\pi)^4}\int_0^1 dx\int_0^1 dw\,(1-w)k^2\left(\frac{1}{\epsilon}-\ln k^2\right)+\cdots,$$

Considering a symmetry factor of $1/6$, the final result is:

$$I_1 = \frac{-i\lambda^2}{12(4\pi)^4}k^2\left(-\frac{1}{\epsilon}+\ln k^2+\cdots\right).$$

For the second diagram, we need the counterterm at 1-loop order.

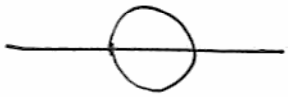
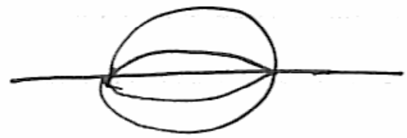
$$I_2 = -i\,\delta_\lambda^{(1)}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2-m^2} = -i\,\delta_\lambda^{(1)}\frac{-i}{(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{\Gamma(1)}(m^2)^{-(1-d/2)}$$

$$= m^2\frac{-\lambda^2}{2(4\pi)^d}\Gamma(2-d/2)\,\Gamma(1-d/2)(m^2)^{-\epsilon/2}\int_0^1 dx\left(\frac{1}{(m^2+4x(x-1)m^2)^{2-d/2}}+\frac{2}{(m^2)^{\epsilon/2}}\right)$$

$$= \frac{-1}{(4\pi)^d}\frac{\lambda^2}{2}\Gamma(2-d/2)\,\Gamma(1-d/2)\,m^{2-2\epsilon}\int_0^1 dx\left(2+\frac{1}{(1+4x(x-1))^{\epsilon/2}}\right),$$

So, as we send $m\to 0$, $I_2 \to 0$.

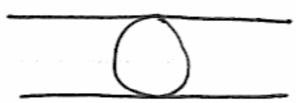
(4) In order to investigate the divergent properties of the given amplitudes, we need to calculate the superficial degree of divergence for each of them.

a) G_2:  , G_2' : 

$L=2$, $P=3$ $L=4$, $P=5$, $V=2$

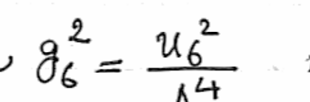
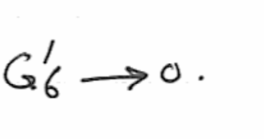
$D=4\cdot2-3\cdot2=2$, $D=4\cdot4-5\cdot2-2\cdot2=2.$

Since, they don't have nested divergence, D for both of them is indeed the same.

b) G_4 :  , G_4': 

$D=4\cdot1-2\cdot2=0$, $D=4\cdot3-4\cdot2-2\cdot2=0.$

So, they are logarithmically divergent.

c) G_6:  , G_6': 

$D=4\cdot2-5\cdot2=-2 \Rightarrow$ Convergent $D=4\cdot2-3\cdot2-2\cdot2=-2$

For G_6': $\sim g_6^2=\dfrac{u_6^2}{\Lambda^4}$ therefore, as $\Lambda\to\infty$, $G_6'\to0$.

d) The theory is renormalizable. Because G_6' (and similar ones which involve g_6) vanish, the divergent amplitudes are:

 ,  ,  ,  ,

which can be absorbed by the usual counterterms as in ϕ^4 theory.