Physics 332, Spring 2008
Final Exam Solutions

E.1

Problem 1: The two β functions are related by the chain rule:

$$\bar\beta(\bar\alpha(\alpha)) = \mu \frac{\partial \bar\alpha(\alpha)}{\partial \mu} = \mu \frac{\partial \alpha}{\partial \mu} \frac{\partial \bar\alpha}{\partial \alpha}$$

$$\Rightarrow \boxed{\bar\beta(\bar\alpha(\alpha)) = \frac{\partial \bar\alpha}{\partial \alpha}\beta(\alpha)}$$

Equate Sue vs. Bob:
$$\bar\beta(\bar\alpha(\alpha)) = \bar b_0 \bar\alpha^2 + \bar b_1 \bar\alpha^3 + \bar b_2 \bar\alpha^4 + \dots$$
$$= \bar b_0 (\alpha + c_1\alpha^2 + c_2\alpha^3)^2$$
$$+ \bar b_1 (\alpha + c_1 \alpha^2 + \dots)^3$$
$$+ \bar b_2 (\alpha + \dots)^4 + \dots$$
$$= \bar b_0 \alpha^2 + (\bar b_1 + 2c_1 \bar b_0)\alpha^3$$
$$+ \left(\bar b_2 + 3c_1 \bar b_1 + (2c_2 + c_1^2)\bar b_0\right)\alpha^4$$
$$+ \dots$$

chain rule

$$= \frac{\partial \bar\alpha}{\partial \alpha}\beta(\alpha) = (1 + 2c_1\alpha + 3c_2\alpha^2 + \dots)(b_0\alpha^2 + b_1\alpha^3 + b_2\alpha^4 + \dots)$$
$$= b_0\alpha^2 + (b_1 + 2c_1 b_0)\alpha^3$$
$$+ (b_2 + 2c_1 b_1 + 3c_2 b_0)\alpha^4 + \dots$$

Equating the $\alpha^2, \alpha^3, \alpha^4$ coefficients gives:

$$\boxed{\bar b_0 = b_0}$$

$$\boxed{\bar b_1 = b_1} \Leftarrow \bar b_1 + \cancel{2c_1 \bar b_0} = b_1 + \cancel{2c_1 b_0}$$

equal

$$\bar b_2 + 3c_1\bar b_1 + (2c_2 + c_1^2)\bar b_0 = b_2 + 2c_1 b_1 + 3c_2 b_0$$

$$\Rightarrow \boxed{\bar b_2 = b_2 - c_1 b_1 + (c_2 - c_1^2) b_0} \Leftarrow \text{Sue's } \bar b_2 \text{ in terms of Bob's } b_i, c_1 \text{ and } c_2$$

$\boxed{\text{Problem 1 (CONT.)}}$

Ralph vs. Bob: $\tilde{\beta}(\tilde{\alpha}(\alpha)) = \tilde{b}_0\tilde{\alpha}^2 + \tilde{b}_1\tilde{\alpha}^3 + \ldots$

$$= \tilde{b}_0\left(\alpha + d_1\alpha^2\ln\alpha + d_2\alpha^2\right)^2 + \tilde{b}_1(\alpha + \ldots)^3 + \ldots$$

$$= \tilde{b}_0\alpha^2 + 2\tilde{b}_0 d_1\alpha^3\ln\alpha + (\tilde{b}_1 + 2\tilde{b}_0 d_2)\alpha^3 + \mathcal{O}(\alpha^4\ln^2\alpha)$$

chain rule

$$= \frac{\partial\tilde{\alpha}}{\partial\alpha}\beta(\alpha) = \left(1 + d_1\alpha + 2d_1\alpha\ln\alpha + 2d_2\alpha\right)\left(b_0\alpha^2 + b_1\alpha^3 + \ldots\right)$$

$$= b_0\alpha^2 + 2b_0 d_1\alpha^3\ln\alpha + (b_1 + (d_1 + 2d_2)b_0)\alpha^3 + \mathcal{O}(\alpha^4\ln\alpha)$$

- Matching coefficients at $\mathcal{O}(\alpha^2) \Rightarrow \boxed{\tilde{b}_0 = b_0}$ still

$\Rightarrow \mathcal{O}(\alpha^3\ln\alpha)$ coefficient matches.

At $\mathcal{O}(\alpha^3)$, we find $\tilde{b}_1 + 2\tilde{b}_0 d_2 = b_1 + d_1 b_0 + 2d_2 b_0$

$$\Rightarrow \boxed{\tilde{b}_1 = b_1 + d_1 b_0}$$

- So Ralph begins to differ from Bob (and from Sue) at 2-loop order ($\tilde{b}_1 \neq b_1$).

- In fact, supersymmetric theories can be renormalized in such a way that the only nonvanishing term in the β function is at 1-loop. However, this so-called Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β function differs from the $\overline{MS}$-scheme β function at 2-loops, precisely because of a "non-analytic" relation of the form

$$\tilde{\alpha} = \alpha + d_1\alpha^2\ln\alpha + \mathcal{O}(\alpha^3)$$

between the NSVZ and $\overline{MS}$ couplings.

Problem 2 Also known as the <u>Banks-Zaks</u> fixed point.

We have

$$\beta(\alpha) = -\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\frac{\alpha^2}{2\pi} - \left(\frac{17}{3}N_c^2 - \left(\frac{13}{6}N_c - \frac{1}{2N_c}\right)N_f\right)\frac{\alpha^3}{4\pi^2} + \mathcal{O}(\alpha^4)$$

- For $N_f = \frac{11}{2}N_c - \delta$, $\delta \ll 1$, $N_c, N_f \gg 1$,

we keep the δ-dependence in the 1-loop term, but drop it in the second term:

$$\beta(\alpha) = -\left(\frac{11}{3}N_c - \frac{2}{3}\left(\frac{11}{2}N_c - \delta\right)\right)\frac{\alpha^2}{2\pi}$$

$$-\left(\frac{17}{3}N_c^2 - \left(\frac{13}{6}N_c - \frac{1}{2N_c}\right)\frac{11}{2}N_c\right)\frac{\alpha^3}{4\pi^2} + \ldots$$

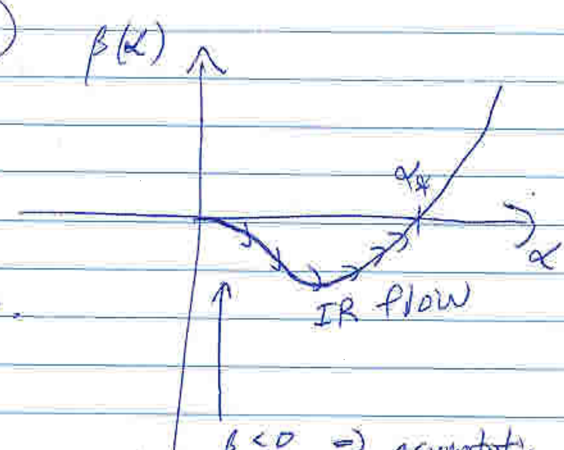
drop as $N_c \gg 1$

$$= -\delta \cdot \frac{\alpha^2}{3\pi} - \left(\frac{17}{3}\frac{(4)}{(4)} - \frac{143}{12}\right)N_c^2\frac{\alpha^3}{4\pi^2} + \ldots$$

$$\boxed{\beta(\alpha) = -\delta \cdot \frac{\alpha^2}{3\pi} + \frac{25N_c^2}{16}\frac{\alpha^3}{\pi^2} + \ldots}$$

$$\beta(\alpha_*) = 0 \Rightarrow \boxed{\alpha_* = \frac{16\pi}{75N_c^2}\cdot\delta}$$

α_* fixed-point is <u>infrared attractive</u>.



$\beta < 0 \Rightarrow$ asymptotic freedom here

$\Rightarrow$ coupling <u>increases</u> in IR

- Out of gauge fields, A_μ, and massless quarks in the fundamental $\underset{\sim}{N_c}$ representation, ψ, the only <u>relevant</u> (dim < 4), gauge-invariant operator (besides the identity) is $\bar\psi\psi$, with engineering dimension 3, $[-\frac{1}{4}F_{\mu\nu}^2$ and $\bar\psi D_\mu\gamma^\mu\psi$ are dimension 4 marginal$]$

(Problem 2 (CONT.))

- ~~Q~~ Technically, there are <u>many</u> such operators, $\bar{\Psi}_i \psi_j$ for $i, j = 1, 2, \ldots, N_f$, the number of flavors.

However, they are all basically equivalent, because of the <u>flavor</u> symmetry of the Lagrangian before adding this operator:

$\psi_i \to U_{ij} \psi_j$

$\Rightarrow$ can diagonalize $\bar{\Psi}_i \psi_j \Rightarrow \bar{\psi}_i \psi_i$ for any particular i.

- This is nothing more than the quark <u>mass</u> operator.

- The anomalous dimension of this operator is actually computed in Chapter 18 of Peskin & Schroeder for QCD. Here we follow that computation [which is also basically a QED computation] adjusting the color factors.
- We will compute a 2-point function with an insertion of $\bar{\Psi}\psi$.
- <u>First</u>, we need the <u>quark wave-function renormalization</u> ~~constant~~ counterterm δ_2. This computation was done in Chapter 16, or for QED in chapters 10 and 12. Eq. (10.43) in particular gave ~~for~~

$$\delta_2 = \frac{d}{d\not p} \underbrace{\Sigma_2'(m)}_{\text{from } \sim\sim} = \frac{-e^2}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma(2-d/2)}{((1-x)^2 m^2 + x\mu^2)^{2-d/2}} * \left[(2-\epsilon) x + m^2 [\ldots] \right]$$

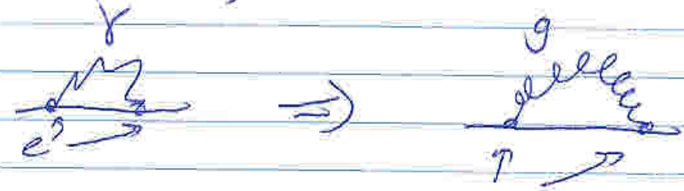
- As $m^2 \to 0$, and replacing the on-shell denominator with an off-shell renormalization at scale M^2, we get

$$\delta_2 = -\frac{e^2}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} \left[\int_0^1 dx \cdot 2x + O(\epsilon) \right]$$

Problem 2 (CONT.) $\qquad 2-d/2 = +\epsilon/2 \qquad$ (E.5)

$$\Rightarrow \delta_2^{QED} = -\frac{e^2}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} = -\frac{e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} - \ln M^2 + \dots\right]$$

In our case, all we need to do is substitute



with group theory factor $C_F = \dfrac{\sum_{a=1}^{(N_c^2-1)} (T^a T^a)_{fund.}}{\mathbb{1}_{fund}} = \dfrac{N_c^2-1}{2N_c}$

$$\Rightarrow \delta_2 = -\frac{N_c^2-1}{2N_c}\frac{g^2}{(4\pi)^2}\left[\frac{2}{\epsilon} - \ln M^2 + \dots\right]$$

Now we compute

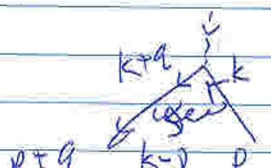
and adjust the operator counterterm $\delta_O = \delta_{\bar q q}$

to maintain

$= 1$ (tree-level value)

(as in eqs. (12.113), (18.7))

(18.10) $\qquad$ (operator $\bar\psi\psi$)



$$= \int \frac{d^dk}{(2\pi)^4}(ig)^2 t^a \gamma^\mu \frac{i(\not k+\not q)}{(k+q)^2}\cdot 1 \cdot \frac{i\not k}{k^2} t^a \gamma_\mu \frac{-i}{(k-p)^2}$$

$k \to \infty$

$$\simeq -iC_F g^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{\not k}{k^2}\frac{\not k}{k^2}\gamma_\mu \frac{1}{k^2}$$

$$\simeq -iC_F g^2 \cdot d \cdot \int \frac{d^{4-\epsilon}k}{(2\pi)^4}\frac{1}{(k^2)^2}$$

$$\simeq +C_F g^2 \cdot \frac{4}{(4\pi)^2}\Gamma(2-d/2)$$

$$\Rightarrow \boxed{\delta_O = \delta_{\bar q q} = -C_F g^2 \cdot 4 \cdot \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} = -\frac{4C_F g^2}{(4\pi)^2}\left[\frac{2}{\epsilon} - \ln M^2 + \dots\right]}$$

(Problem 2 (cont.))

· Now the anomalous dimension is:

$$\gamma_{\mathcal{O}} = \gamma_{\bar{q}q} = \mu \frac{\partial}{\partial\mu}\left(-\delta_{\bar{q}q} + \delta_2\right) \qquad \left[\begin{array}{l} \text{eqs. } \overset{\sigma(m=2)}{(12.112)} \\ \text{or } (18.8) \end{array}\right]$$

$$= -8\frac{C_F\, g^2}{(4\pi)^2} + 2C_F\frac{g^2}{(4\pi)^2}$$

$$\gamma_{\bar{q}q} = -\frac{6C_F\, g^2}{(4\pi)^2} = -\frac{3C_F}{2\pi}\alpha \simeq -\frac{3N_c}{4\pi}\alpha \qquad \left(\alpha = \frac{g^2}{4\pi}\right) \qquad \left(C_F \simeq \frac{N_c}{2},\ N_c \gg 1\right)$$

· At the fixed point, $\alpha = \alpha_* = \frac{16\pi}{75N_c^2}\delta$

$$\Rightarrow \gamma_{\bar{q}q}(\alpha_*) = -\frac{\cancel{3}N_c}{\cancel{4}\cancel{\pi}}\cdot\frac{\overset{4}{\cancel{16}}\cancel{\pi}}{\underset{25}{\cancel{75}}N_c^{\cancel{2}}}\delta$$

$$\gamma_{\bar{q}q}(\alpha_*) = -\frac{4}{25N_c}\delta$$

· To work out how physical masses scale when we add this operator, we return to the considerations of Chapter 12 (the end of it), where we found

$$\xi \sim (T-T_c)^{-\nu}$$

· But ξ = correlation length = $\frac{1}{\text{physical mass}}$

· And "$T-T_c$" is the shift away from the critical point, which is related linearly to the quark mass perturbation coefficient:

$$\Rightarrow (\text{physical mass}) \sim [\text{coefficient of } \bar{\psi}\psi]^{+\nu}$$

· But now we need to compute ν.

Problem 2 (CONT') (E.7)

To do this, we use the running-coupling formula (12.126),

$$\frac{d}{d\ln(p/M)}\bar{\rho}_i = \beta_i(\bar{\rho},\bar{\lambda})$$

where the function β_i has an "engineering term" $d_i - d = 3-4 = -1$ in our case (from how far from marginal it is) plus a term from the one-loop anomalous dimension: [eq. (12.131)]

$$\beta_i = \left[d_i - d + \gamma_i^{(4)}\right]\rho_i$$

$$= \left[-1 - \frac{4}{25N_c}\cdot\delta\right]\rho_i \quad \text{for } \alpha \simeq \alpha_*$$

$$\Rightarrow \frac{d}{d\ln(p/M)}\bar{\rho}_m = \left[-1-\frac{4}{25N_c}\delta\right]\bar{\rho}_m \quad \text{[eq. (12.134)]}$$

$$\Rightarrow \bar{\rho}_m = \rho_m\left(\frac{M}{p}\right)^{1+\frac{4}{25N_c}\delta} = \rho_m\left(\frac{M}{p}\right)^{1/\nu}$$

where $\boxed{\nu = \frac{1}{1+\frac{4}{25N_c}\cdot\delta}}$

We already computed δ_2. The quark anomalous dimension is

$$\gamma_q = \frac{1}{2}M\frac{\partial}{\partial M}\delta_2 = C_F\frac{g^2}{(4\pi)^2} = C_F\frac{\alpha}{4\pi}$$

for $N_c \gg 1$

$$\simeq \frac{N_c}{2}\frac{\alpha_*}{4\pi} = \frac{N_c}{2}\,\frac{\frac{16\pi}{75N_c^2}}{4\pi}\,\delta$$

$$\boxed{\gamma_q = \left(\frac{2}{75N_c}\right)\delta}$$

Problem 2 (CONT.)

Callan-Szymanzik eq. for quark propagator

Let's write the quark propagator as

$$G^2(p) = \frac{i\not{p}}{p^2}\, g(-p^2/M^2)$$

$\left(p\frac{\partial}{\partial p} = -1\right)$

(dimensionless, so $\left(M\frac{\partial}{\partial M} + p\frac{\partial}{\partial p}\right) g = 0$

only Dirac structure allowed in massless limit

$$\Rightarrow \left(M\frac{\partial}{\partial M} + p\frac{\partial}{\partial p} + 1\right) G^{(2)} = 0$$

The CS equation is $\cancel{p\frac{\partial}{\partial p}\, 2\beta(\alpha)\frac{\partial}{\partial \alpha} + 2}$ (substitute in)

$$\left[M\frac{\partial}{\partial M} + \beta(\alpha)\frac{\partial}{\partial \alpha} + 2\gamma_q\right] G^{(2)}(p; M, \alpha) = 0$$

$$\Rightarrow \left[p\frac{\partial}{\partial p} - \beta(\alpha)\frac{\partial}{\partial \alpha} + 1 - 2\gamma_q\right] G^{(2)}(p; M, \alpha) = 0$$ (analog of eq (12.66))

At the fixed-point, $\beta = 0$,

so $$\left(\frac{\partial}{\partial \ln p} + (1 - 2\gamma_q)\right) G^{(2)}(p; M, \alpha) = 0$$

(constant, at fixed point)

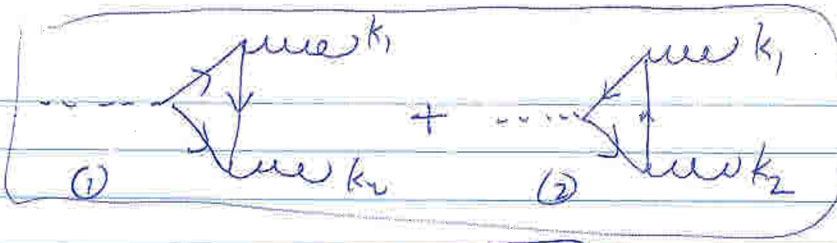
$G^{(2)} \sim \exp[-(1-2\gamma_q)\ln p]$

$$\Rightarrow \boxed{G^{(2)}(p; M, \alpha) = \frac{i\not{p}}{(p^2)^{1-\gamma_q}}}$$

$$\boxed{= \frac{i\not{p}}{(p^2)^{1 - \frac{20}{75 N_c}\delta}}}$$

$\boxed{\text{Problem 3}}$

$h \to gg$
via quark loop



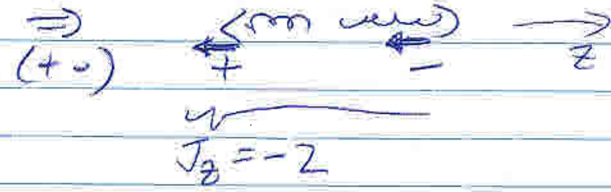
$\boxed{\text{Note}}$ 

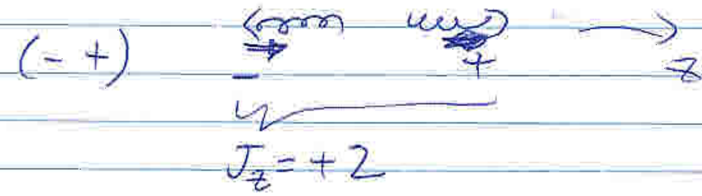
$= 0$ by color $(\mathrm{Tr}\, T^a = 0)$

⇒ Two Feynman diagrams.

- The Higgs boson is a scalar $(J=0)$.
  - The $(+-)$ and $(-+)$ helicities vanish using angular-momentum conservation, for the component along the gluon flight direction:

h $(J_z = 0\ (J=0))$





$(++)$ and $(--)$ are allowed, but related by parity (which exchanges all $+ \leftrightarrow -$)

- Let's compute

$A(h \to g_1^+ g_2^+)$ with $q_1 = k_2$, $q_2 = k_1$

$$\Rightarrow \quad \varepsilon_1^{+\mu} = \frac{\langle 2^+|\gamma^\mu|1^+\rangle}{\sqrt{2}\,\langle 21\rangle} \qquad \varepsilon_2^{+\mu} = \frac{\langle 1^+|\gamma^\mu|2^+\rangle}{\sqrt{2}\,\langle 12\rangle}$$

- Note that $\varepsilon_1^+\cdot k_1 = \varepsilon_1^+\cdot k_2 = \varepsilon_2^+\cdot k_1 = \varepsilon_2^+\cdot k_2 = 0$
- So only ~~the~~ terms with $\varepsilon_1^+\cdot\varepsilon_2^+ = -\frac{1}{2\langle 12\rangle^2}\langle 2^+|\gamma^\mu|1^+\rangle\langle 1^+|\gamma_\mu|2^+\rangle$

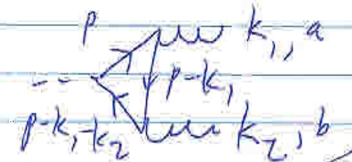
$$= -\frac{2}{2\langle 12\rangle^{\cancel{2}}}[21]\cancel{\langle 12\rangle}$$

$$\Rightarrow \quad \varepsilon_1^+\cdot\varepsilon_2^+ = \frac{[12]}{\langle 12\rangle}$$

will survive (with this reference momentum choice)

- Actually we will only make this explicit substitution toward the end.

Problem 3 (CONT)

① 

[Diagram ② is exactly the same as ①, due to charge-conjugation (arrow reversal)]

Diagram ① + ②

$$\mathcal{A}(h\to gg) = 2(-1)\int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}\Big[\frac{im_f}{v}\,\frac{i(\not p+m_f)}{p^2-m_f^2}\, ig t^a \gamma^{\mu_1}\, \frac{i(\not p-\not k_1+m_f)}{(p-k_1)^2-m_f^2}\, ig t^b \gamma^{\mu_2}$$

closed fermion loop

$$\times \frac{i(\not p-\not k_1-\not k_2+m_f)}{(p-k_1-k_2)^2-m_f^2}\Big]\, \varepsilon_1^{+\mu_1}\varepsilon_2^{+\mu_2}$$

$$= -2i\frac{m_f}{v} g^2 \underbrace{\mathrm{tr}_{fund}[t^a t^b]}_{\frac{1}{2}\delta^{ab}} \int\frac{d^dp}{(2\pi)^d}\, \frac{\mathrm{tr}[(\not p+m_f)\not\varepsilon_1^+(\not p-\not k_1+m_f)\not\varepsilon_2^+(\not p-\not k_1-\not k_2+m_f)]}{(p^2-m_f^2)((p-k_1)^2-m_f^2)((p-k_1-k_2)^2-m_f^2)}$$

$$= -i\frac{m_f}{v} g^2\delta^{ab}\int\frac{d^dp}{(2\pi)^d}\, \frac{1}{(p^2-m_f^2)((p-k_1)^2-m_f^2)((p-k_1-k_2)^2-m_f^2)}$$

$$\cdot\Big\{4m_f^3\,\varepsilon_1^+\cdot\varepsilon_2^+ + 4m_f\big[\varepsilon_1^+\cdot(p-k_1)\,\varepsilon_2^+\cdot(p-k_1-k_2) - \varepsilon_1^+\cdot\varepsilon_2^+\,(p-k_1)\cdot(p-k_1-k_2)$$
$$+\varepsilon_1^+\cdot(p-k_1-k_2)\,\varepsilon_2^+\cdot(p-k_1)$$
$$+\varepsilon_1^+\cdot(p-k_1)\,\varepsilon_2^+\cdot(p-k_1-k_2) - \varepsilon_1^+\cdot(p-k_1-k_2)\,\varepsilon_2^+\cdot p + \varepsilon_1^+\cdot\varepsilon_2^+\, p\cdot(p-k_1-k_2)$$
$$+\varepsilon_1^+\cdot(p-k_1)\,\varepsilon_2^+\cdot(p-k_1) + \varepsilon_1\cdot(p-k_1)\,\varepsilon_2\cdot p - \varepsilon_1^+\cdot\varepsilon_2^+\, p\cdot(p-k_1)\Big\}$$

(some algebra) using $\varepsilon_1\cdot k_1 = \varepsilon_2\cdot k_2 = 0$

$$= -4i\frac{m_f^2}{v} g^2\delta^{ab}\int\frac{d^dp}{(2\pi)^d}\, \frac{1}{(p^2-m_f^2)((p-k_1)^2-m_f^2)((p-k_1-k_2)^2-m_f^2)}$$

$$\Big\{\varepsilon_1^+\cdot\varepsilon_2^+\Big(m_f^2-(p-k_1)^2-\frac{m_h^2}{2}\Big) + 4\varepsilon_1^+\cdot(p-k_1)\,\varepsilon_2^+\cdot(p-k_1)$$
$$+\varepsilon_1^+\cdot k_2\,\varepsilon_2^+\cdot k_1\Big\}$$

· Now we Feynman parametrize $\Rightarrow$

$$\frac{1}{(p^2-m_f^2)((p-k_1)^2-m_f^2)((p-k_1-k_2)^2-m_f^2)} = \int_0^1 dx_1dx_2dx_3\,\delta\Big(\sum_i x_i-1\Big)\frac{2}{[x_1p^2+x_2(p-k_1)^2+x_3(p-k_1-k_2)^2-m_f^2]^3}$$

Complete the square by letting

$$p = l + x_2 k_1 + x_3(k_1+k_2)$$

or $p-k_1 = l - k_1 + x_2 k_1 + x_3(k_1+k_2)$

$$p-k_1 = l - x_1 k_1 + x_3 k_2$$

$$x_1 p^2 + x_2(p-k_1)^2 + x_3(p-k_1-k_2)^2 - m_f^2$$

$$= l^2 + x_3\underbrace{(k_1+k_2)^2}_{m_h^2} - \underbrace{(x_2 k_1 + x_3(k_1+k_2))^2}_{(x_2+x_3)x_3 m_h^2} - m_f^2$$

$$= l_0^2 - \underbrace{\left[m_f^2 - x_1 x_3 m_h^2\right]}_{\Delta}$$

$$\Delta = m_f^2 - x_1 x_3 m_h^2$$

The numerator becomes:

$$\varepsilon_1^+ \cdot \varepsilon_2^+ \left(m_f^2 - (p-k_1)^2 - \frac{m_h^2}{2}\right) + 4\varepsilon_1^+ \cdot (p-k_1)\, \varepsilon_2^+ \cdot (p-k_1)$$
$$+ \varepsilon_1^+ \cdot k_2\, \varepsilon_2^+ \cdot k_1$$

$$= \varepsilon_1^+ \cdot \varepsilon_2^+ \left(m_f^2 - l^2 - (x_1 k_1 - x_3 k_2)^2 - \frac{m_h^2}{2}\right)$$
$$+ 4\varepsilon_1^+ \cdot l\, \varepsilon_2^+ \cdot l + 4\varepsilon_1^+ \cdot (x_1 k_1 - x_3 k_2)\, \varepsilon_2^+ \cdot (x_1 k_1 - x_3 k_2)$$
$$+ \varepsilon_1^+ \cdot k_2\, \varepsilon_2^+ \cdot k_1 + [\text{terms linear in } l, \text{ vanish under integration}]$$

$$= \varepsilon_1^+ \cdot k_2\, \varepsilon_2^+ \cdot k_1 (1 - 4x_1 x_3)$$
$$+ \varepsilon_1^+ \cdot \varepsilon_2^+ \left[m_f^2 - \frac{m_h^2}{2} + x_1 x_3 m_h^2 + \left(\frac{4}{d} - 1\right) l^2\right]$$

from $4\varepsilon_1^+ \cdot l\, \varepsilon_2^+ \cdot l = \frac{4}{d} \varepsilon_1^+ \cdot \varepsilon_2^+ l^2$ after integration

We cannot set $d = 4-\epsilon \to 4$ yet, because of a cancelling $\frac{1}{\epsilon}$ pole in the l^2 integral.

Now we use

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^3} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)} \left(\frac{1}{\Delta}\right)^{3-\frac{d}{2}}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^3} = \frac{+i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-d/2}$$

$$= \underbrace{-\frac{d}{2} \cdot \Delta \cdot \frac{\Gamma(2-d/2)}{\Gamma(3-d/2)} \frac{\Gamma(3)}{\Gamma(2)}}_{-\frac{d}{2(2-d/2)}\Delta \,=\, -\frac{d}{4-d}\Delta} \cdot \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^3}$$

So the $\left(\frac{4}{d}-1\right) l^2$ term can be replaced by

$$-\frac{d}{4-d} \Delta \left(\frac{4-d}{d}\right) = -\Delta = -m_f^2 + x_1 x_3 m_h^2$$

$\Rightarrow$ The numerator becomes

$$\varepsilon_1^+ \cdot k_2\, \varepsilon_2^+ \cdot k_1 (1-4x_1x_3) + \varepsilon_1^+ \cdot \varepsilon_2^+ \left(\cancel{m_f^2} - \frac{m_h^2}{2} + x_1 x_3 m_h^2 - \cancel{m_f^2} + x_1 x_3 m_h^2\right)$$

$$= \underbrace{\left(\varepsilon_1^+ \cdot k_2\, \varepsilon_2^+ \cdot k_1 - \varepsilon_1^+ \cdot \varepsilon_2^+ \, k_1 \cdot k_2\right)}_{\text{Now manifestly transverse:}} (1-4x_1x_3)$$

Now manifestly transverse:
$\varepsilon_1 \to k_1 \Rightarrow k_1 \cdot k_2\, \varepsilon_2 \cdot k_1 - \varepsilon_2 \cdot k_1\, k_1 \cdot k_2 = 0$

And we can now set $d=4$ and assemble factors.

$$A(h\to gg) = -4i\frac{m_f^2}{v} g^2 \delta^{ab} \frac{-i}{(4\pi)^2} \frac{\Gamma(1)}{\Gamma(3)} \cdot 2 \left(\varepsilon_1^+ \cdot k_2\, \varepsilon_2^+ \cdot k_1 - \varepsilon_1^+ \cdot \varepsilon_2^+ \, k_1 \cdot k_2\right) \int_0^1 dx_1 \int_0^{1-x_1} dx_3 \frac{1-4x_1x_3}{m_f^2 - x_1 x_3 m_h^2}$$

Now we let $\varepsilon_i^+ \cdot k_j = 0$, $\varepsilon_1^+ \cdot \varepsilon_2^+ = \frac{[12]}{\langle 12 \rangle}$ $k_1 \cdot k_2 = \frac{m_h^2}{2}$

$$\Rightarrow \boxed{A(h \to g^+ g^+) = 2 \frac{m_h^2}{v} \frac{g^2}{(4\pi)^2} \delta^{ab} \underbrace{\frac{[12]}{\langle 12 \rangle}}_{\text{pure phase}} \cdot I(m_f/m_h)}$$

where

$$\boxed{I(m_f/m_h) \equiv m_f^2 \int_0^1 dx_1 \int_0^{1-x_1} dx_3 \frac{1 - 4x_1 x_3}{m_f^2 - x_1 x_3 m_h^2}}$$

· For very large m_f, neglect this denominator term

$$I(m_f/m_h) \to \int_0^1 dx_1 \int_0^{1-x_1} dx_3 (1 - 4x_1 x_3)$$

$$= \int_0^1 dx_1 \left[1 - x_1 - 2x_1(1-x_1)^2\right]$$

$$= \int_0^1 dx_1 \left[1 - 3x_1 + 4x_1^2 - 2x_1^3\right]$$

$$= 1 - \frac{3}{2} + \frac{4}{3} - \frac{2}{4}$$

$$\boxed{I(m_f/m_h) \simeq \frac{1}{3} \quad \text{for } m_f \gg m_h}$$

· For very small m_f, there is an infrared divergence leading to a large logarithm, $\ln(m_h^2/m_f^2)$

· First write $\frac{1 - 4x_1 x_3}{m_f^2 - x_1 x_3 m_h^2} = \frac{1 - 4\left(-\frac{1}{m_h^2}\right)\left[m_f^2 - x_1 x_3 m_h^2\right] - \frac{4m_f^2}{m_h^2}}{m_f^2 - x_1 x_3 m_h^2}$

$$= \frac{1 - \frac{4m_f^2}{m_h^2}}{m_f^2 - x_1 x_3 m_h^2} + \frac{4}{m_h^2}$$

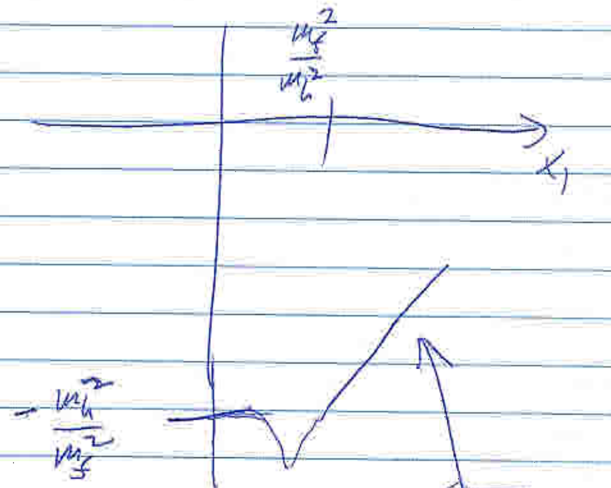
And $\int_0^{1-x_1} dx_3 \left[\frac{1 - \frac{4m_f^2}{m_h^2}}{m_f^2 - x_1 x_3 m_h^2} + \frac{4}{m_h^2}\right]$

$$= \frac{4}{m_h^2}(1 - x_1) - \frac{\left(1 - 4m_f^2/m_h^2\right)}{m_h^2} \frac{1}{x_1} \ln\left(1 - \frac{m_h^2}{m_f^2} x_1 (1 - x_1)\right)$$

$$\Rightarrow I(m_f/m_h) = \frac{m_f^2}{m_h^2}\left\{ 2 - \left(1 - \frac{4m_f^2}{m_h^2}\right)\int_0^1 \frac{dx_1}{x_1} \ln\left(1 - \frac{m_h^2}{m_f^2}x_1(1-x_1)\right)\right\}$$

For $m_f \ll m_h$, the integrand here

$\frac{1}{x_1}\ln\left(1 - \frac{m_h^2}{m_f^2}x_1(1-x_1)\right)$ looks like this:



$x_1 \to 0$

$$\simeq \frac{-\frac{m_h^2}{m_f^2}x_1}{x_1}$$

$$\Rightarrow \int_0^1 \frac{dx_1}{x_1}\ln\left(1 - \frac{m_h^2}{m_f^2}x_1(1-x_1)\right)$$

(finite)

$$\simeq \int_{\frac{m_f^2}{m_h^2}} \frac{dx_1}{x_1}\left[\ln\left(\frac{m_h^2}{m_f^2}\right) + \ln x \pm i\pi\right] \qquad \simeq \ln\left(\frac{m_h^2}{m_f^2}x_1\right) \pm i\pi$$

$$\simeq -\ln\left(\frac{m_h^2}{m_f^2}\right)\ln\left(\frac{m_f^2}{m_h^2}\right) - \frac{1}{2}\ln^2\left(\frac{m_f^2}{m_h^2}\right) \mp i\pi\ln\left(\frac{m_f^2}{m_h^2}\right)$$

$$= \frac{1}{2}\ln^2\left(\frac{m_h^2}{m_f^2}\right) \pm i\pi\ln\left(\frac{m_h^2}{m_f^2}\right)$$

$$\Rightarrow \boxed{I(m_f/m_h) \simeq \frac{m_f^2}{m_h^2}\left\{2 - \frac{1}{2}\ln^2\left(\frac{m_h^2}{m_f^2}\right) \pm i\pi\ln\left(\frac{m_h^2}{m_f^2}\right)\right\}}$$

$-\frac{\pi^2}{2}$ ← (in limit of exact answer)

for $m_f \ll m_h$

parity $\leftrightarrow (h \to g^- g^-)$

The decay width:

$$\Gamma(h \to gg) = \left| A(h \to g^+ g^+) \right|^2 \, 2 \cdot \frac{1}{2m_h} \frac{1}{8\pi} \cdot \frac{1}{2}$$

massless 2-body final state phase space

Bose statistics for identical gluons

$$= \frac{1}{8\pi} \cdot 8 \cdot \frac{4 m_h^4}{2m_h v^2} \left(\frac{\alpha_s}{4\pi}\right)^2 \left| \sum_f I(m_f/m_h) \right|^2$$

$\delta^{aa} = N_c^2 - 1$ # of gluon colors

$$\boxed{\Gamma(h \to gg) = \frac{m_h^3}{8\pi v^2} \left(\frac{\alpha_s}{\pi}\right)^2 \left| \sum_f I(m_f/m_h) \right|^2}$$

Let's now compare $\dfrac{A(h \to gg)\big|_{b \text{ quark}}}{A(h \to gg)\big|_{t \text{ quark}}}$

$$= \frac{I(m_b/m_h)}{I(m_t/m_h)}$$

$$\simeq \frac{m_b^2}{m_h^2} \left[\cancel{0} - \frac{1}{2} \ln^2 \left(\frac{m_h^2}{m_f^2}\right) \pm i\pi \overbrace{\ln\left(\frac{m_h^2}{m_f^2}\right)}^{6.3} + 2 - \frac{\pi^2}{2} \right] \Big/ \tfrac{1}{3}$$

$$\simeq \frac{3 \, 5^2}{120^2} \left[-\frac{1}{2} (6.3)^2 + 2 - \frac{\pi^2}{2} \pm i\pi \, 6.3 \right]$$

$$\simeq 0.005 \left[-19.8 + 2 - 4.9 \pm i \, 19.8 \right]$$

$$\frac{A(h \to gg)\big|_{b \text{ quark}}}{A(h \to gg)\big|_{t \text{ quark}}} \simeq -0.12 \pm i \, 0.10$$

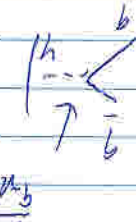
As a contribution to the width, ~~i.e.~~ in $\left| \sum_f I(m_f/m_h) \right|^2$, the b quark contributes, in interference with the t quark, about $2(-0.12) \simeq -20\%$

Numerical value of $\Gamma(h \to gg)$: Let's take $\alpha_s = \alpha_s(M_Z) \simeq 0.12$

$$\Gamma(h \to gg) = \frac{120^3}{8\pi (246)^2}\,\mathrm{GeV}\left(\frac{0.12}{\pi}\right)^2 \left| \underbrace{\frac{1}{3}}_{t}\big(1 \underbrace{- 0.12 \pm i\,0.10}_{b}\big)\right|^2$$

$$\boxed{\Gamma(h \to gg) = 0.14\ \mathrm{MeV}}$$

Next we need $\Gamma(h \to b\bar{b}) = \sum_{\text{colors}} |A(h \to b\bar{b})|^2 \frac{1}{2m_h}\frac{1}{8\pi}$

$$|A(h \to b\bar{b})|^2 = \Bigg|$$  $$\Bigg|^2 = \left(\frac{m_b}{v}\right)^2 \mathrm{tr}\big[\underbrace{k_b\, k_{\bar{b}}}_{4k_b \cdot k_{\bar{b}} = 2m_h^2}\big]$$

(for $m_b \to 0$ except in Yukawa coupling)

$$\Rightarrow \boxed{\Gamma(h \to b\bar{b}) = \frac{3m_h}{8\pi}\frac{m_b^2}{v^2} = 3\,\frac{120}{8\pi}\left(\frac{5}{246}\right)^2 \mathrm{GeV} = 5.91\ \mathrm{MeV}}$$

The branching ratio to gluons, assuming that $Br(h \to b\bar{b}) \approx 1$ (i.e. $\Gamma_{tot} = \Gamma(h \to b\bar{b})$) is

$$\boxed{Br(h \to gg) = \frac{\Gamma(h \to gg)}{\Gamma(h \to b\bar{b})} = \frac{0.14\ \mathrm{MeV}}{5.91\ \mathrm{MeV}} \simeq 0.024}$$

For $m_t \gg m_h$, the hgg amplitude is

$$A(h \to g^+ g^+) = \frac{2}{3}\frac{m_h^2}{v}\frac{g^2}{(4\pi)^2}\delta^{ab}\frac{[12]}{\langle 12\rangle}$$

$$= \Big(\underbrace{\varepsilon_1^+ \cdot \varepsilon_2^+}_{\frac{[12]}{\langle 12\rangle}}\, \underbrace{k_1 \cdot k_2}_{\frac{m_h^2}{2}} - \varepsilon_1^+ \cdot k_2\, \varepsilon_2^+ \cdot k_1\Big)\frac{4}{3v}\frac{g^2}{(4\pi)^2}\delta^{ab}$$

The $\frac{1}{v}$ suggests that we are looking for a dimension-5, gauge-invariant operator coupling h to 2 gluons.

The obvious candidate is $\mathcal{O} = h F_{\mu\nu}^a F^{\mu\nu a}$

The hgg matrix element of this operator is indeed of the right form

k_2, b k_1, a

$$(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)$$

$$\varepsilon_1 \cdot \varepsilon_2\, k_1 \cdot k_2 - \varepsilon_1 \cdot k_2\, \varepsilon_2 \cdot k_1$$

$$\to -2(\varepsilon_1 \cdot \varepsilon_2\, k_1 \cdot k_2 - \varepsilon_1 \cdot k_2\, \varepsilon_2 \cdot k_1)$$

(($\mathcal{O}(A^2)$ terms in $F_{\mu\nu}$) $\Rightarrow$ hggg couplings)

$\Rightarrow$ the coefficient is

$$C = \pm\frac{1}{2}\frac{4g^2}{3v(4\pi)^2} = \frac{\alpha_s}{6\pi v}$$

$\Rightarrow$ A Lagrangian term $\frac{\alpha_s}{6\pi v} h F_{\mu\nu}^a F^{\mu\nu a}$