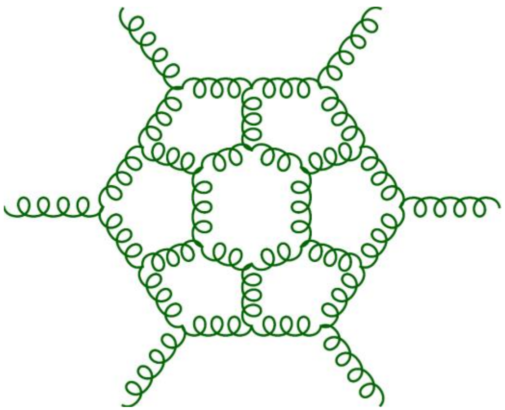
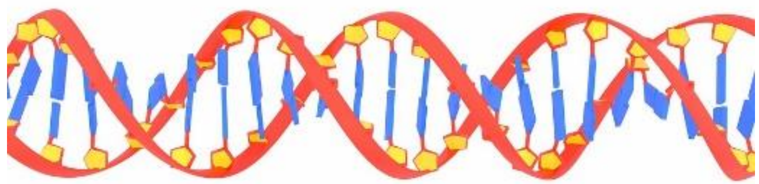
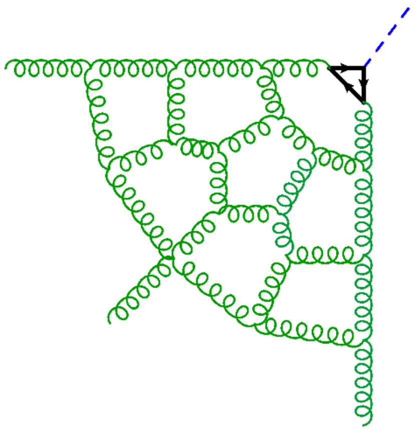
The DNA of Particle Scattering







Lance Dixon (SLAC)

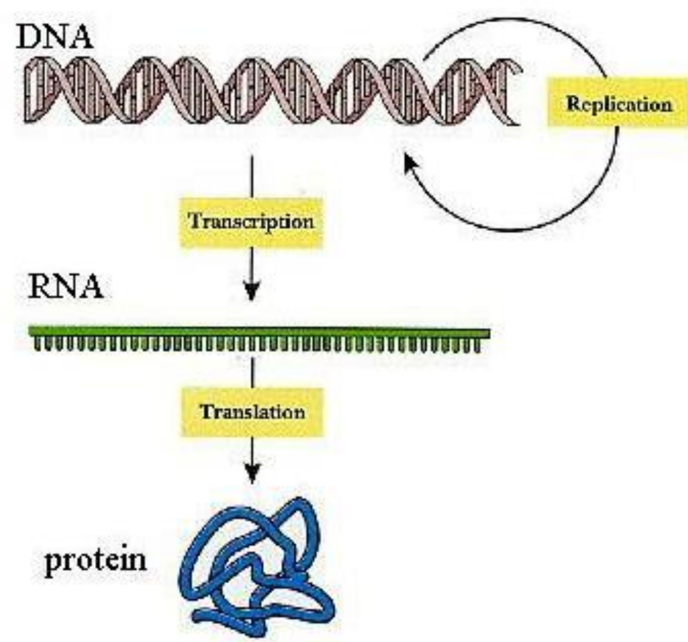
Physics Colloquium, UC Davis
November 7, 2022

SLAC
NATIONAL ACCELERATOR LABORATORY

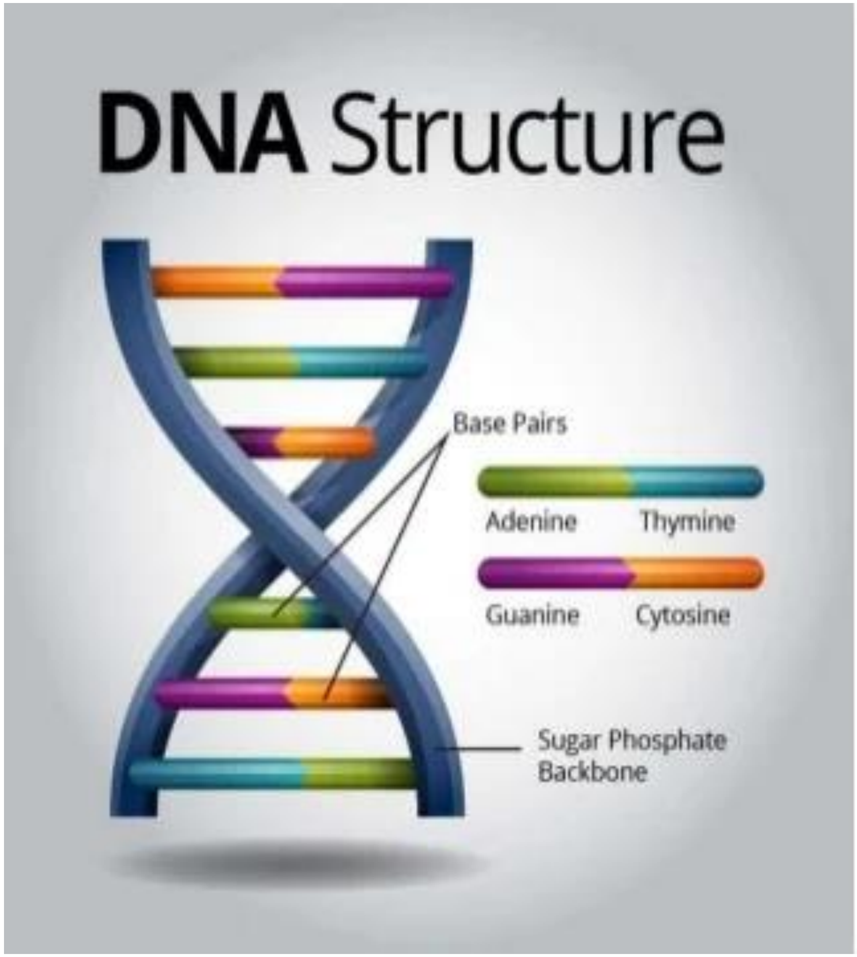
“Don’t know much biology…”

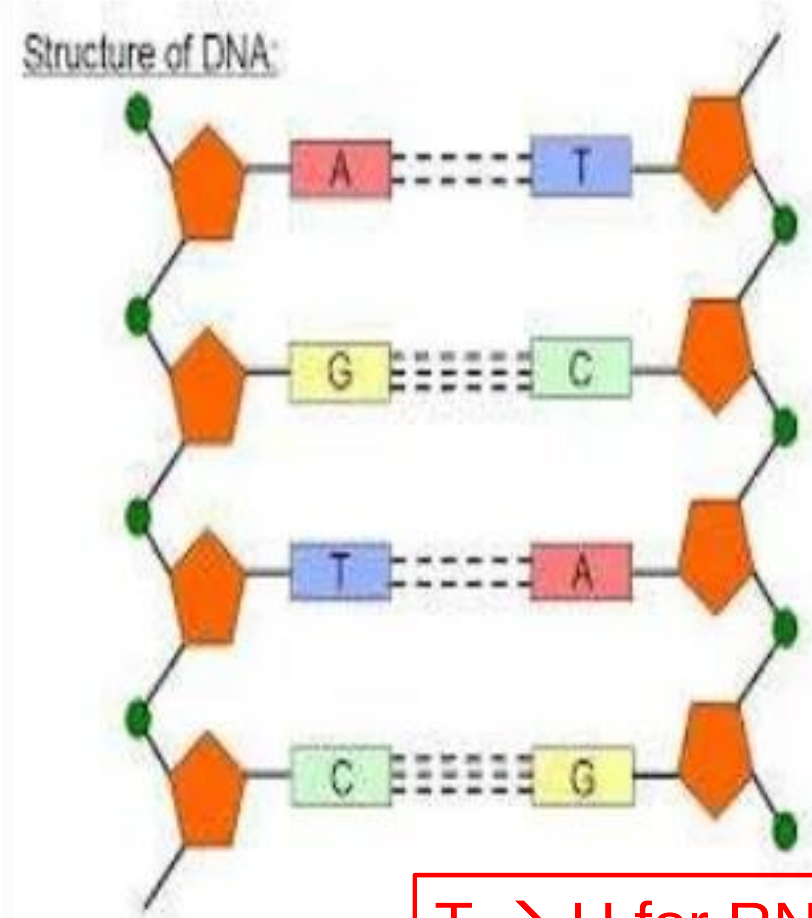
- Particle physics has its Standard Model
- Molecular biology has its Central Dogma

instructions for life ⟶



Code of life carried by nucleotides

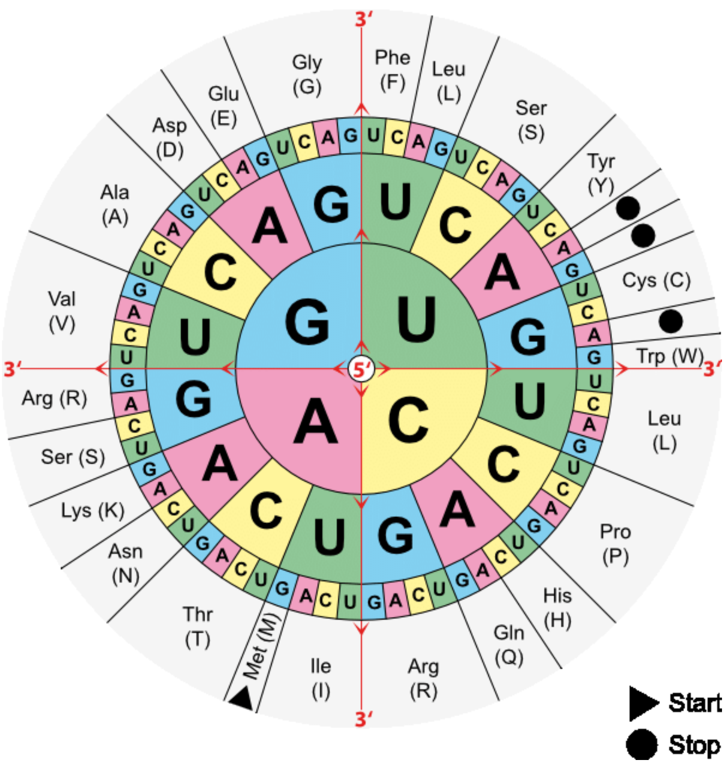




T → U for RNA

Codon redundancy

Amino acids are coded by triplets or codons of base pairs, and the code has redundancies



- There are only 20 amino acids, and **stop** and "start" codons
- Information content is not

$$4^{\text{number of base pairs}}$$

but only

$$20^{(\text{number of base pairs})/3}$$

- In fact, this is still a vast overcount, but the number of foldable proteins that do interesting things is still huge!

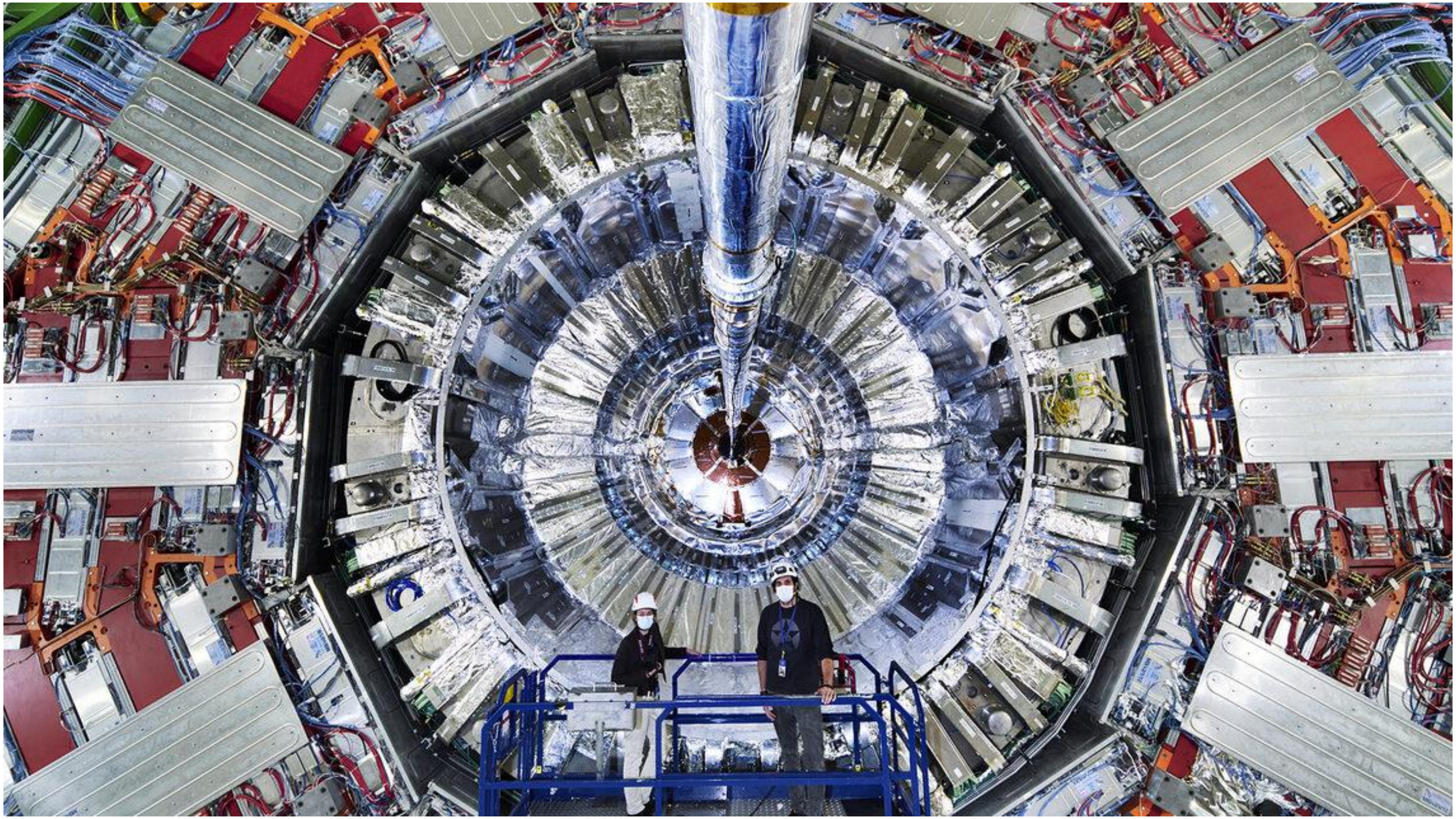
Analogous code for particle scattering

- Instruction set for quantum mechanical scattering amplitudes, building blocks for all reaction probabilities (cross sections).
- The code is not as universal as ATGC, it depends on the process
- We have only seen 16 letter sequences so far
- Still, cracking the code lets us understand scattering at a deeper level than with traditional methods, like Feynman diagrams, and sometimes do more precise computations

Large Hadron Collider

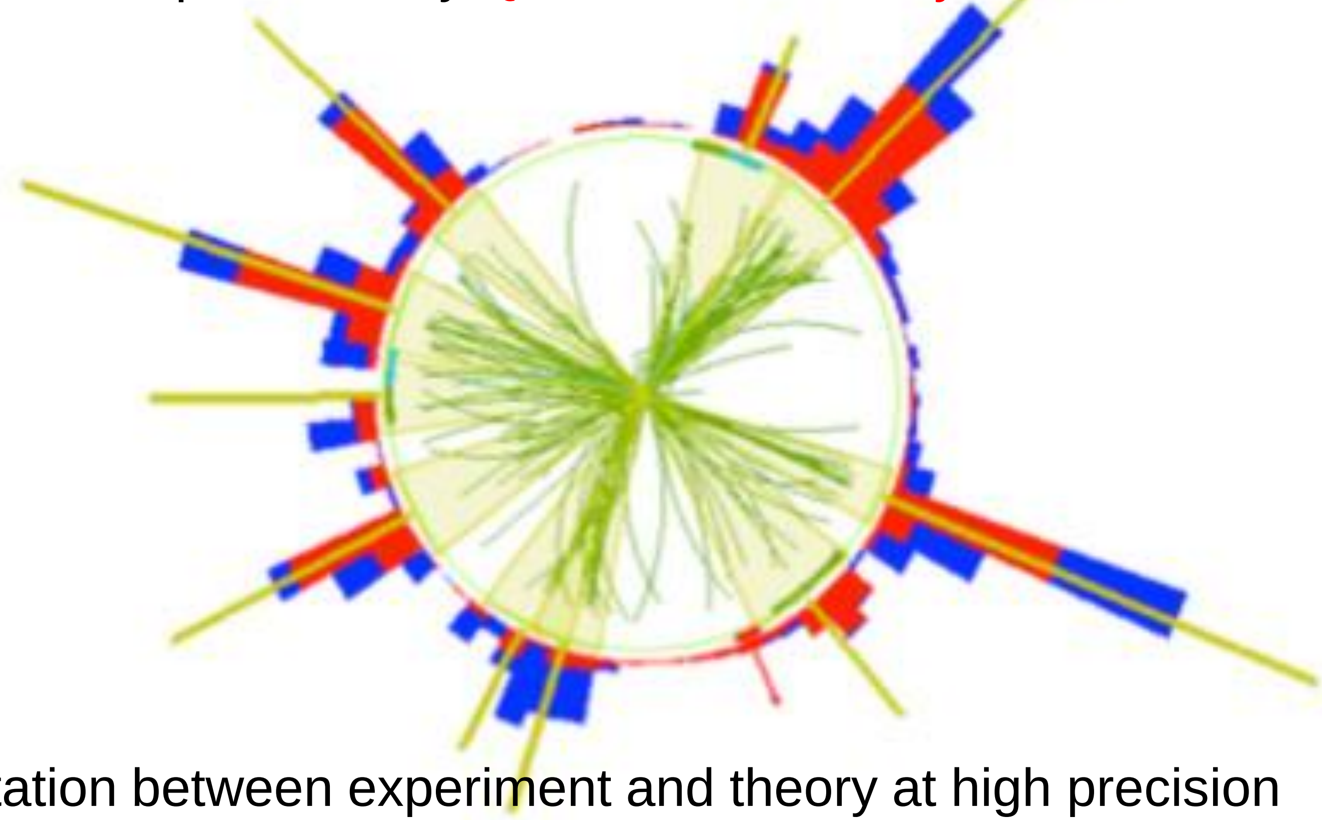
World's most powerful particle collider, 27 km around CERN, Geneva, Switzerland
Collides protons at 13,600 GeV. 2008 → 2038

CMS detector



LHC is QCD Machine

- Copious production of quarks and gluons, materialize as collimated jets of hadrons, predicted by Quantum Chromodynamics



- Confrontation between experiment and theory at high precision requires higher order corrections in the strong coupling α_s

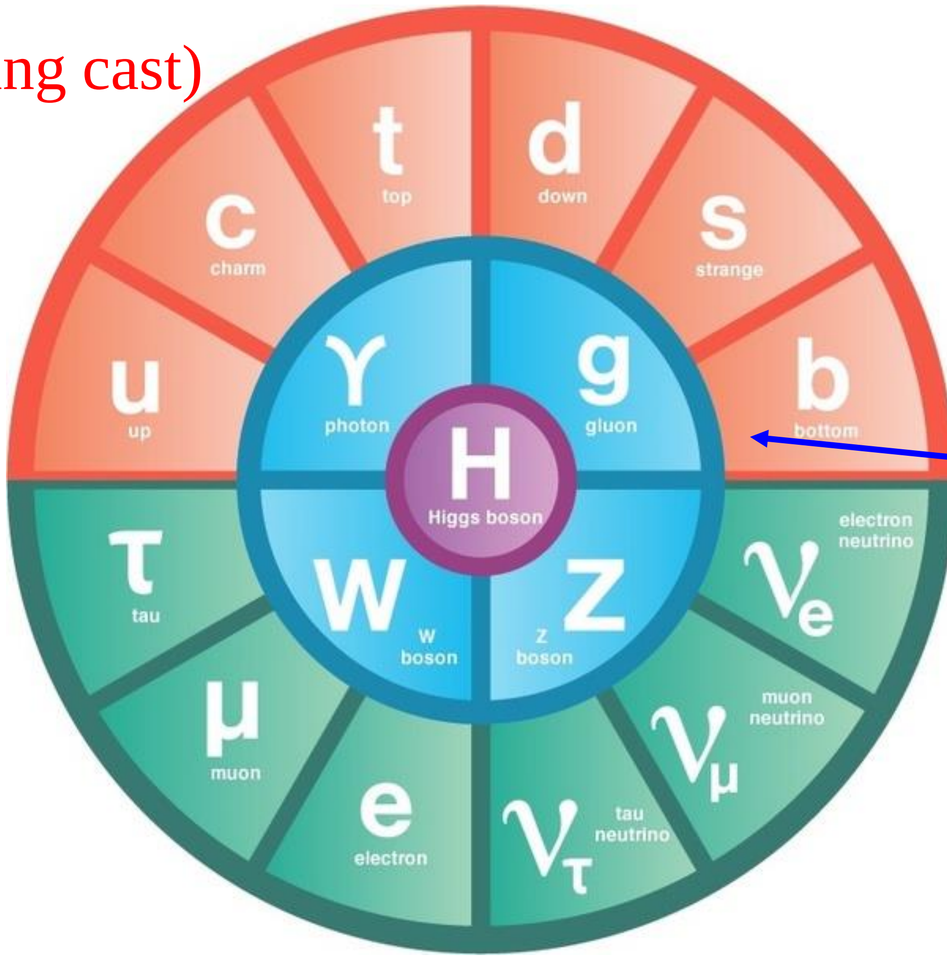
All Standard Model particles are produced at LHC

6 quarks (supporting cast)

$S = \frac{1}{2}$

4 force carriers

$S = 1$



gluon plays starring role in QCD

6 leptons (light ones)

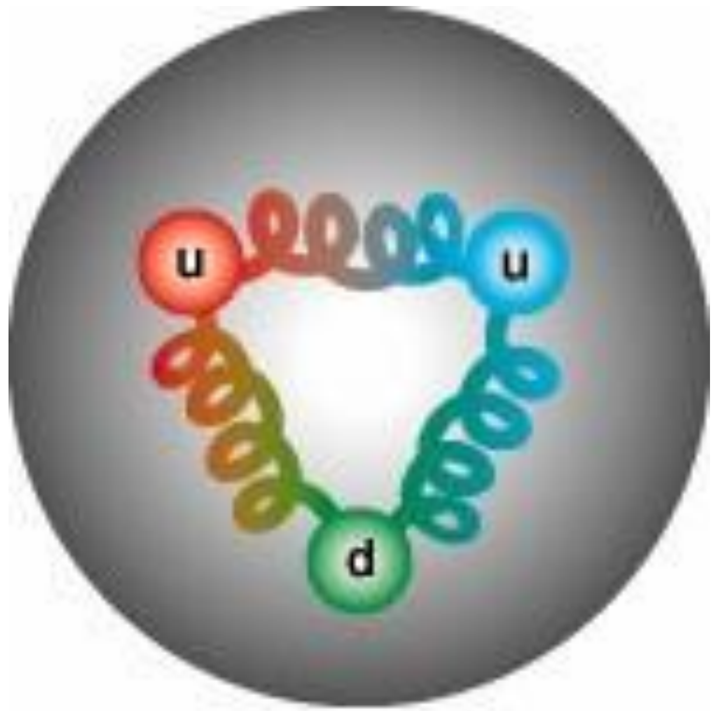
$S = \frac{1}{2}$

1 Higgs boson

$S = 0$

Gluon binds quarks into proton

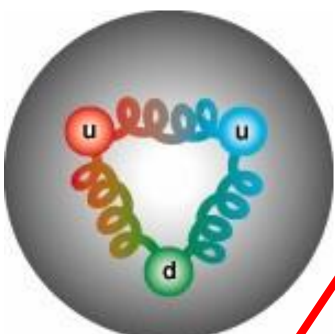
p =



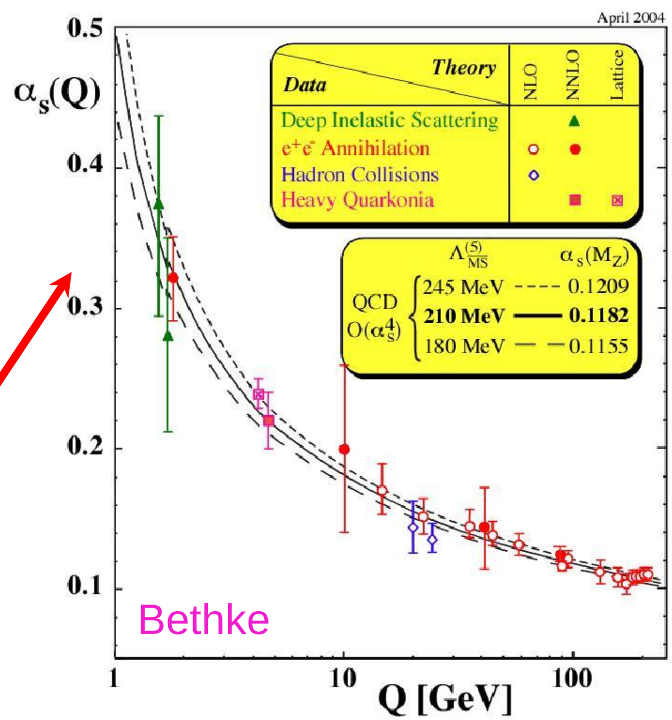
and all other strongly interacting particles (hadrons): $\pi, K, \eta, \varrho, n, \Lambda, \Xi, \ldots$

The Key of Asymptotic Freedom

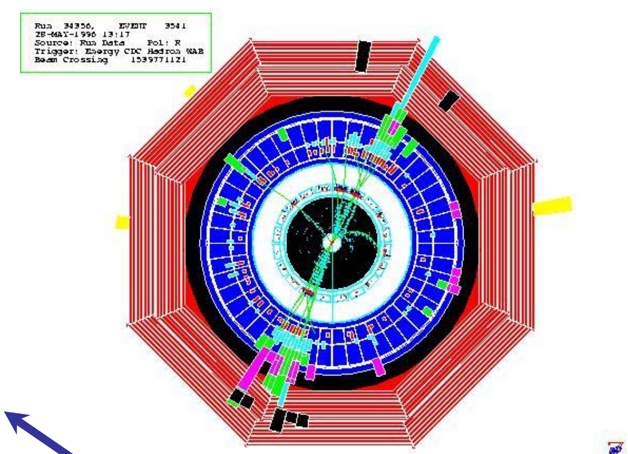
- Gluons anti-screen charge: Interaction strength α_s for QCD is small at short distances, so we can series expand in α_s
  → perturbation theory → Feynman diagrams



confining



Gross, Wilczek, Politzer (1973)
Nobel 2004



calculable

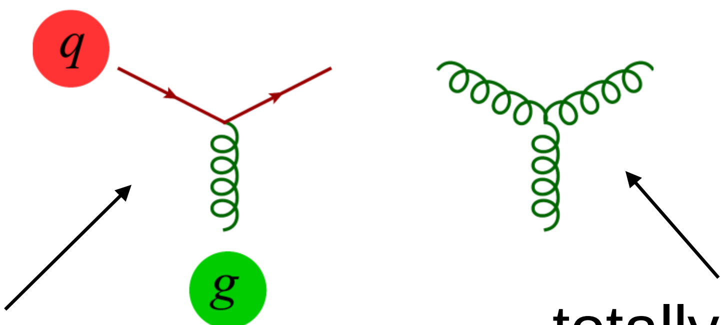
Feynman rules for QCD

- Lines (propagators) for each particle

quarks spin ½, ~ like electrons

gluons massless spin 1, ~ like photons

- Vertices for nonlinear interactions



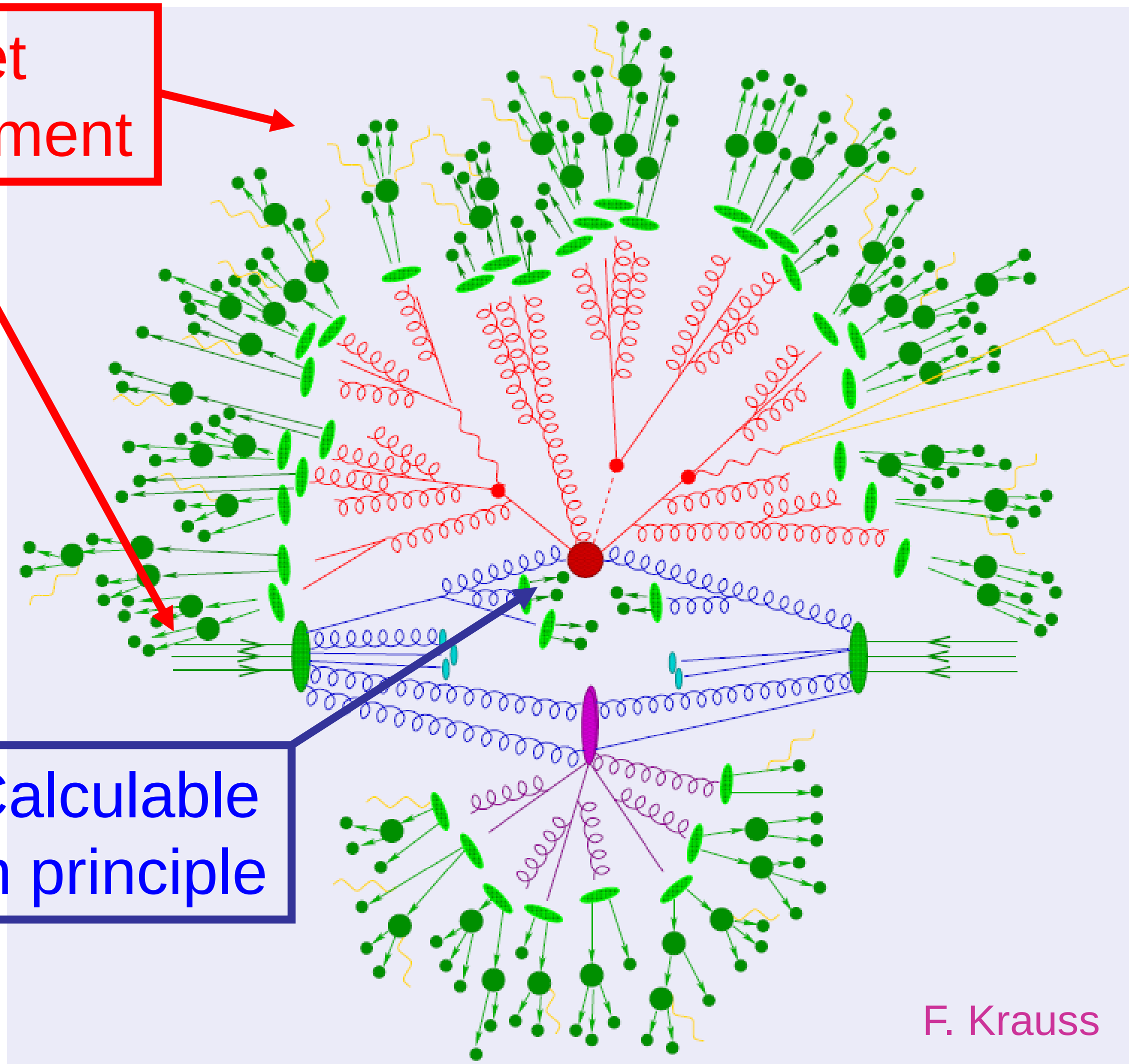
~ QED

totally unlike QED; force carrier charged;
causes anti-screening

Typical LHC Collision

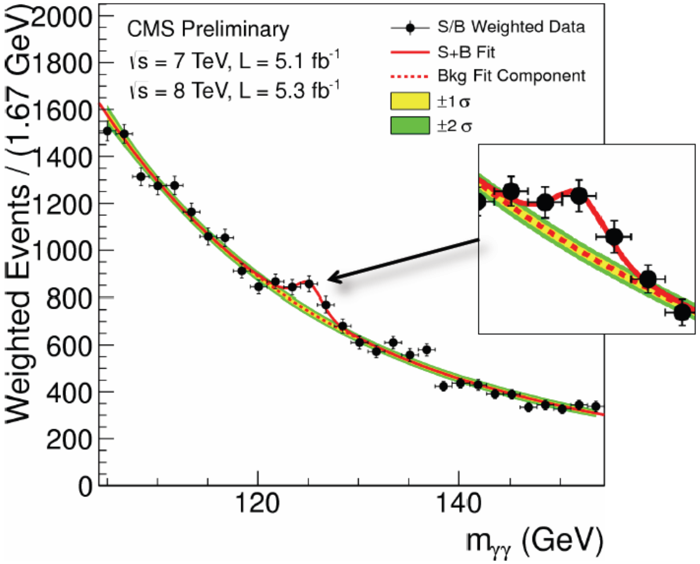
Model or get from experiment

Calculable in principle

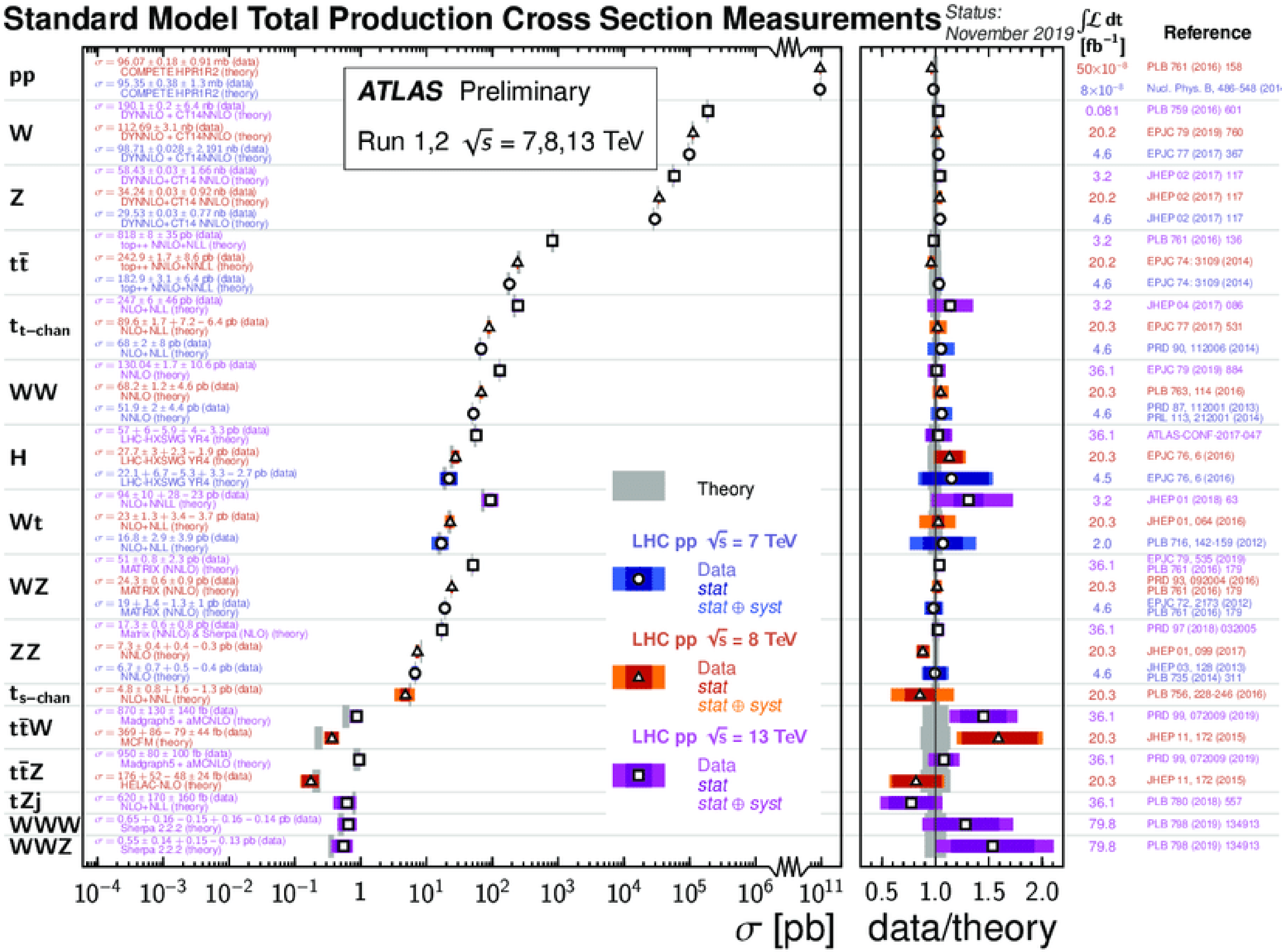


After 50 year hunt, Higgs boson discovered at LHC a decade ago

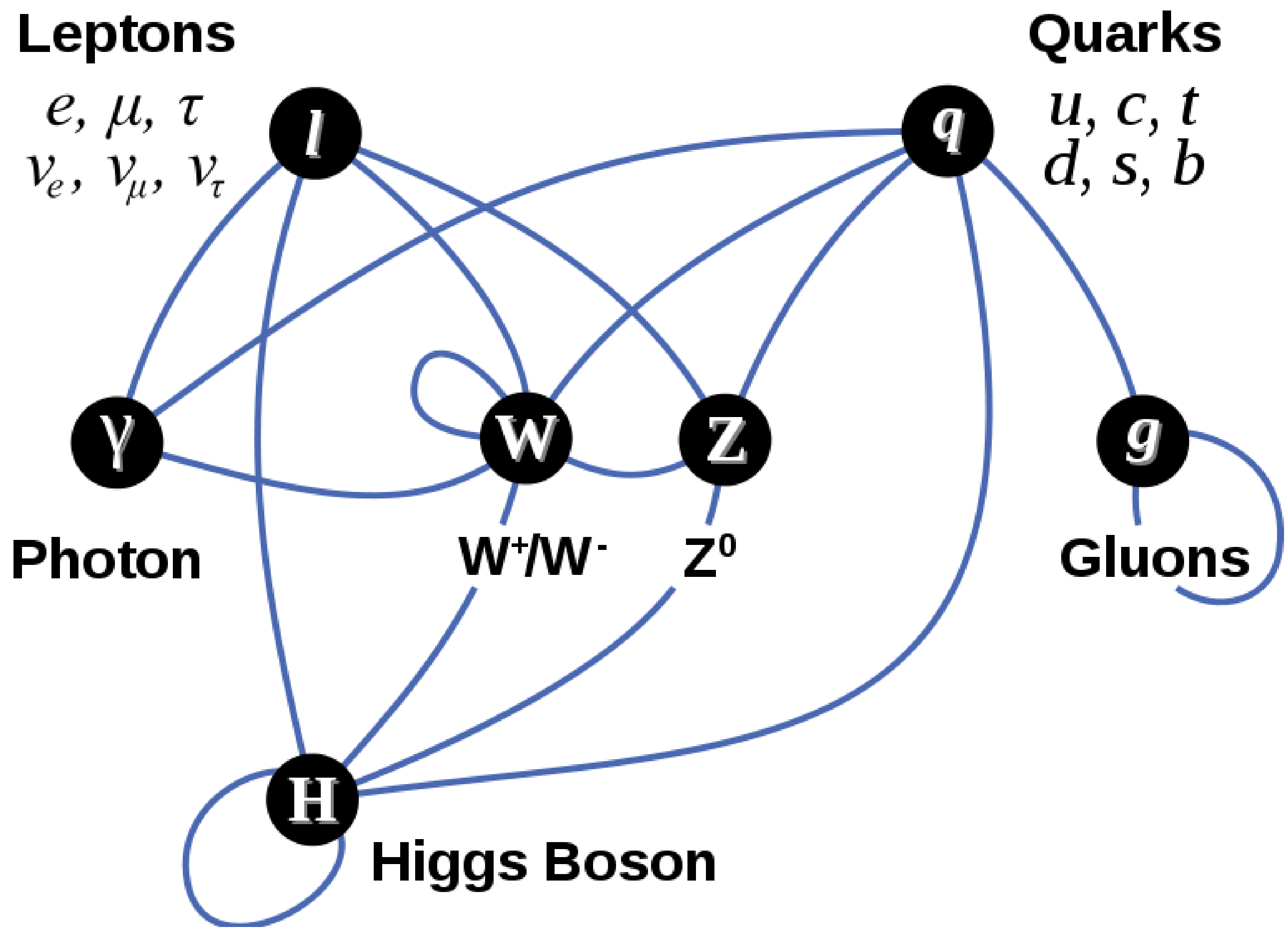
July 4, 2012



- No other elementary particles discovered at LHC yet
- **Precision** phase: measuring Higgs and SM properties to look for a crack in the SM
- High statistics data vs. precise "multi-loop" theory


Standard Model Total Production Cross Section Measurements
Status: November 2019
∫ℒ dt [fb⁻¹]
Reference
ATLAS Preliminary
Run 1,2 √s = 7,8,13 TeV
pp
W
Z
tt̄
t t-chan
WW
H
Wt
WZ
ZZ
t s-chan
tt̄W
tt̄Z
tZj
WWW
WWZ
Theory
LHC pp √s = 7 TeV
Data
stat
stat ⊕ syst
LHC pp √s = 8 TeV
Data
stat
stat ⊕ syst
LHC pp √s = 13 TeV
Data
stat
stat ⊕ syst
σ [pb]
data/theory

Higgs talks to almost everyone

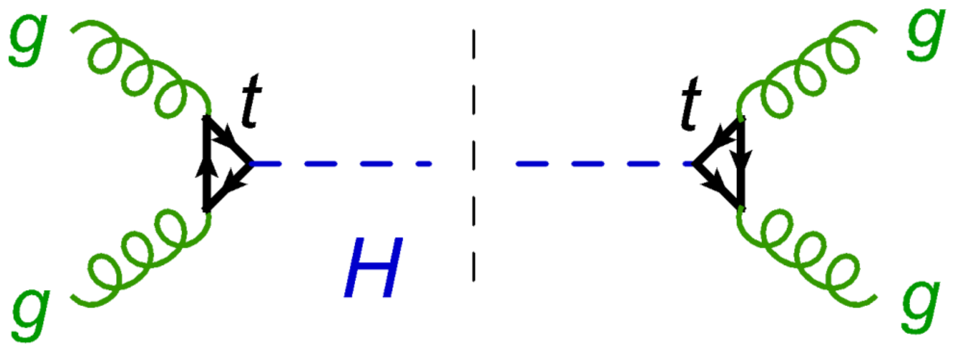


…but not equally, and this doesn't include quantum effects

Producing Higgs bosons at LHC

Higgs boson dominantly produced by **gluon fusion**, a quantum process at “one loop”, mediated by **top quark,** because ***t*** couples strongly to both gluons and Higgs

Leading Order (LO) cross section
$= |\mathrm{one-loop\ amplitude}|^2$



- Since $\boldsymbol{2m_{top}} = 350\ \mathrm{GeV}$
  $\gg \boldsymbol{m_{Higgs}} = 125\ \mathrm{GeV}$,

interaction between gluons and Higgs is approximately local (mediated by an operator $H G_{\mu\nu}^{a} G^{\mu\nu\, a}$)

Perturbative Short-Distance Cross Section

$$\hat{\sigma}(\alpha_s, \mu_F, \mu_R) \;=\; [\alpha_s(\mu_R)]^{n_\alpha} \left[\hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi} \hat{\sigma}^{(1)}(\mu_F, \mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}(\mu_F, \mu_R) + \cdots \right]$$

LO NLO NNLO

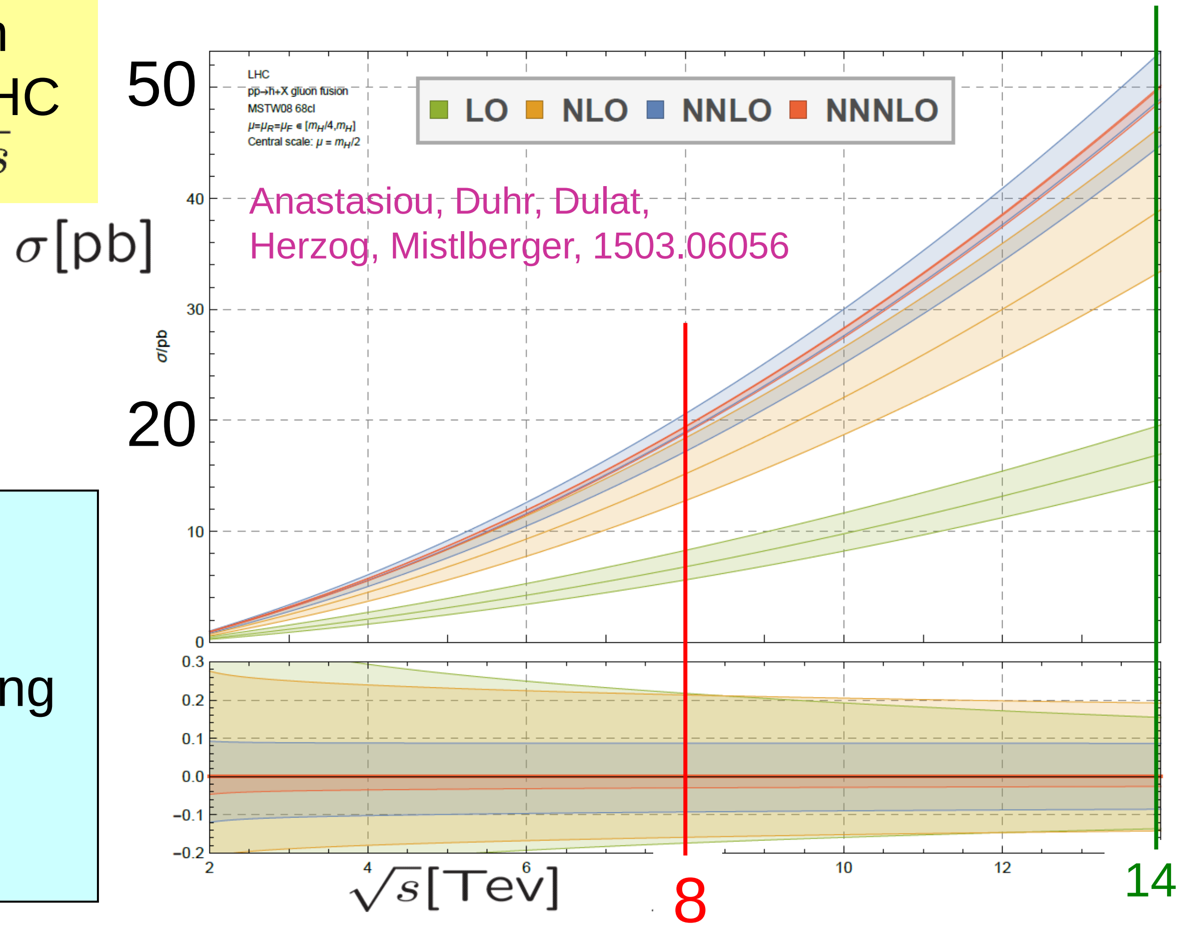
Higgs gluon fusion cross section at LHC vs. CM energy $\sqrt{s}$

LO approx. is terrible!
LO → NNNLO
→ factor of 2 or 3 increase!

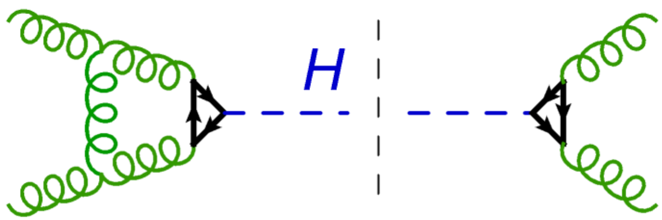
Poor convergence of expansion in $\alpha_s(\mu)$

Uncertainty bands from varying $\mu_R = \mu_F = \mu$

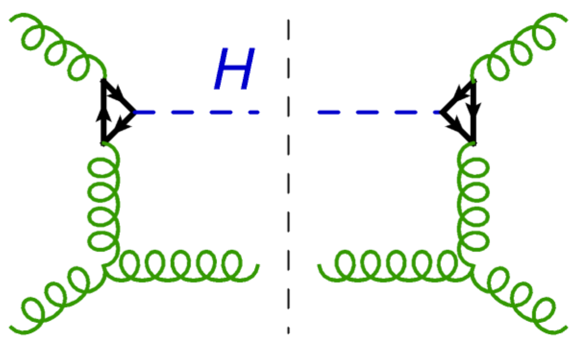
Necessitates high orders!



Some NLO QCD Feynman diagrams

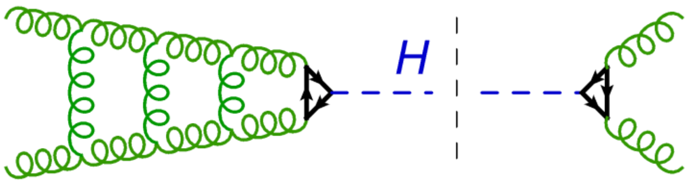


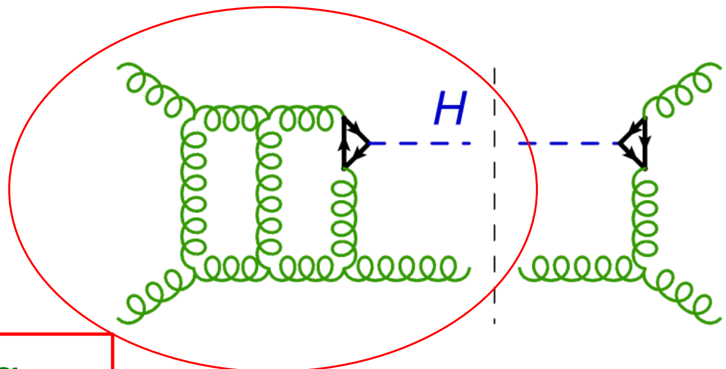
virtual $gg \rightarrow H$

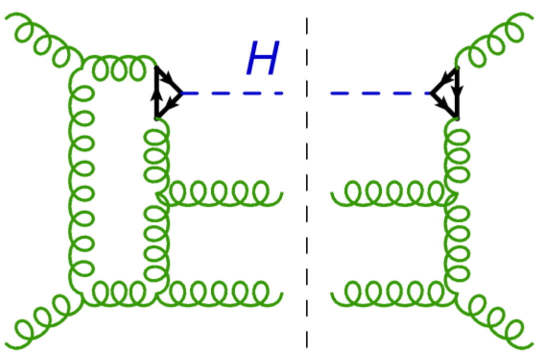


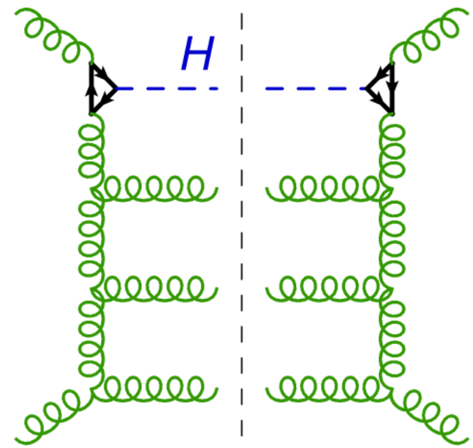
real, $gg \rightarrow Hg$

Very few of the NNNLO QCD diagrams







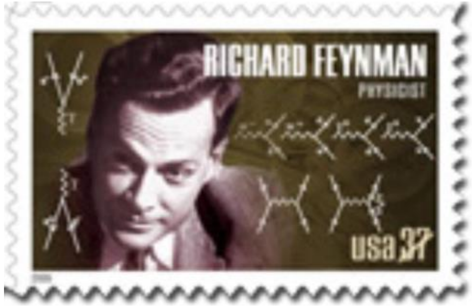


gg → *Hg*
@ 2 loops,
state of art
in QCD

+ …
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions

Scattering amplitudes are underlying building blocks

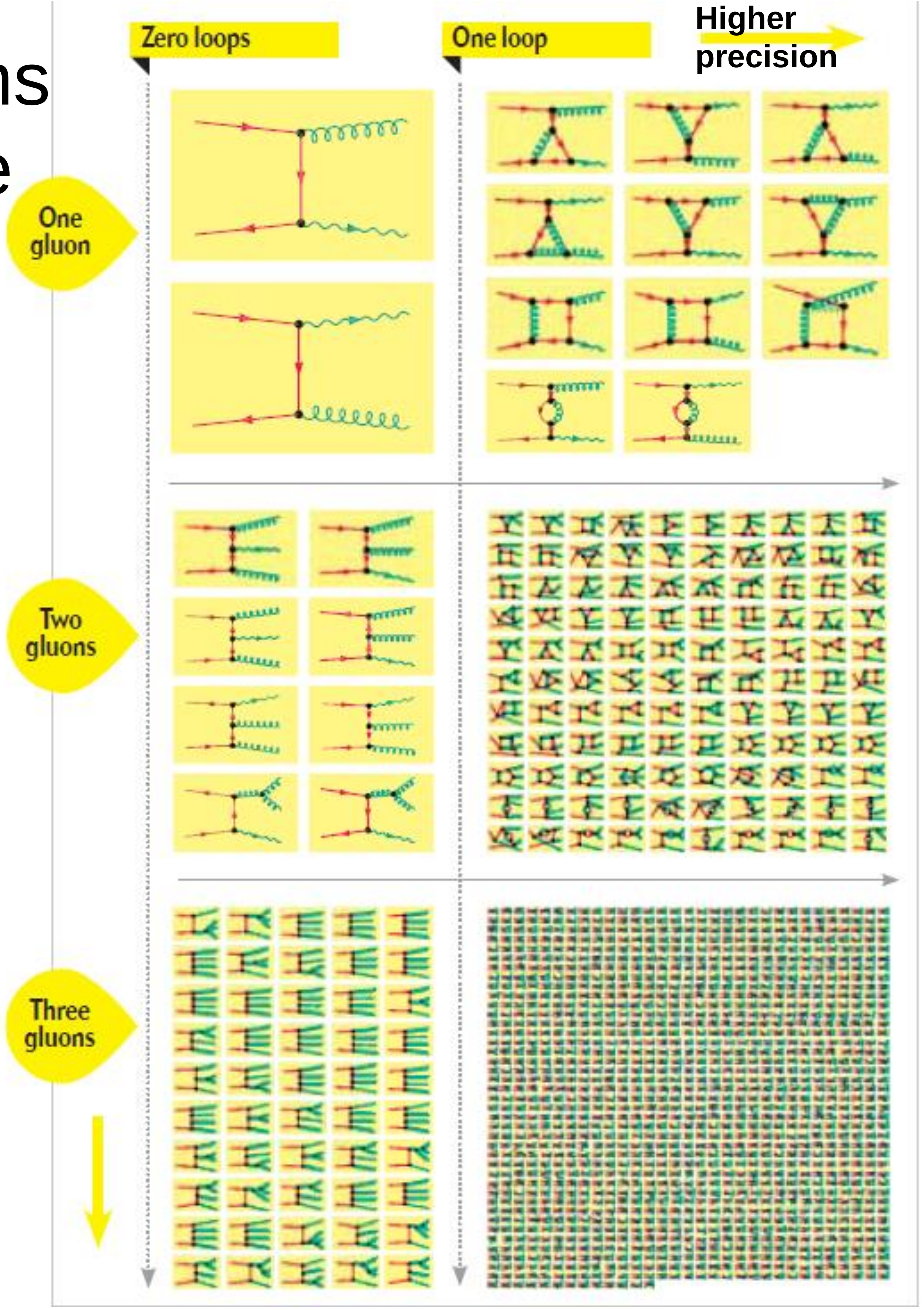
Number of Feynman diagrams grows exponentially with the loop order, and with the complexity of the process



Feynman diagrams for

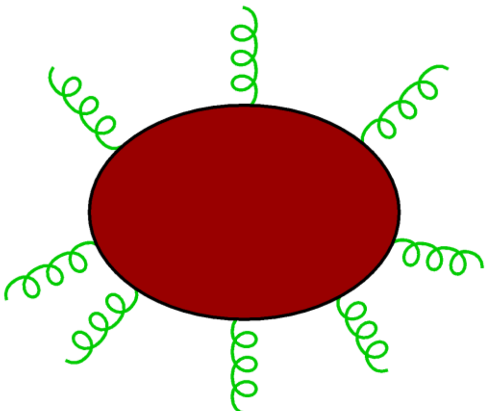
$$q\bar{q} \to W + 1, 2, 3 \text{ gluons}$$

A background to searches for supersymmetry at LHC

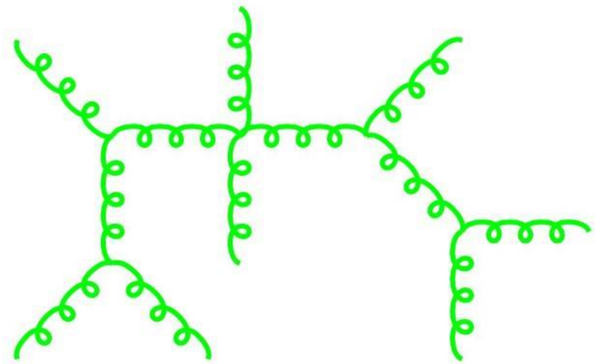


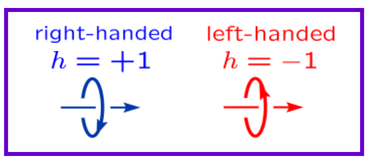
Whole more than sum of its parts

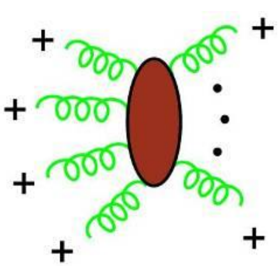
Simplicity often hidden from individual Feynman diagrams



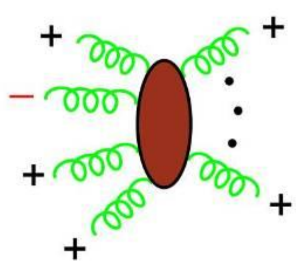
$$= \sum_{\text{trees}}$$



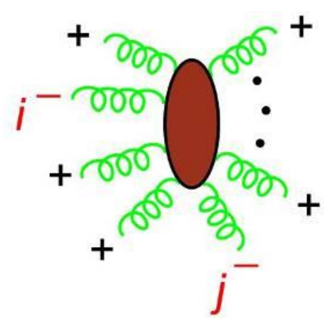




=



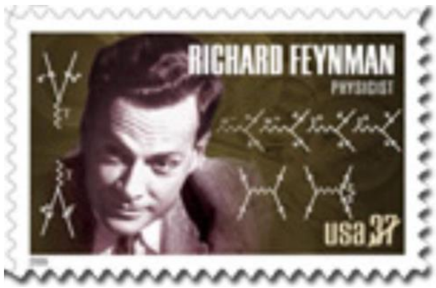
= 0

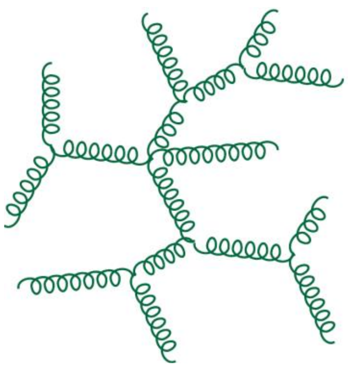


$$= \frac{< ij >^4}{< 12 >< 23 > \cdots < n1 >}$$

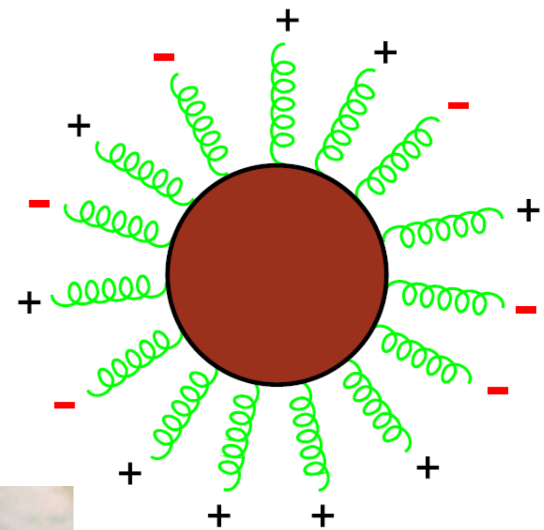
Parke-Taylor (1986)

Granularity vs. Fluidity





+ …

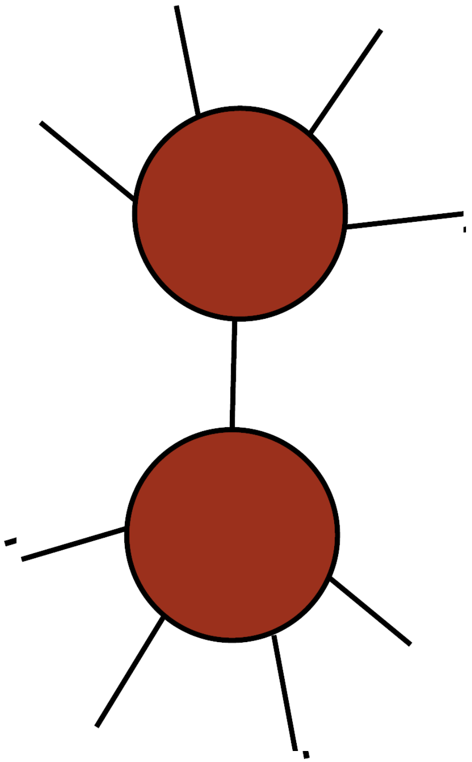


Fluid Tree Amplitudes

Tree amplitude is a rational function of kinematic variables.
Falls apart into simpler tree amplitudes in special limits

Picture leads directly to BCFW (on-shell) recursion relations:
Reconstruct amplitude from poles in complex plane, where it factorizes into simpler amplitudes

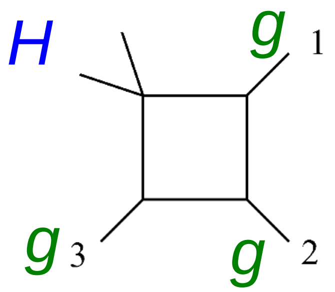
Britto, Cachazo, Feng, Witten, hep-th/0501052



Beyond tree level

- Loop level Feynman diagrams come with an instruction to integrate over all loop momenta

- For example, at one loop the amplitude for $gg \rightarrow Hg$ involves the "scalar box" integral



$$= \int \frac{d^4 p}{p^2 (p-p_1)^2 (p-p_1-p_2)^2 (p-p_1-p_2-p_3)^2}$$

$$= \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

where the dilogarithm is $\mathrm{Li}_2(x) \equiv -\int_0^x \frac{dt}{t} \ln(1-t)$

One loop not too bad

- For any number of external particles, all one-loop integrals (even in dimensional regularization, $D = 4 - 2\epsilon$) can be reduced to scalar box integrals + simpler

Brown-Feynman (1952), Melrose (1965), ‘t Hooft-Veltman (1974), Passarino-Veltman (1979), van Neerven-Vermaseren (1984), Bern, LD, Kosower (1992)

→ combinations of $\mathrm{Li}_2(x) \equiv -\int_0^x \frac{dt}{t} \ln(1-t)$

where x is (many different) functions of the kinematic variables (Mandelstam invariants), plus logarithms

Multi-loop much more complex

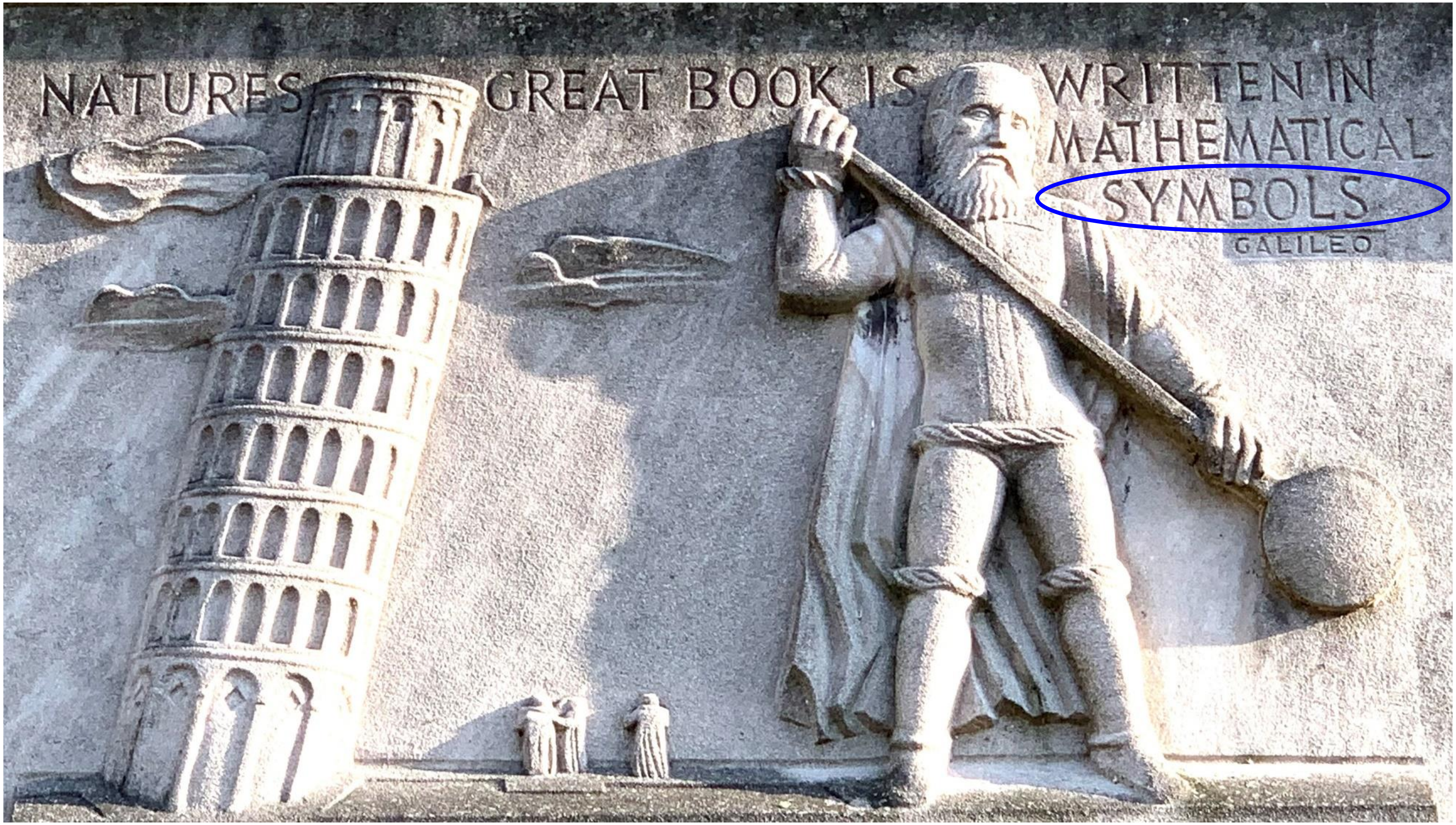
- At L loops, instead of just Li_2's, get special functions with up to $2L$ integrations
- Weight $2L$ "iterated integrals"
- **Best case**: generalized polylogarithms, defined iteratively by

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

and $G(\vec{0}_n, x) = \frac{(\ln x)^n}{n!}$

- Still very intricate multi-variate functions

Complexity tamed by “the symbol”



Entrance to Northwestern Physics Department

Symbol ~ DNA Code

Complexity **encoded** in words written in an alphabet defined by iterative differentiation

- Code is analog of ATGC code for DNA.
- Characterizes answer fairly completely
- We can learn to read it
- We can use that understanding to go to ever higher loop orders (at least in simple theories)

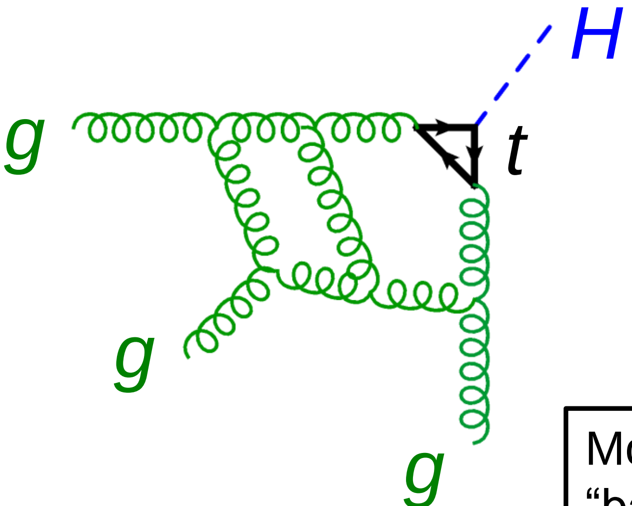
"Goldilocks Process": $gg \rightarrow Hg$

QCD state of art is two loops (not counting top quark loop)

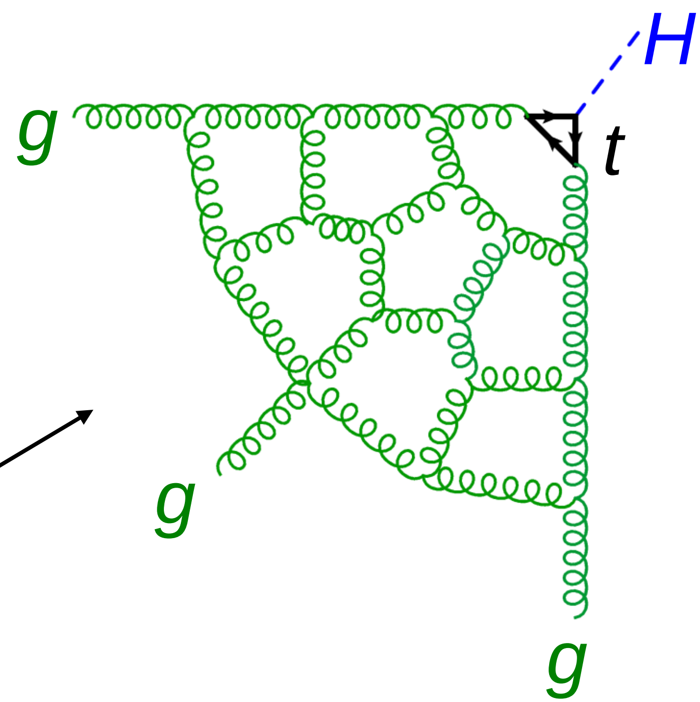
Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554

We can get to **eight** loops – in a simpler cousin of QCD called "planar N=4 SYM"

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm 2012.12286, 2112.06243, 2204.11901



Model system, like "bacteriophage λ" for DNA in 1950s



Defining the symbol

- Suppose $F(x_i)$ is a linear combination of generalized polylogarithms of weight n
- Then the partial derivatives of F with respect to underlying coordinates x_i are given by

$$\frac{\partial F}{\partial x_i} = \sum_{s_k \in \mathcal{L}} F^{s_k} \frac{\partial \ln s_k}{\partial x_i}$$

where F^{s_k} are in same space, but have weight $n-1$, and s_k are letters in the symbol alphabet $\mathcal{L}$

Now iterate

Define F^{s_k,s_j} via derivatives of F^{s_j}:

$$\frac{\partial F^{s_j}}{\partial x_i} \equiv \sum_{s_k \in \mathcal{S}} \; F^{s_k,s_j} \; \frac{\partial \ln s_k}{\partial x_i}$$

- Iterating, n times for weight n function, gives symbol $\mathcal{S}[F]$,

$$\sum_{s_{i_1},\dots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\dots,s_{i_n}} \, d\ln s_{i_1} \dots d\ln s_{i_n}$$

→ $$\mathcal{S}[F] \equiv \; \sum_{s_{i_1},\dots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\dots,s_{i_n}} \; s_{i_1} \otimes \dots \otimes s_{i_n}$$

where now $F^{s_{i_1},\dots,s_{i_n}}$ are just rational numbers

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example: Classical polylogarithms

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

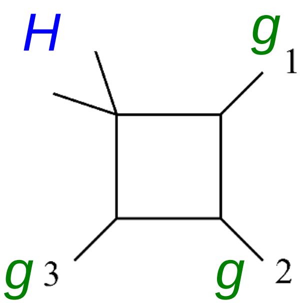
$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

- Regular at $x = 0$, branch cut starts at $x = 1$.
- Iterated differentiation gives the symbol:

$$\mathcal{S}[\mathrm{Li}_n(x)] = \mathcal{S}[\mathrm{Li}_{n-1}(x)] \otimes x$$
$$= \cdots = -(1-x) \otimes x \otimes x \ \ldots \otimes x$$

- Branch cut discontinuities displayed in first entry of symbol, e.g. clip off leading $(1-x)$ to compute discontinuity at $x = 1$.
- Derivatives computed from symbol by clipping last entry, multiplying by that $d\ln(\ldots)$. **Alphabet** $\mathcal{L} = \{x, 1-x\}$

Our one-loop $gg \to Hg$ example



$$F = \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

$$= \mathrm{Li}_2\left(1 - \frac{1}{u}\right) + \mathrm{Li}_2\left(1 - \frac{1}{v}\right) + \frac{1}{2}\ln^2\left(\frac{u}{v}\right) + \cdots$$

where $u = \frac{s_{12}}{s_{123}}$ and $v = \frac{s_{23}}{s_{123}}$ are only 2 dimensionless variables $\left(w = \frac{s_{13}}{s_{123}} = 1 - u - v\right)$

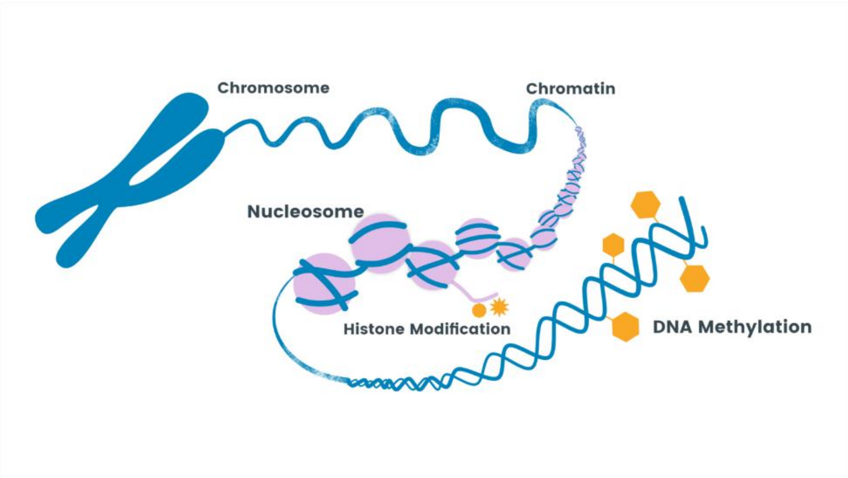
symbol $\mathcal{S}[F] = u \otimes (1-u) + v \otimes (1-v) - u \otimes v - v \otimes u$

Six letter alphabet: $\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$

Symbol as scaffolding for function

- Taking n derivatives drops constants like $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$
- Restore by tracking values at some boundary point(s)

- In (crude) DNA analogy, "beyond-the-symbol" information corresponds to epigenetics: stuff close to DNA that influences its expression e.g. methylation



Better alphabet for *gg* → *Hg*

These 6 letters are equivalent but "diagonalize" things better:

$$\mathcal{L}' = \{\, a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \,\}$$

- Symbols of *gg*→*Hg* amplitude $F_3^{(L)}$ simplify (remarkably) at $L = 1$ and 2 loops, to just 6 and 12 terms:

$$\mathcal{S}\left[F_3^{(1)}\right] = (-1)[\, b\otimes d + c\otimes e + a\otimes f + b\otimes f + c\otimes d + a\otimes e]$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 4[\, b\otimes d\otimes d\otimes d + c\otimes e\otimes e\otimes e + a\otimes f\otimes f\otimes f + b\otimes f\otimes f\otimes f + c\otimes d\otimes d\otimes d + a\otimes e\otimes e\otimes e] + 2\,[b\otimes b\otimes b\otimes d + c\otimes c\otimes c\otimes e + a\otimes a\otimes a\otimes f + b\otimes b\otimes b\otimes f + c\otimes c\otimes c\otimes d + a\otimes a\otimes a\otimes e]$$

(really only 1 and 2 terms, plus images under dihedral symmetry)

- 3 loops: 636 terms, …

$gg \to Hg$ symbol terms per loop

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

Not advisable to look at full symbol directly without eye protection!

However, there is a lot of data to be mined from it!

Ripe potential area for machine learning (Cranmer, Charton, LD, Wilhelm, in progress)

Examples of patterns

- Every term in the symbol starts with a, b, c; never d, e, f
- Analogous to “start codon” in DNA.
- Physical reason related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- 12 pairs of adjacent letters are forbidden:

~ DNA codon redundancy

$$\ldots a \otimes d \ldots, \quad \ldots b \otimes e \ldots, \quad \ldots c \otimes f \ldots$$
$$\ldots d \otimes a \ldots, \quad \ldots e \otimes b \ldots, \quad \ldots f \otimes c \ldots$$
$$\ldots d \otimes e \ldots, \quad \ldots e \otimes f \ldots, \quad \ldots f \otimes d \ldots$$
$$\ldots e \otimes d \ldots, \quad \ldots f \otimes e \ldots, \quad \ldots d \otimes f \ldots$$

- Resemble constraints from causality called the Steinmann relations Steinmann, Helv. Phys. Acta (1960)
- But not quite, which mystified us for a while…

Symbol alphabets for n-gluon amplitudes in planar N=4 SYM

$n = 4{,}5$ trivial in this theory

parity-odd letters, algebraic in $\hat{u}, \hat{v}, \hat{w}$

$n = 6$ has 9 letters: $\mathcal{S}_6 = \{\hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w\}$

(3 kinematic variables)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

$n = 7$ has 42 letters

(6 var's)

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289

$n = 8$ has at least 222 letters, could even be infinite as $L \to \infty$

(9 var's)

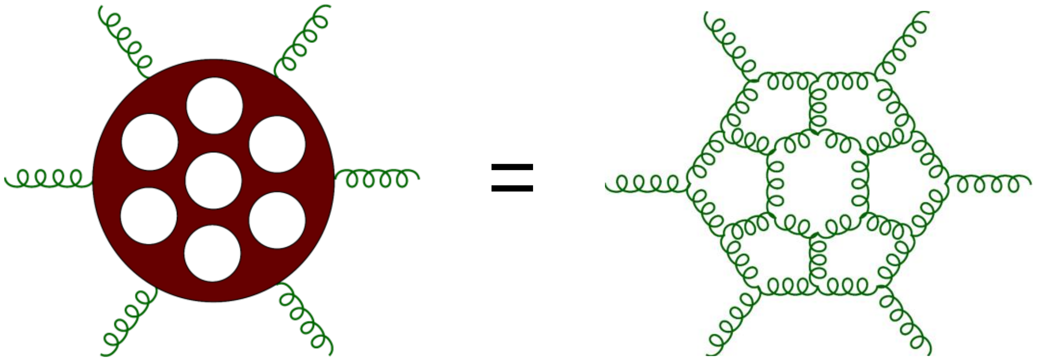
Arkani-Hamed, Lam, Spradlin, 1912.08222;
Drummond, Foster, Gürdoğan, Kalousios, 1912.08217
Henke, Papathanasiou 1912.08254; Z. Li, C. Zhang, 2110.00350

Structure closely tied to cluster algebras Fomin, Zelevinsky (2001)

tropical geometry & polytopes

We know $n = 6$ amplitude to 7 loops

Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890



L	A_6 no. of terms
1	6
2	84
3	5,034
4	243,000
5	15,534,750

There's a "parity-preserving" surface, $\Delta(\hat{u},\hat{v},\hat{w}) = 0$, where

$\hat{y}_u = \hat{y}_v = \hat{y}_w = 1$

→ any word containing them is 0,

→ 9 letters drops to 6 letters

L	$A_6\|_{\Delta=0}$ no. of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568

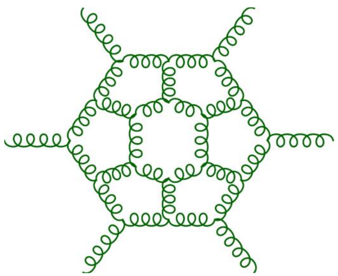
Q: Where have we seen these numbers before? A: In gg→Hg !!!

Strange new "antipodal" duality

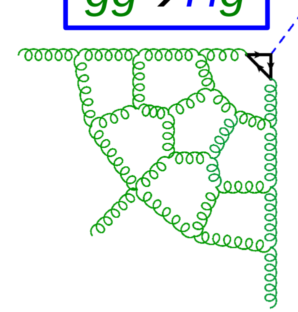
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

Relates two seemingly unrelated processes!

(MHV) $gg \to gggg$



$gg \to Hg$



$$A_6^{(L)}(\hat{u},\hat{v},\hat{w})|_{\Delta=0} = S(F_3^{(L)}(u,v,w))$$

"Antipode map" S
reverses order of all entries in symbol!!!

L	$A_6\vert_{\Delta=0}$ no. of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568

8 now also checked; LD, Liu

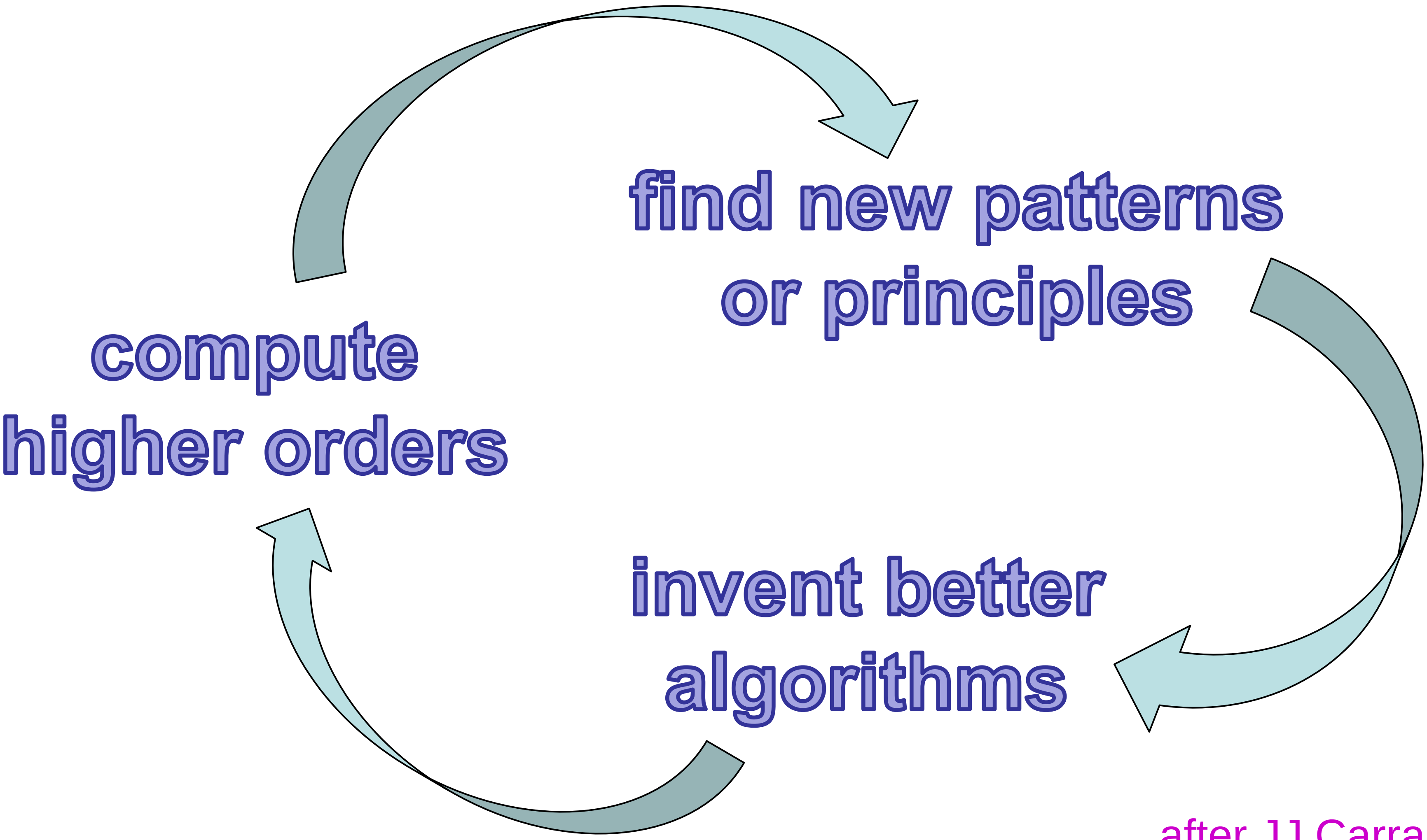
$$\sqrt{\hat{a}} \Leftrightarrow d$$
$$\sqrt{\hat{b}} \Leftrightarrow e$$
$$\sqrt{\hat{c}} \Leftrightarrow f$$
$$\hat{d} \Leftrightarrow a$$
$$\hat{e} \Leftrightarrow b$$
$$\hat{f} \Leftrightarrow c$$

L	F_3 no. of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

Not clear why antipodal duality holds

- We have some clues, but far from a physical understanding
- But it does explain the mystery of the "Steinmann-like" adjacency constraints:
    - They are **actual** Steinmann constraints for the 6 gluon amplitude!
- Also, many other previously empirical "final entry relations" become part of the causal construction of the 6 gluon function space.
- Important to try to uncover the physical reason, besides just "it's in their genes!"
- Looking at other cases now

Amplitudes virtuous circle

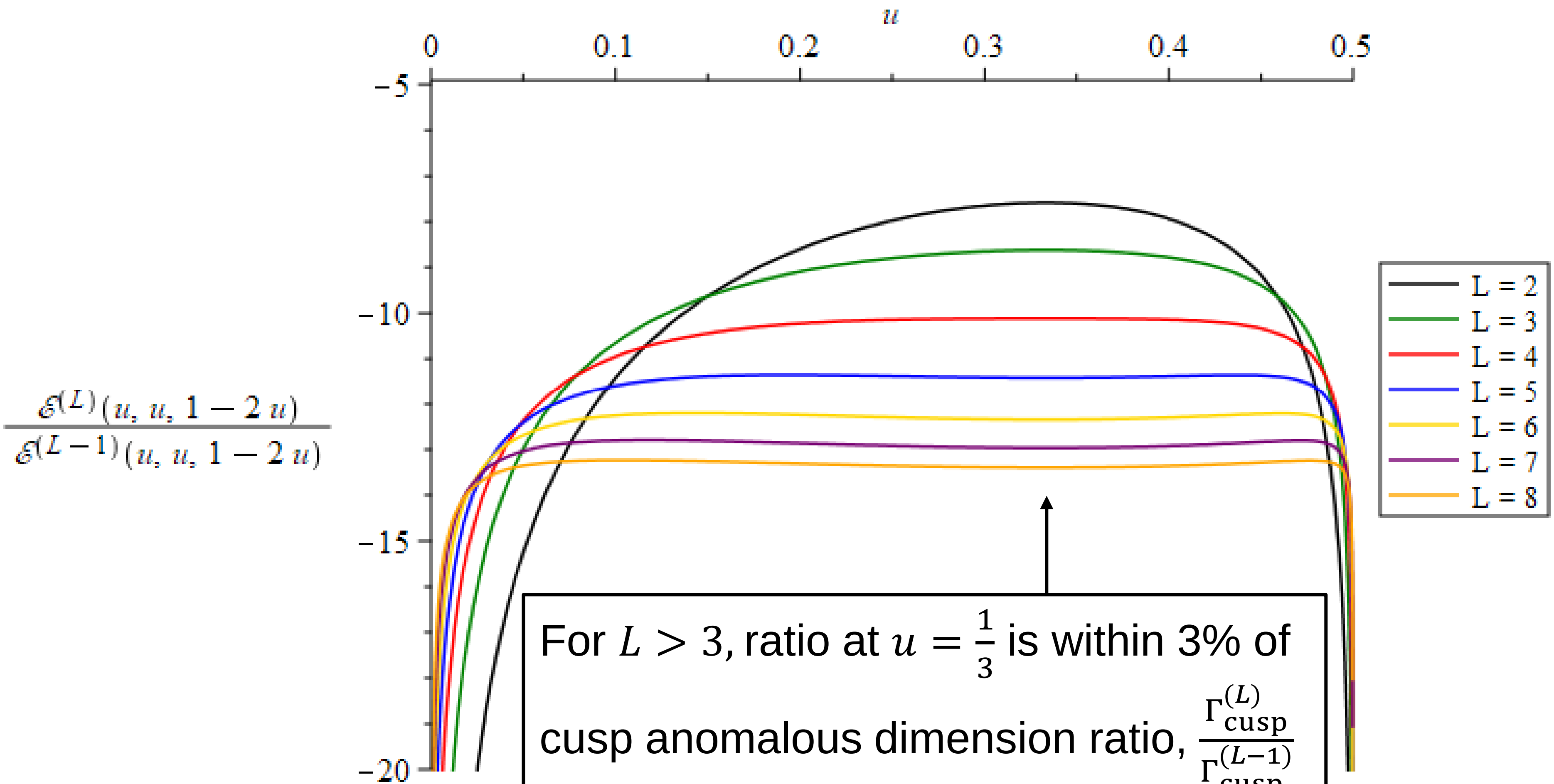


after JJ Carrasco

Summary & Outlook

- There is a code based on the symbol of generalized polylogs, which is analogous to the code of DNA
- The symbol of amplitudes is a linear combination of code words of maximal length $2L$ at L loops, and the coefficients are highly correlated, ~ “quantum DNA”
- The symbol is a general tool for disentangling the structure of scattering amplitudes in gauge theory, not just planar N=4 SYM
- Comparing 3-gluon form factor to 6-gluon amplitude, a strange new antipodal duality swaps the role of branch cuts and derivatives
- How much more can we exploit the symbol, to learn more about why and how particles scatter, in gauge theory and in (quantum) gravity?

Symbology enables plots to 8 loops

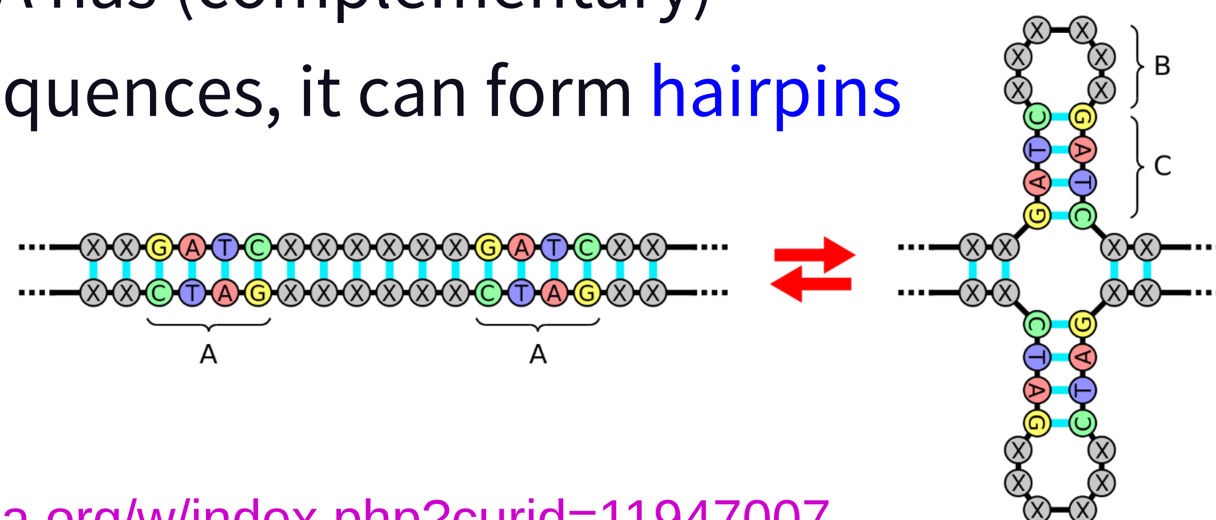


For $L > 3$, ratio at $u = \frac{1}{3}$ is within 3% of cusp anomalous dimension ratio, $\frac{\Gamma_{\text{cusp}}^{(L)}}{\Gamma_{\text{cusp}}^{(L-1)}}$

→ same finite radius of convergence

Antipodal Duality
←→ ? Palindromic DNA ?

- DNA & RNA normally read in only one direction
- Complementary strand normally carries no more information
- However, if RNA has (complementary) palindromic sequences, it can form hairpins

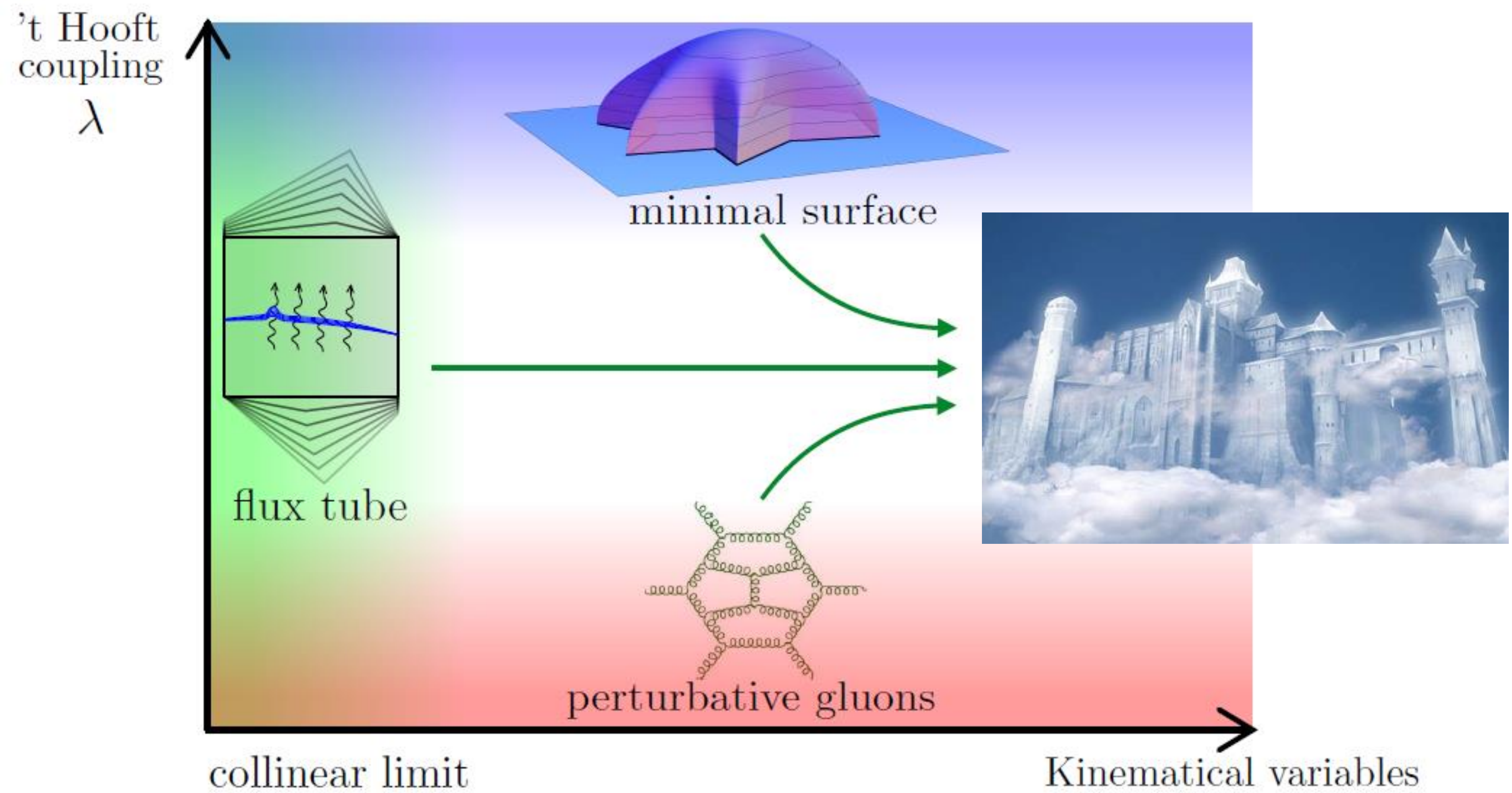


Acdx,
https://commons.wikimedia.org/w/index.php?curid=11947007

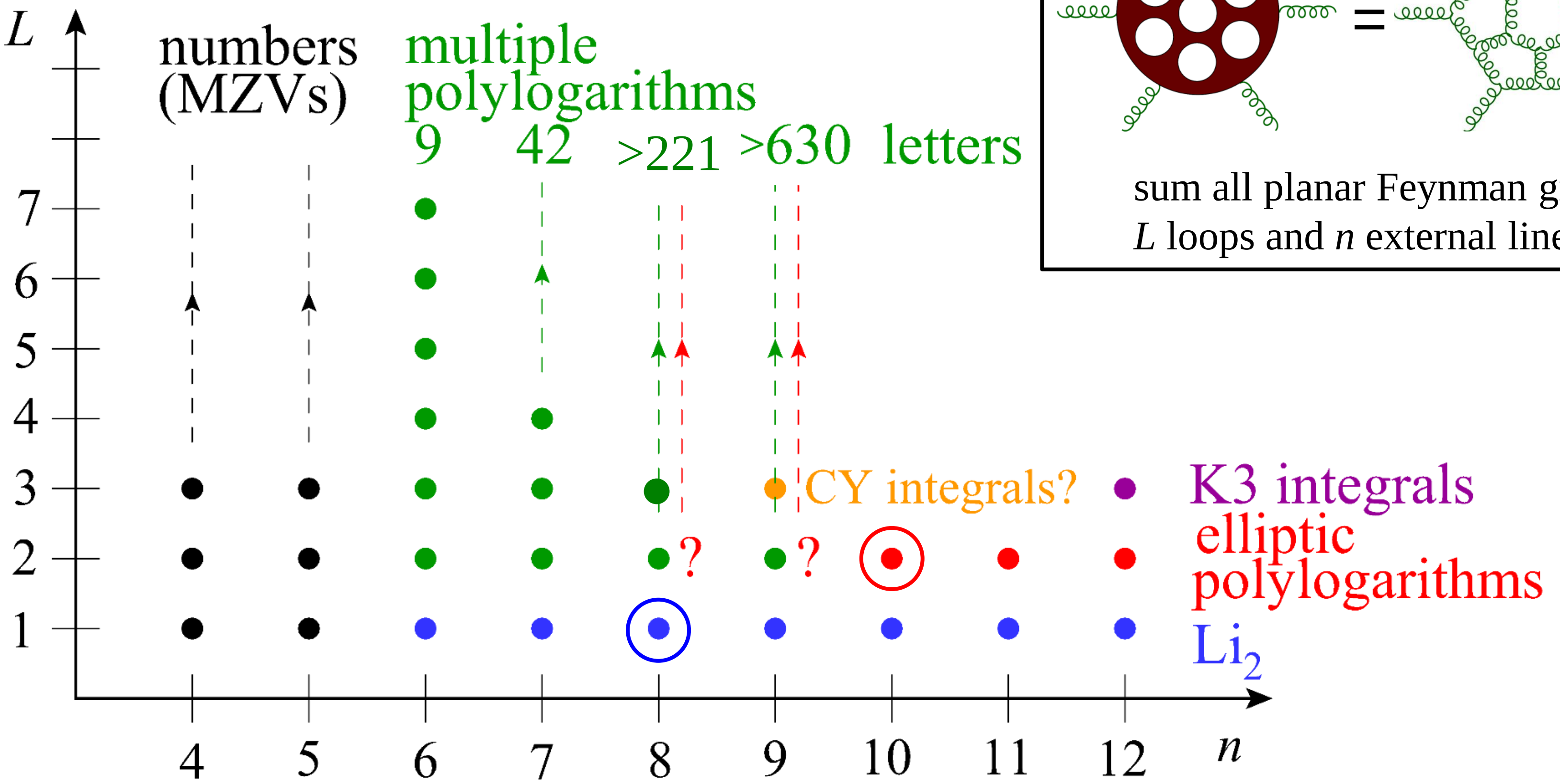
Extra Slides

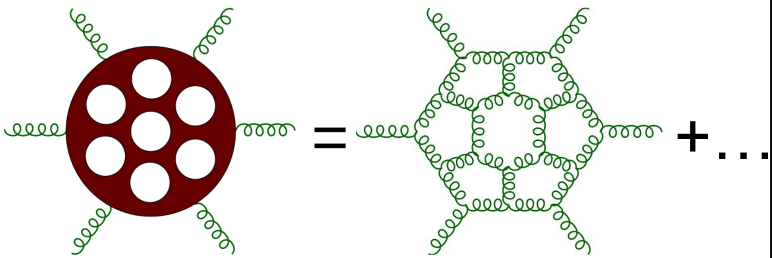
Solving for Planar N=4 SYM Amplitudes

Images: A. Sever, N. Arkani-Hamed



Beyond $n = 8$





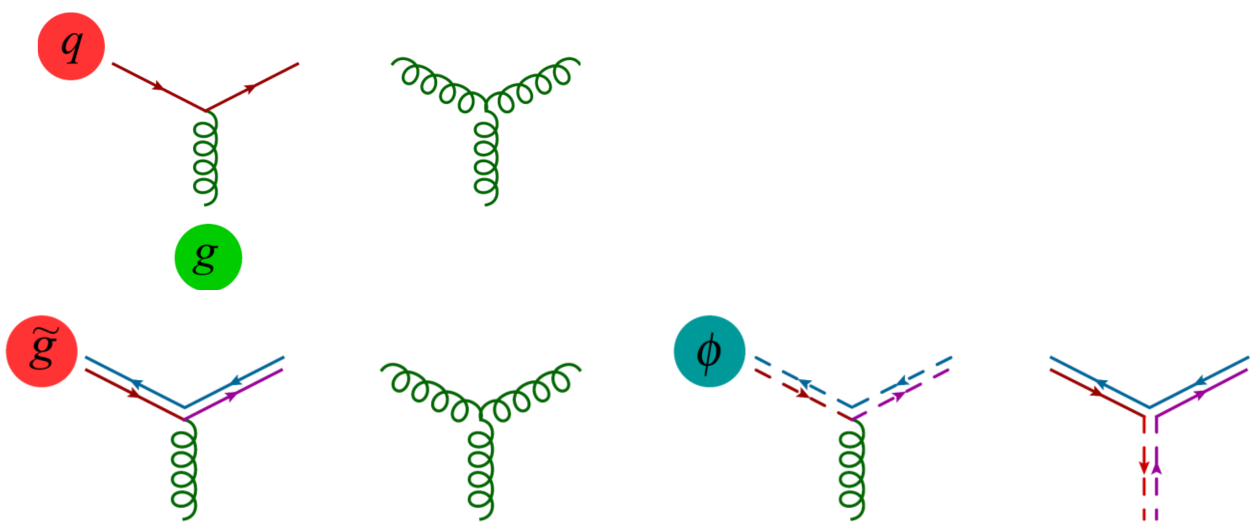
sum all planar Feynman graphs with L loops and n external lines

QCD vs. N=4 SYM

- QCD: gluons and quarks in fundamental rep. of $SU(N_c)$
- N=4: Replace quarks with 4 copies of fermions in adjoint rep. (gluinos) and add 6 real adjoint scalars
- All in same supermultiplet
- Feynman vertices:

QCD

N=4 SYM



Planar N=4 SYM, testing ground for QCD amplitudes

- QCD's maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, in large N_c (planar) limit
- Structure very rigid:

$$\text{Amplitudes} = \sum_i rational_i \times transcendental_i$$

- For planar N=4 SYM, rational structure well understood, focus on transcendental functions.
- Furthermore, at least three dualities hold:

1. AdS/CFT
2. Amplitudes dual to Wilson loops
3. New "antipodal" duality between amplitudes and form factors

Finite radius of convergence

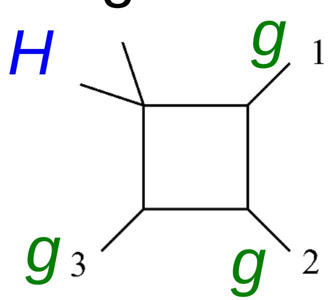
- Planar N=4 SYM has no renormalons ($\beta(g)=0$) and no instantons ($e^{-1/g_{\mathrm{YM}}^2} = e^{-N_c/\lambda}$)
- Perturbative expansion can have finite radius of convergence, unlike QCD, QED, whose perturbative series are asymptotic.
- For cusp anomalous dimension, using coupling $g^2 \equiv \frac{N_c g_{\mathrm{YM}}^2}{16\pi^2} = \frac{\lambda}{16\,\pi^2}$, radius is $\frac{1}{16}$

Beisert, Eden, Staudacher (BES), 0610251

- Ratio of successive loop orders $\frac{\Gamma_{\mathrm{cusp}}^{(L)}}{\Gamma_{\mathrm{cusp}}^{(L-1)}} \rightarrow -16$
- Find same radius of convergence in high-loop-order behavior of amplitudes and form factors, in most kinematic regions.

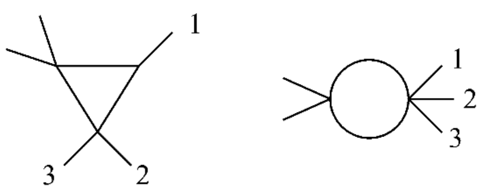
N=4 SYM very special

- At one loop, cancellation of loop momenta in numerator → only scalar box integrals
  Bern, LD, Dunbar, Kosower, hep-ph/9403226
- Weight 2 functions – dilogs. E.g., *gg* → *Hg* @ 1 loop ⊃



$$= \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

- QCD results also contain single log's and rational parts from (tensor) triangle + bubble integrals



$$= \frac{1}{\epsilon} - \ln(s_{123})$$

Higher loops

- N=4 SYM amplitudes have "uniform weight" (transcendentality) $2L$ at loop order L
- Weight ~ number of integrations, e.g.

$$\ln(s) = \int_1^s \frac{dt}{t} = \int_1^s d\ln t \qquad 1$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x d\ln t \cdot [-\ln(1-t)] \qquad 2$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad n$$

- QCD amps typically all weights from 0 to $2L$

Different routes to perturbative amplitudes

Draw all Feynman graphs G_i

↓

Evaluate all Feynman rules: I_i

↓

Perform all loop integrations: A_i

↓

$$A = \sum_i A_i$$

Evaluate all unitarity cuts C_α

↓

Construct **local**: integrand I

↓

Perform all loop integrations: A_α

↓

$$A = \sum_\alpha A_\alpha$$

Bootstrap: Guess $A = \sum_m c_m F_m$

F_m known functions

$c_m \in \mathbb{Q}$ unknown constants

↓

Solve constraints, linear equations for $c_m \to r_m$

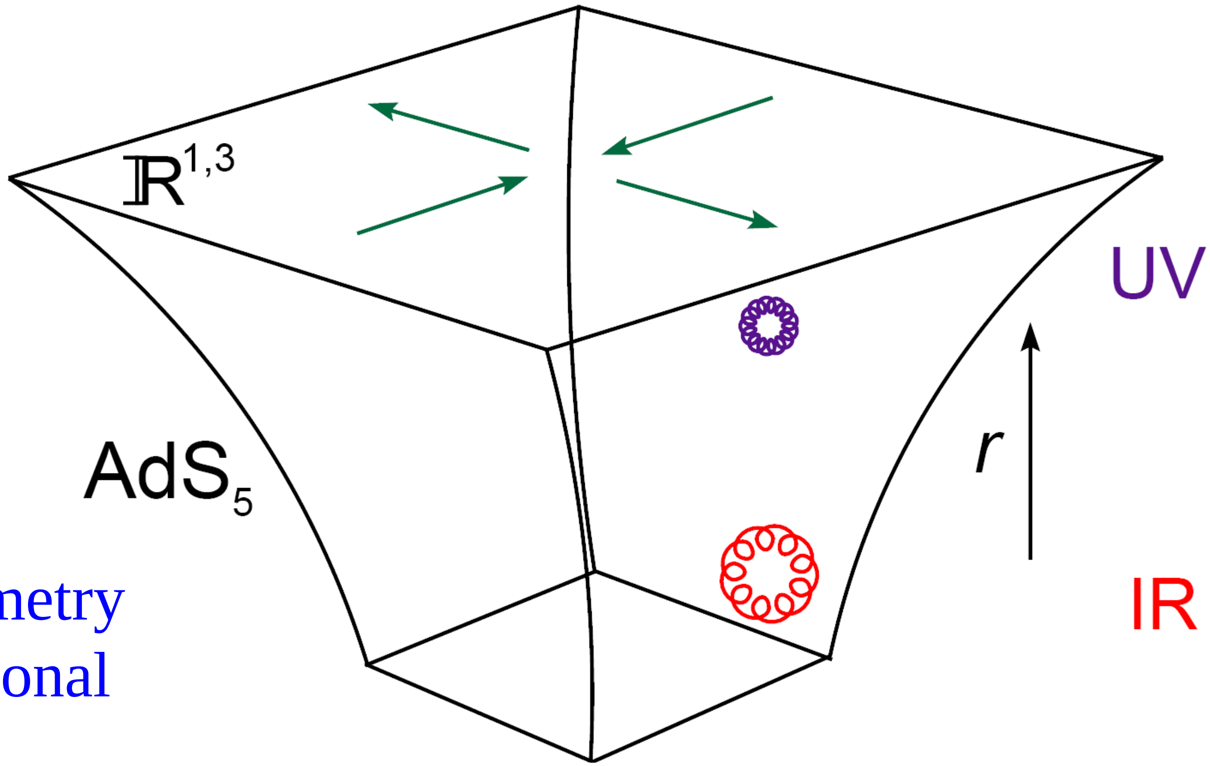
↓

$$A = \sum_m r_m F_m$$

AdS/CFT

Maldacena (1997)

Conformal field theory, N=4 SYM
dual to strings moving in $AdS_5 \times S^5$



SO(4,2) isometry
of 5 dimensional
space-time
←→ 4d conformal symmetry

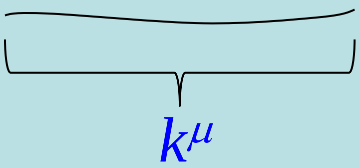
weak-strong duality

T-duality symmetry of string theory

Alday, Maldacena, 0705.0303

- Exchanges string world-sheet variables σ, τ
- $X^\mu(\tau,\sigma) = x^\mu + k^\mu \tau$ + oscillators

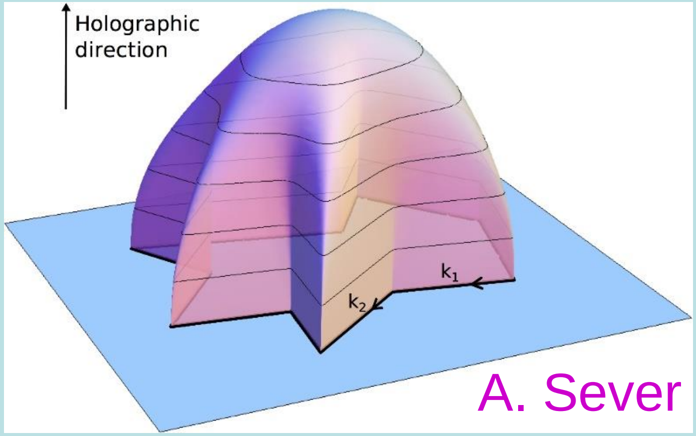
→ $X^\mu(\tau,\sigma) = x^\mu + k^\mu \sigma$ + oscillators



- Strong coupling limit of planar N=4 SYM

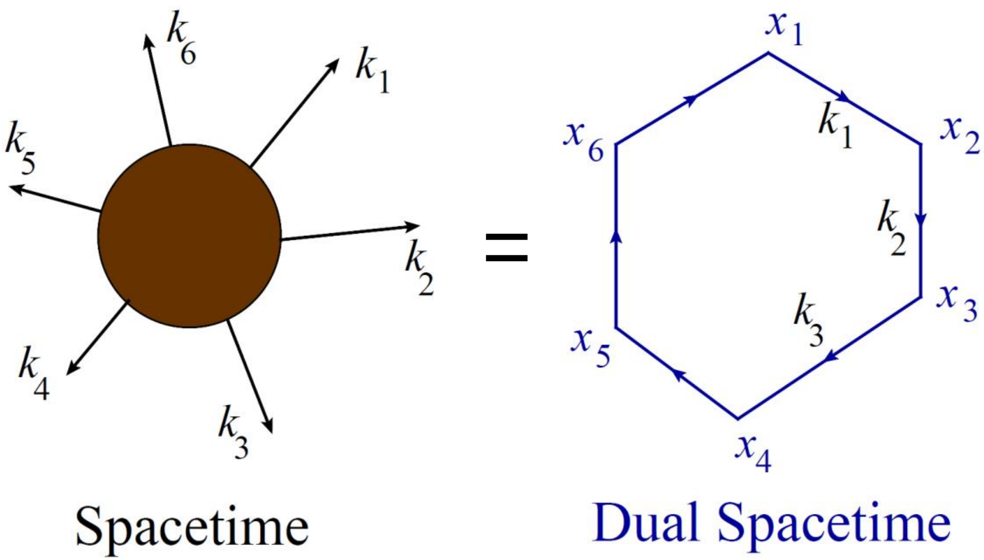
is semi-classical limit of string theory: world-sheet stretches tight around minimal area surface in AdS.

- Boundary determined by momenta of external states: light-like polygon with null edges = momenta k^μ



A. Sever

Amplitudes = Wilson loops



- Polygon vertices x_i are not positions but dual momenta,
  $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Duality verified to hold at weak coupling too

weak-weak duality, holds order-by-order

Dual conformal invariance

- Wilson *n*-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

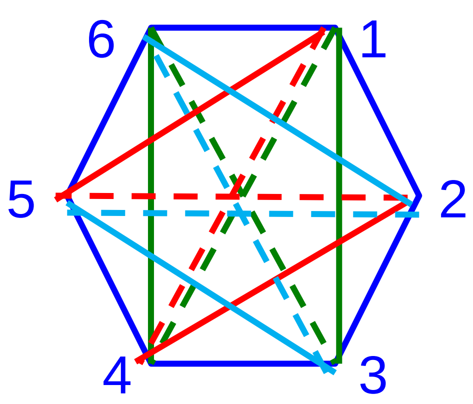
- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for *n* = 4,5

n = 6 → precisely 3 ratios:

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

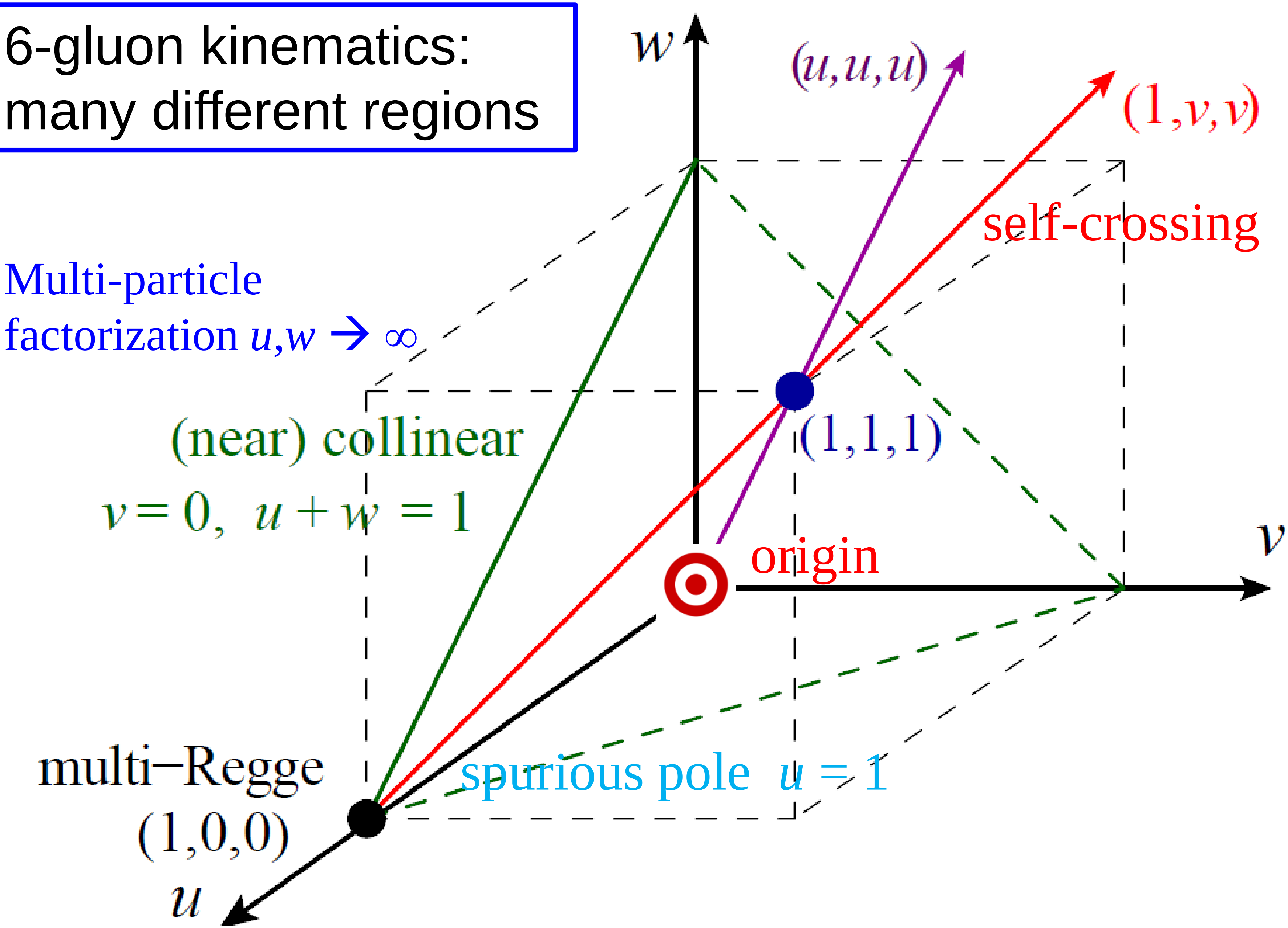
$$v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$



n = 7 → 6 ratios.
In general, 3*n*-15 ratios.

6-gluon kinematics: many different regions



Rich theoretical “data mine”

- Rare to have perturbative results to 7 loops
- Usually high loop order → single numbers such as β functions or anomalous dimensions
- Here we have analytic functions of 3 variables
- Many limits to study (and exploit)

Hggg kinematics is two-dimensional

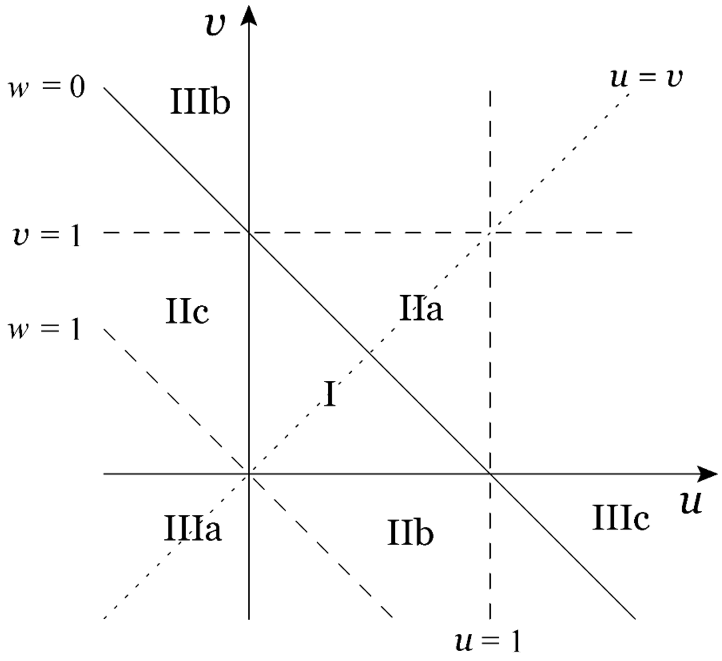
$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

$$u + v + w = 1$$



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or
$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

A two-loop story

- *Hggg* computed in QCD at 2 loops
  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554
- Stress tensor 3-point form factor $\mathcal{F}_3$ in N=4 SYM computed next Brandhuber, Travaglini, Yang, 1201.4170
- Highest weight part of QCD result was **same as N=4 result!!**
- "Principle of maximal transcendentality"
  Kotikov, Lipatov, Velizhanin, hep-ph/0301021, hep-ph/0611204
- Does it hold here beyond two loops?

2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

Space graded by weight n. Every function F obeys:

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$

$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{w} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{1-w}$$

$$w = 1 - u - v$$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight n-1 2d HPLs.

To **bootstrap** *Hggg* amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight

Symbol alphabet $\mathcal{S}$ for *Hggg*

Gehrmann, Remiddi, hep-ph/0008287

- Comparing

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^{u}}{u} - \frac{F^{w}}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$

$$\frac{\partial F(u,v)}{\partial v} = \frac{F^{v}}{v} - \frac{F^{w}}{w} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{1-w}$$

$$w = 1 - u - v$$

with $\quad dF = \sum_{s_k \in \mathcal{S}} F^{s_k} \, d \ln s_k$

we see that the alphabet is

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w\}$$

Example: Harmonic Polylogarithms in one variable (HPL{0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Generalize classical polylogs
- Define HPLs by iterated integration:

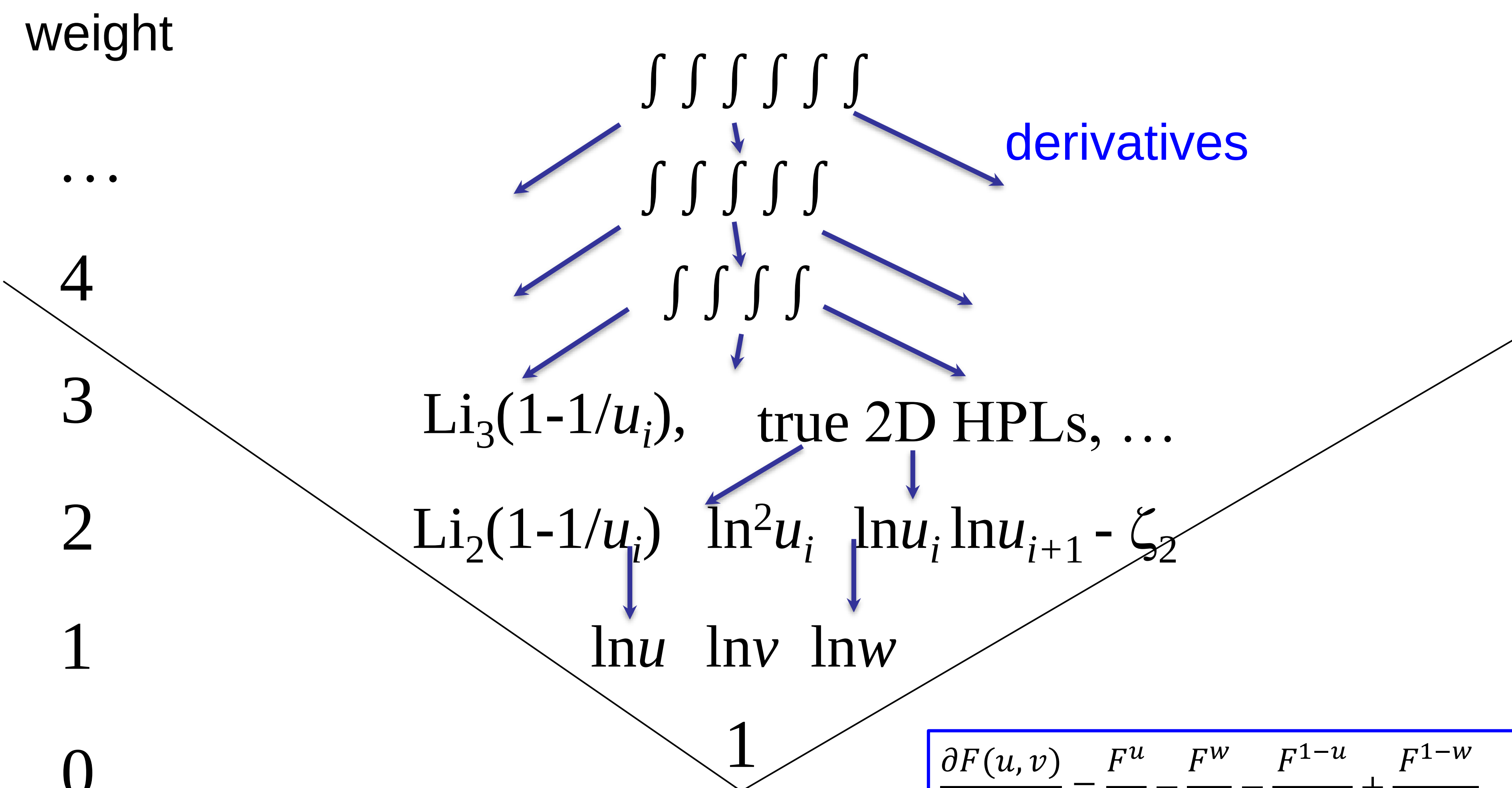
$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives:

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x) d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

- Symbol alphabet: $\mathcal{S} = \{x, 1-x\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$ is number of binary strings: 2^{2L}
- Branch cuts dictated by first integration/entry in symbol
- Derivatives dictated by last integration/entry in symbol

Heuristic view of function space



$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$

More bootstrapping details

- Four lectures at "The Amplitude Games", Mainz Institute for Theoretical Physics, July 2021

- https://indico.mitp.uni-mainz.de/event/204/sessions/875/

3-gluon form factor dihedral symmetry

dihedral cycle: $a \to b \to c \to a, \quad d \to e \to f \to d$

dihedral flip: $a \leftrightarrow b, \quad d \leftrightarrow \mathrm{e}$

Simplest analytic form is for $v \to \infty$

→ Harmonic polylogarithms $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{u})$

$$
\begin{aligned}
F_3^{(1)}(v \to \infty) &= 2H_{0,1} + 6\zeta_2 \\
F_3^{(2)}(v \to \infty) &= -8H_{0,0,0,1} - 4H_{0,1,1,1} + 12\zeta_2 H_{0,1} + 13\zeta_4 \\
F_3^{(3)}(v \to \infty) &= 96H_{0,0,0,0,0,1} + 16H_{0,0,0,1,0,1} + 16H_{0,0,0,1,1,1} + 16H_{0,0,1,0,0,1} + 8H_{0,0,1,0,1,1} \\
&\quad + 8H_{0,0,1,1,0,1} + 16H_{0,1,0,0,0,1} + 8H_{0,1,0,0,1,1} + 12H_{0,1,0,1,0,1} + 4H_{0,1,0,1,1,1} \\
&\quad + 8H_{0,1,1,0,0,1} + 4H_{0,1,1,0,1,1} + 4H_{0,1,1,1,0,1} + 24H_{0,1,1,1,1,1} \\
&\quad - \zeta_2(32H_{0,0,0,1} + 8H_{0,0,1,1} + 4H_{0,1,0,1} + 52H_{0,1,1,1}) \\
&\quad - \zeta_3(8H_{0,0,1} - 4H_{0,1,1}) - 53\zeta_4 H_{0,1} - \frac{167}{4}\zeta_6 + 2(\zeta_3)^2
\end{aligned}
$$

8 loop result has $\sim 2^{2\times 8-2} = 16{,}384$ terms

6-gluon amplitude is simplest for $(\hat{u}, \hat{v}, \hat{w}) = (1, \hat{v}, \hat{v})$

- Let $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{\hat{v}})$

$$A_6^{(1)}(1, \hat{v}, \hat{v}) = 2H_{0,1}$$

$$A_6^{(2)}(1, \hat{v}, \hat{v}) = -8H_{0,1,1,1} - 4H_{0,0,0,1} - 4\zeta_2 H_{0,1} - 9\zeta_4$$

$$\begin{aligned} A_6^{(3)}(1, \hat{v}, \hat{v}) = {} & 96H_{0,1,1,1,1,1} + 16H_{0,1,0,1,1,1} + 16H_{0,0,0,1,1,1} + 16H_{0,1,1,0,1,1} + 8H_{0,0,1,0,1,1} \\ & + 8H_{0,1,0,0,1,1} + 16H_{0,1,1,1,0,1} + 8H_{0,0,1,1,0,1} + 12H_{0,1,0,1,0,1} + 4H_{0,0,0,1,0,1} \\ & + 8H_{0,1,1,0,0,1} + 4H_{0,0,1,0,0,1} + 4H_{0,1,0,0,0,1} + 24H_{0,0,0,0,0,1} \\ & + \zeta_2(8H_{0,0,0,1} + 8H_{0,1,0,1} + 48H_{0,1,1,1}) \\ & + 42\zeta_4 H_{0,1} + 121\zeta_6 \end{aligned}$$

Exact map at symbol level, with $\frac{1}{\hat{v}} = 1 - \frac{1}{u}$, $0 \leftrightarrow 1$,
if you also reverse order of symbol entries / HPL indices!!!
Works to 7 loops, where $\sim 2^{2\times 7-2} = 4{,}096$ terms agree

Antipodal duality

weak-weak duality

LD, Ö. Gürdoğan, A. McLeod,
M. Wilhelm, 2112.06243

$$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)$$

Antipode map S, at symbol level, reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \; x_m \otimes \cdots \otimes x_2 \otimes x_1$$

Kinematic map is

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

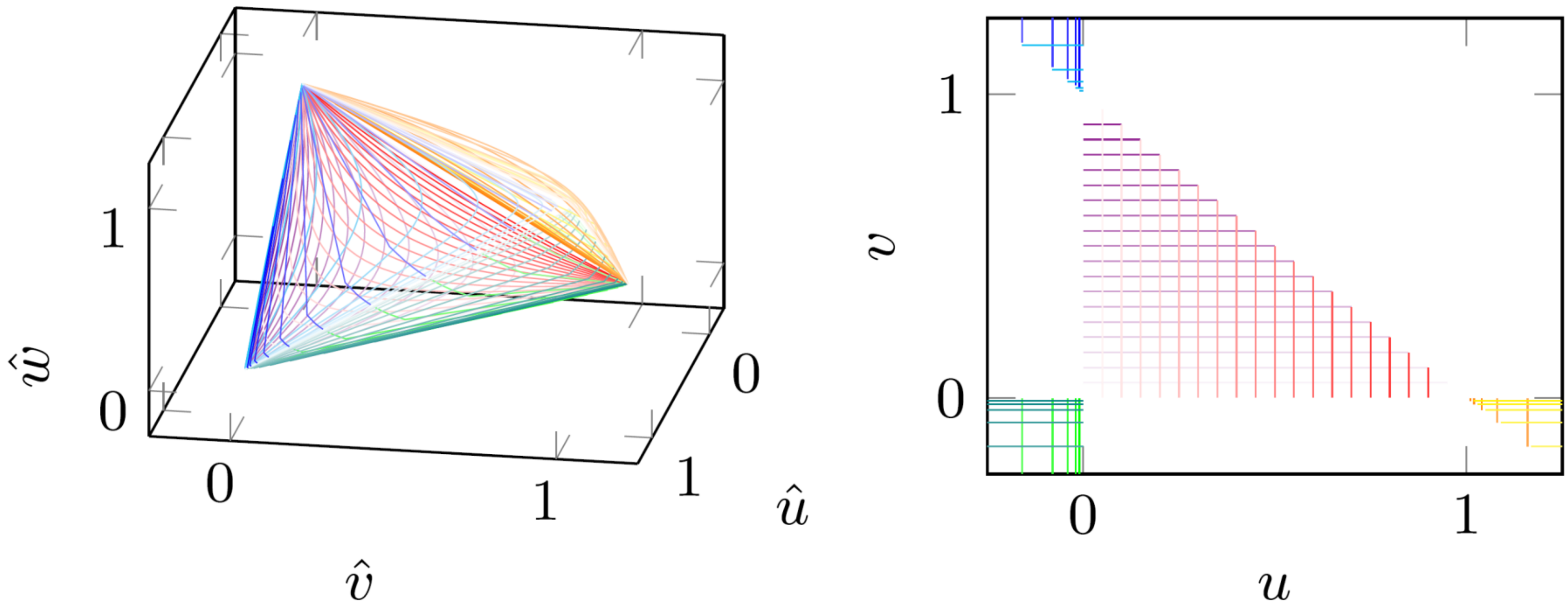
Maps $u + v + w = 1$ to parity-preserving surface

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

also corresponds to "twisted forward scattering":

$$\hat{k}_{i+n}^{\mu} = -\hat{k}_i^{\mu}, \quad i = 1,2,\ldots,n \qquad (n = 3 \; here)$$

Map covers entire phase space for 3-gluon form factor

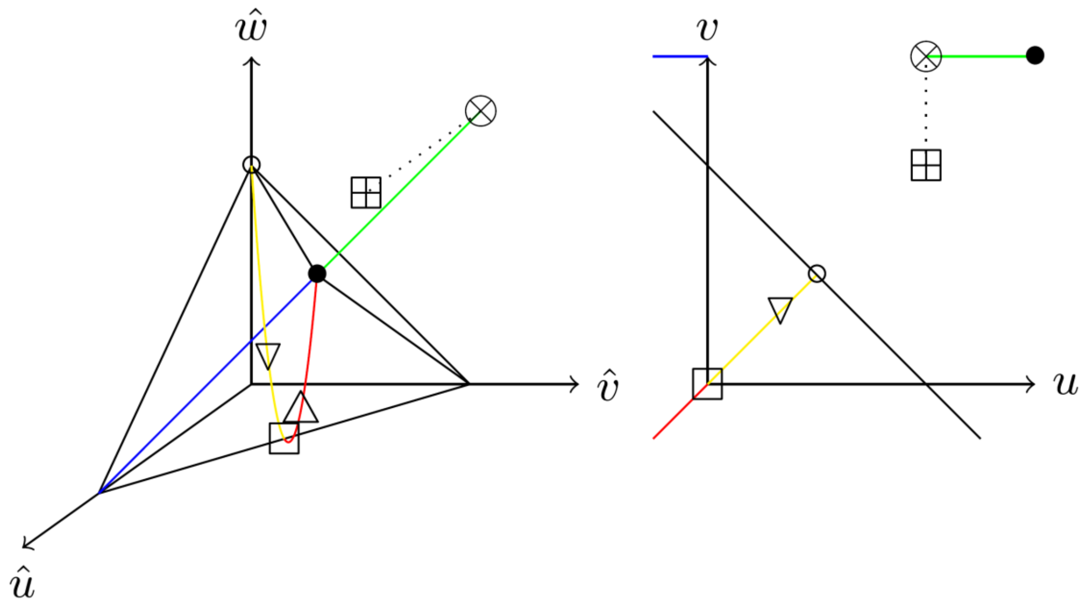


- Soft is dual to collinear; collinear is dual to soft
- White regions in (u, v) map to some of $\hat{u}, \hat{v}, \hat{w} > 1$

Many special dual points

There is an "f" alphabet at all these points: a way of writing multiple zeta values (MZV's) so that coaction is manifest.

F. Brown, 1102.1310; O. Schnetz, HyperlogProcedures



	$(\hat{u},\hat{v},\hat{w})$	(u,v,w)	functions
▽	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\sqrt[6]{1}$
□	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,1)$	$\mathrm{Li}_2(\frac{1}{2})$ + logs
●	$(1,1,1)$	$\lim_{u\to\infty}(u,u,1-2u)$	MZVs
○	$(0,0,1)$	$(\frac{1}{2},\frac{1}{2},0)$	MZVs + logs
△	$(\frac{3}{4},\frac{3}{4},\frac{1}{4})$	$(-1,-1,3)$	$\sqrt[6]{1}$
⊞	(∞,∞,∞)	$(1,1,-1)$	alternating sums
⊗	$\lim_{\hat{v}\to\infty}(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(1,v,-v)$	MZVs
(green–blue line)	$(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(u,v,1-u-v)$	HPL$\{0,1\}$
(red–yellow line)	$(\hat{u},\hat{u},(1-2\hat{u})^2)$	$(u,u,1-2u)$	HPL$\{-1,0,1\}$

Simplest point

- $(\hat{u},\hat{v},\hat{w}) = (1,1,1) \Leftrightarrow u,v \to \infty$
- At this point,

$$A_6^{(1)}(\cdot) = 0 \qquad F_3^{(1)}(\cdot) = 8\zeta_2$$

$$A_6^{(2)}(\cdot) = -9\zeta_4 \qquad F_3^{(2)}(\cdot) = 31\zeta_4$$

$$A_6^{(3)}(\cdot) = 121\zeta_6 \qquad F_3^{(3)}(\cdot) = -145\zeta_6$$

$$A_6^{(4)}(\cdot) = 120 f_{3,5} - 48\zeta_2 f_{3,3} - \frac{6381}{4}\zeta_8 \qquad F_3^{(4)}(\cdot) = 120 f_{5,3} + \frac{11363}{4}\zeta_8$$

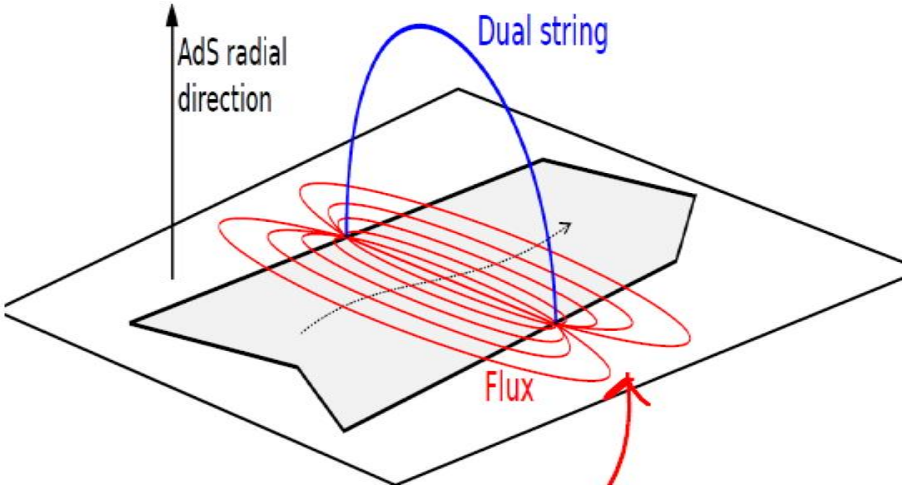
$$A_6^{(5)}(\cdot) = -2688 f_{3,7} - 1560 f_{5,5} + \mathcal{O}(\pi^2) \qquad F_3^{(5)}(\cdot) = -2688 f_{7,3} - 1560 f_{5,5} + \mathcal{O}(\pi^2)$$

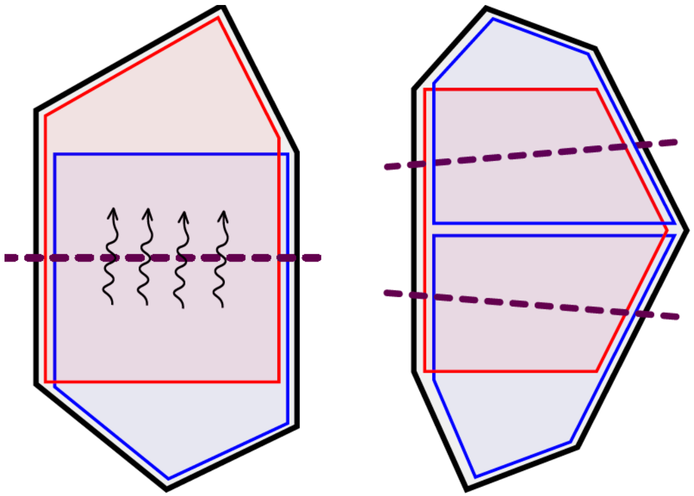
$$A_6^{(6)}(\cdot) = 48528 f_{3,9} + 37296 f_{5,7} + 21120 f_{7,5} + \mathcal{O}(\pi^2) \qquad F_3^{(6)}(\cdot) = 48528 f_{9,3} + 37296 f_{7,5} + 21120 f_{5,7} + \mathcal{O}(\pi^2)$$

- Reversing ordering of letters in f-alphabet, blue values show that antipodal duality holds beyond symbol level, modulo $i\pi$
- modulo $i\pi$ is best we can get from antipode map

Bootstrap boundary data:
Flux tubes at finite coupling

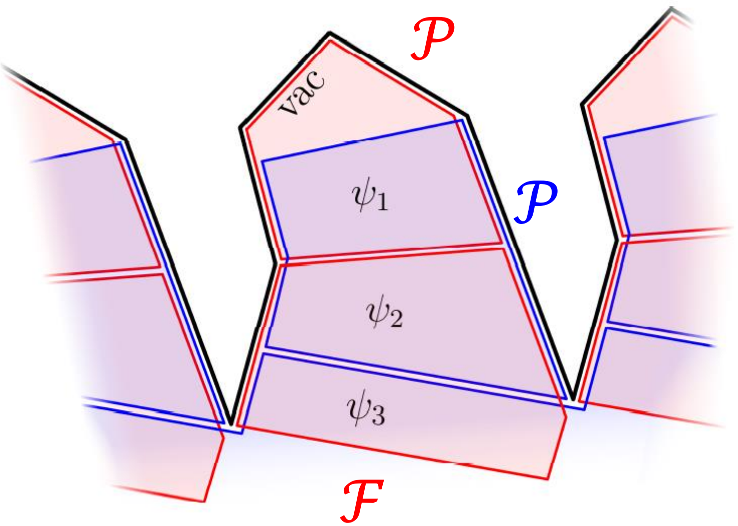
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

A New Form Factor OPE



- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$

Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

OPE representation

- 6-gluon amplitude:

$$\mathcal{W}_{\rm hex} = \sum_{\mathbf{a}} \int d\mathbf{u}\, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{a}}(\bar{\mathbf{u}}|0)\, e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi}$$

$T = e^{-\tau}, S = e^{-\sigma}, F = e^{i\phi}$. $\quad v = \frac{T^2}{1+T^2} \to 0,$

weak-coupling, $E = k + \mathcal{O}(g^2)$ → expansion in T^k

- 3-gluon form factor: $\psi = helicity\ 0\ pairs\ of\ states$

$$\mathcal{W}_3 = \sum\!\!\!\!\!\!\int_{\psi} e^{-E_\psi \tau + i p_\psi \sigma} \mathcal{P}(0|\psi) \mathcal{F}(\psi)$$

weak-coupling → expansion in T^{2k} (no azimuthal angle ϕ)

OPE parametrizations

- Amplitude:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})},$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \qquad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2}$$

$(\hat{F} = 1 \text{ for } \Delta = 0)$

- Form factor:

$$u = \frac{1}{1 + S^2 + T^2}, \qquad v = \frac{T^2}{1 + T^2},$$

$$w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))},$$

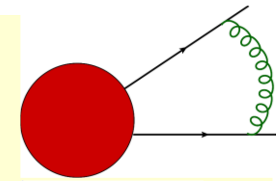
- Apply kinematic map →

$$\hat{T} = \frac{T}{S}, \qquad \hat{S} = \frac{1}{TS}$$

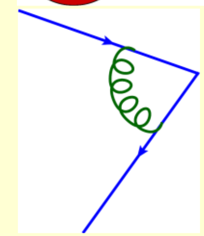
- Apparently some correspondence between single flux tube excitations for the amplitude (T^1) and double (or bound state) excitations for the form factor (T^2)

Removing Amplitude (or Form Factor) Infrared Divergences

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\rm cusp}$



– known to all orders in planar N=4 SYM:
Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\rm BDS-like}(s_{i,i+1},\epsilon)} = \exp\Big[\frac{\Gamma_{\rm cusp}}{4}\mathcal{E}^{(1)} + R_6\Big]$$

remainder function

BDS & BDS-like normalization for $\mathcal{F}_3$

$$\frac{\mathcal{F}_3}{\mathcal{F}_3^{\text{MHV, tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\text{cusp}}^{(L)}}{4} + \mathcal{O}(\epsilon)\right) M^{1-\text{loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u,v,w)\right]\right\}$$

BDS ansatz

remainder function only a function of u, v, w; vanishes in all collinear limits, but no adjac... ...y constraints

split 1-loop amplitude judiciously:

$$\frac{\mathcal{F}_3^{1-\text{loop}}}{\mathcal{F}_3^{\text{MHV, tree}}} \equiv M^{1-\text{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u,v,w)$$

$$M(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{3}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} - \frac{7}{2}\zeta_2 + \sum^{3} \ldots$$

$$\mathcal{E}^{(1)}(u,v \ldots) \ldots \left(1 - \frac{1}{v}\right) + \text{Li}_2\left(1 - \frac{1}{w}\right)\Bigg]$$

$$\mathcal{E}^{(1),u} + \mathcal{E}^{(1),1-u} = 0$$

$\mathcal{E}$ obeys “adjacency constraints”

Now div... ...by:

$$\frac{\mathcal{F}_3^{\text{BDS-like}}}{\mathcal{F}_3^{\text{MHV, tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\text{cusp}}}{4} + \mathcal{O}(\epsilon)\right) M(L\epsilon) + C^{(L)}\right]\right\} \Rightarrow$$

$$\mathcal{E} = \exp\left[\frac{\Gamma_{\text{cusp}}}{4}\mathcal{E}^{(1)} + R\right]$$

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.

  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$

Number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L = 1$	1	3	1														
$L = 2$	1	3	6	3	1												
$L = 3$	1	3	9	12	6	3	1										
$L = 4$	1	3	9	21	24	12	6	3	1								
$L = 5$	1	3	9	21	46	45	24	12	6	3	1						
$L = 6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L = 7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L = 8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- Properly normalized L loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right

Reading DNA backwards?

"Bacteria contain symmetry in their DNA signals that enable them to be read either forwards or backwards, according to new findings at the University of Birmingham [Grainger et al.] which challenge existing knowledge about gene transcription. The new study shows that single-celled organisms have symmetrical DNA signposts. This means that the DNA code can be read in either direction." phys.org May 6, 2021