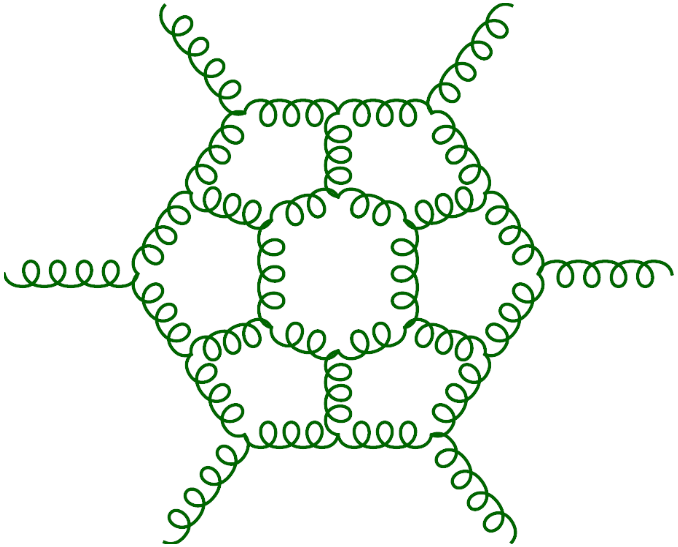
Bootstrapping Amplitudes in Planar N=4 SYM Theory

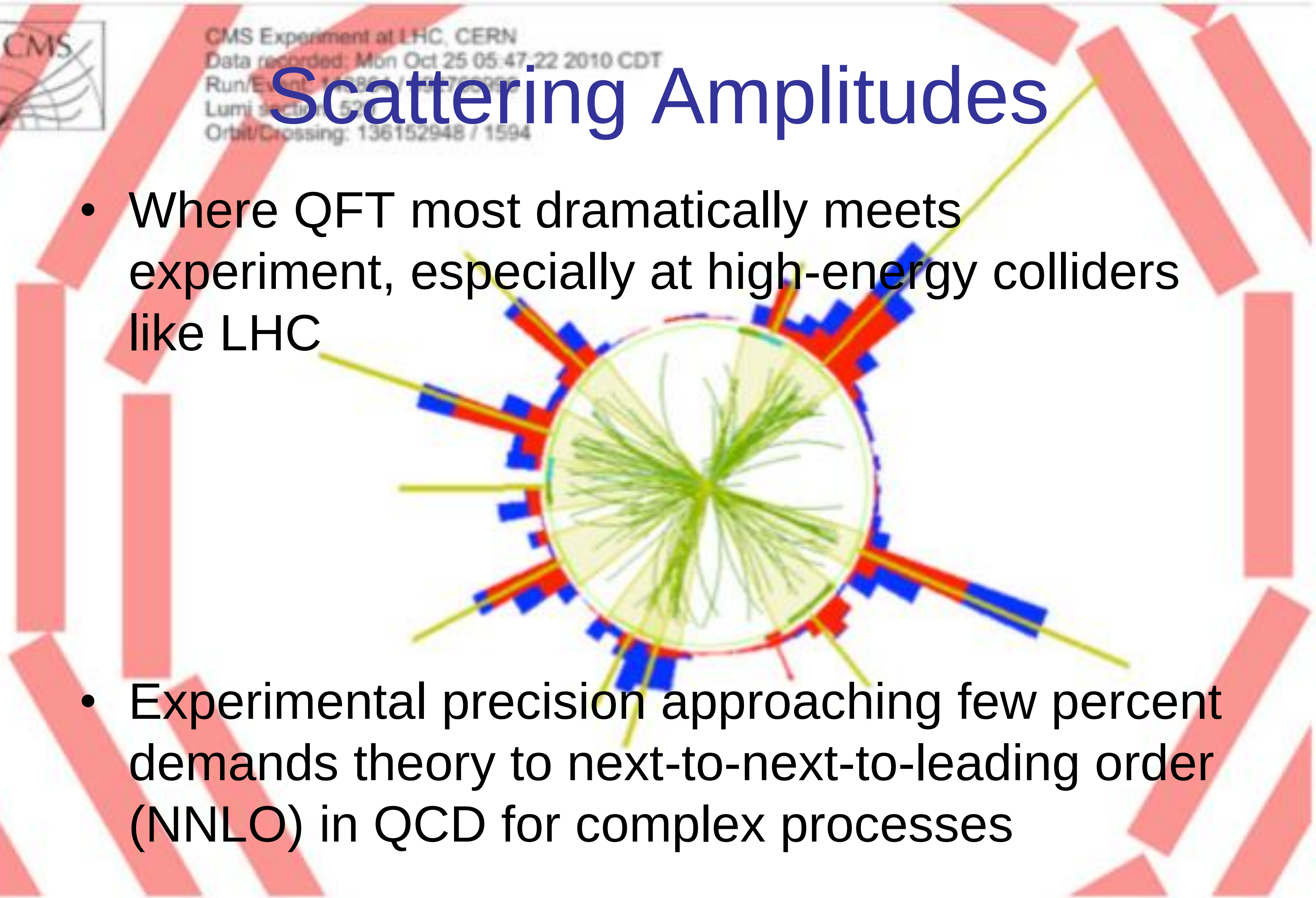


Lance Dixon (SLAC)

QFT Colloquium
Humboldt University, Berlin
8 January 2019

SLAC NATIONAL ACCELERATOR LABORATORY

Scattering Amplitudes

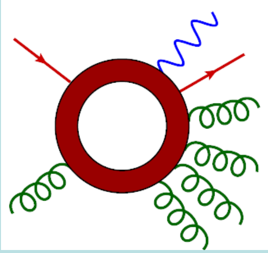


- Where QFT most dramatically meets experiment, especially at high-energy colliders like LHC

- Experimental precision approaching few percent demands theory to next-to-next-to-leading order (NNLO) in QCD for complex processes

QCD Loop Amplitude Bottleneck

- NLO: Have efficient, unitarity-based methods for computing one-loop amplitudes at high multiplicity, e.g. the 2 → 6 process pp → W + 5 jets



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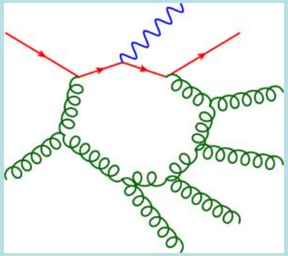
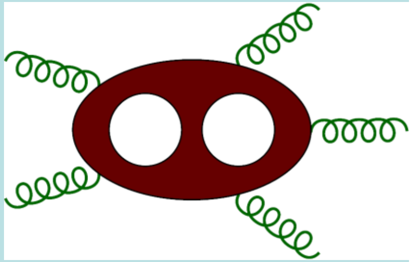


+ 256,264 more

Bern, LD, et al., 1304.1253

- NNLO: Two-loop QCD amplitudes unknown beyond 2 → 2 processes, except for one very recent 2 → 3 case ($gg \to ggg$) in large N_c (planar) limit



Badger et al., 1712.02229; Abreu et al., 1712.03946, 1812.04586

Why is two loops so hard?

- Primarily because two-loop integrals are intricate, transcendental, multi-variate functions
- In contrast, at one loop all integrals are reducible to scalar box integrals + simpler

→ combinations of dilogarithms

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

+ logarithms and rational terms

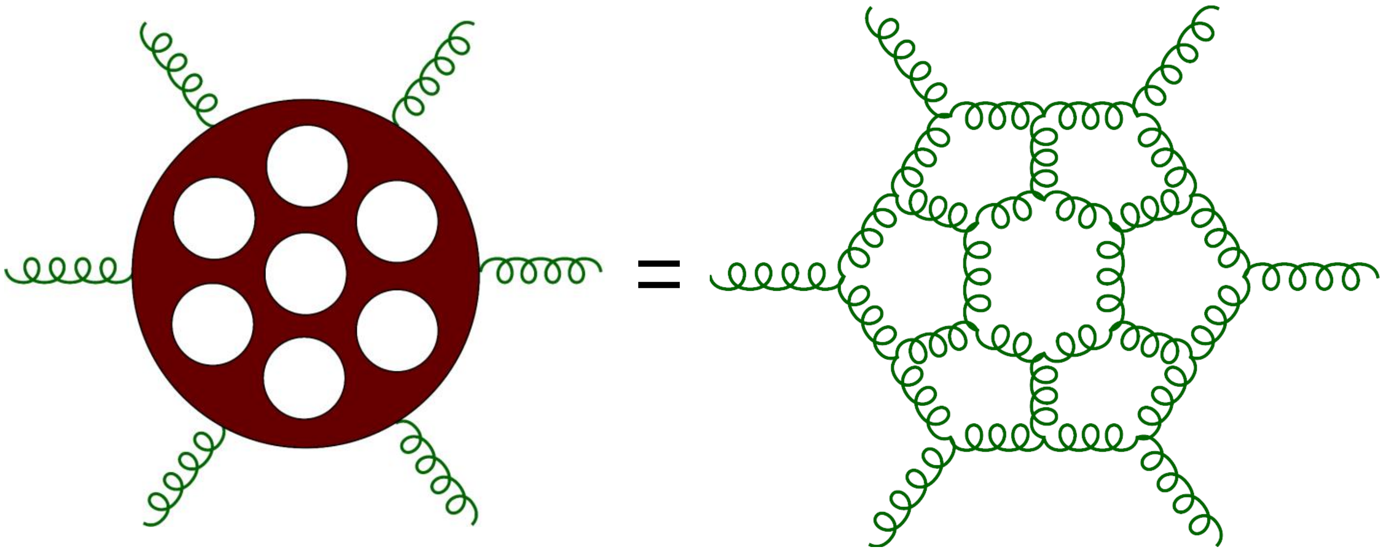
't Hooft, Veltman (1974)

A "toy" model

- Today, explore some of the complexity of multi-loop, multi-leg QCD amplitudes in a controlled setting: QCD's maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, and in the large N_c (planar) limit
- (Some) planar N=4 SYM scattering amplitudes belong to a universal space of transcendental functions, to any loop order.
- The space is so restrictive, and the physical constraints are so powerful, that one can bootstrap the solution: Write down answer as linear combination of functions, rational number coefficients determined by solving a set of linear constraints.

How far can we go?

- So far, through 7 loops (one helicity configuration)

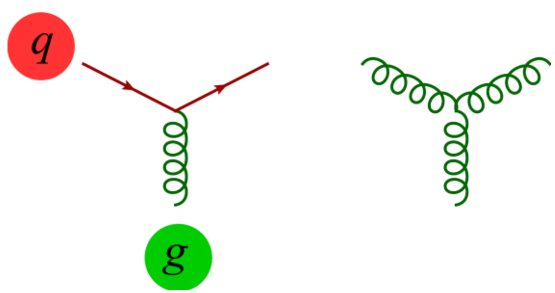


+ ~10^9 more

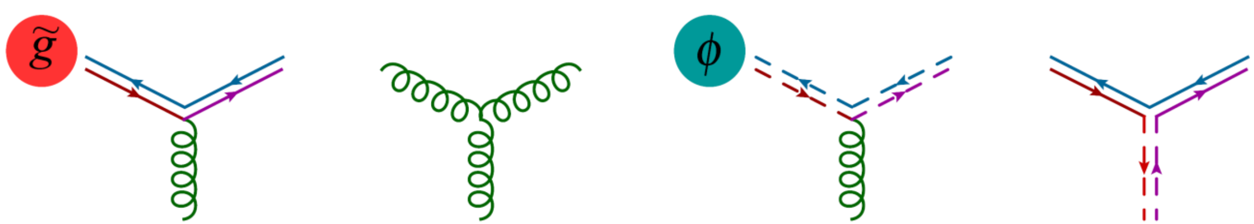
QCD vs. N=4 SYM

- QCD has gluons and quarks in fundamental rep. of $SU(N_c)$
- Replace quarks with 4 copies of fermions in adjoint rep. (gluinos) and add 6 real adjoint scalars
- Feynman vertices:

QCD



N=4 SYM



N=4 SYM particle content

massless spin 1 gluon

4 massless spin 1/2 gluinos

6 massless spin 0 scalars

SUSY
Q_a, a=1,2,3,4
shifts helicity
by 1/2 $\longleftrightarrow$

$\mathcal{N}=4$	1	$\longleftrightarrow$	4	$\longleftrightarrow$	6	$\longleftrightarrow$	4	$\longleftrightarrow$	1
	g^-		$\lambda^-_{\bar{\imath}}$		$\bar{\phi}_{\bar{\imath}\bar{\jmath}}, \phi_{ij}$		λ^+_i		g^+
helicity	-1		$-\frac{1}{2}$		0		$\frac{1}{2}$		1

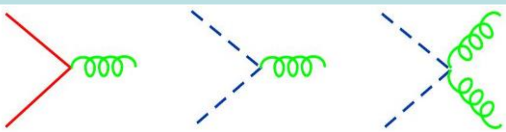
all in adjoint representation

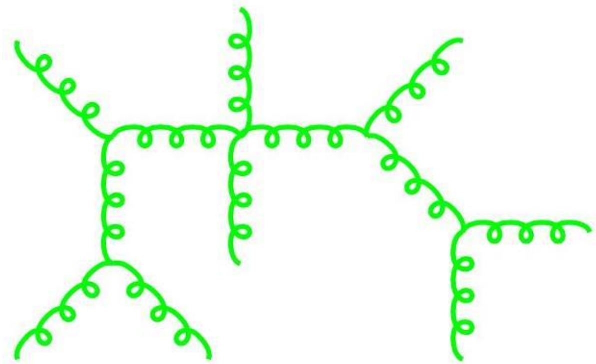
QCD vs. N=4 SYM at tree level

At tree-level essentially identical

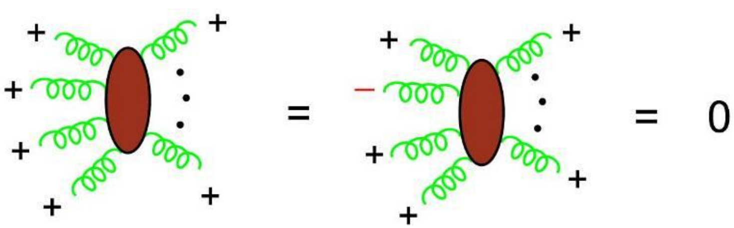
Consider a tree amplitude for *n* gluons.

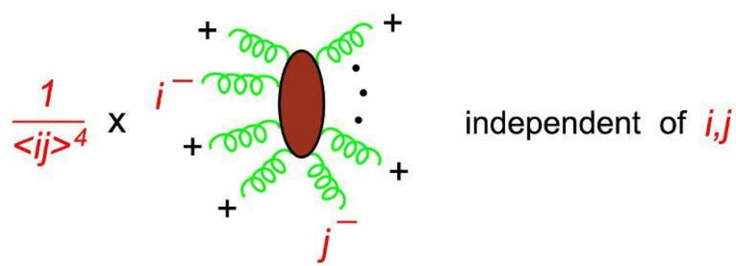
Fermions and scalars cannot appear because they are produced in pairs





Hence the amplitude is the same in QCD and N=4 SYM.
The QCD tree amplitude “secretly” obeys all identities of N=4 supersymmetry:





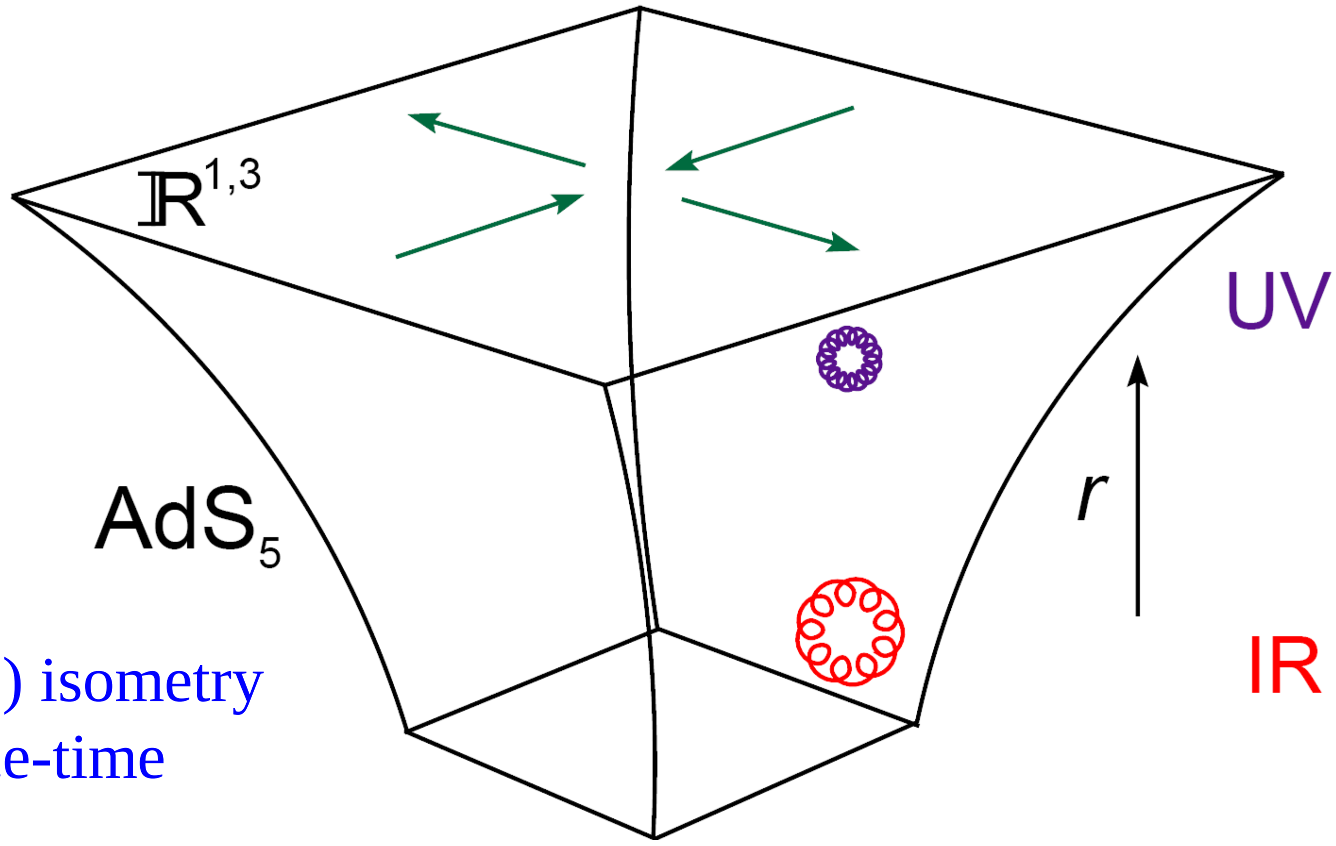
Quantum Symmetries

- Massless QCD has classical scale + conformal symmetry: SO(3,1) → SO(4,2)
- Spoiled at quantum level by nonvanishing β function (asymptotic freedom).
- N=4 SYM has $\beta = 0$ → full (position space) SO(4,2), actually full N=4 superconformal algebra, PSU(2,2|4)

- Planar N=4 SYM also has momentum-space version of SO(4,2) [PSU(2,2|4)] → dual N=4 superconformal invariance

Dual conformal invariance from AdS/CFT + T-duality

Alday, Maldacena, 0705.0303



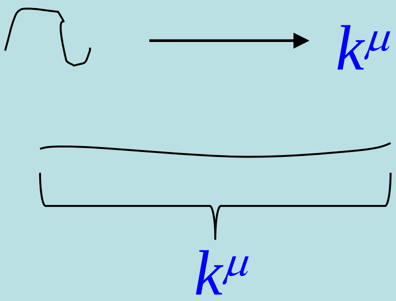
SO(4,2) isometry of space-time

T-duality symmetry of string theory

Alday, Maldacena, 0705.0303

- Exchanges string world-sheet variables σ, τ
- $X^\mu(\tau,\sigma) = x^\mu + k^\mu \tau$ + oscillators

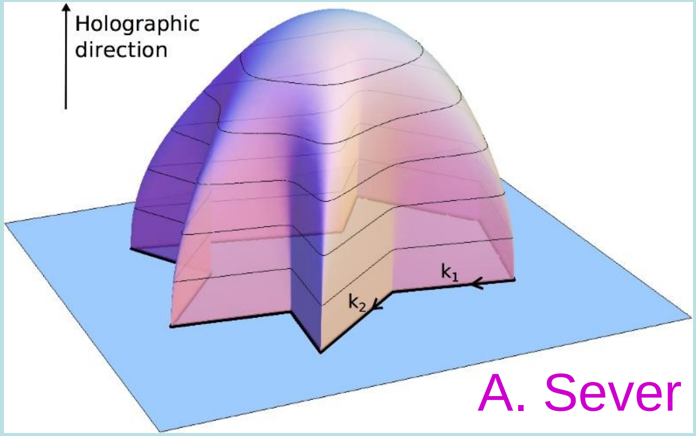
$\rightarrow$ $X^\mu(\tau,\sigma) = x^\mu + k^\mu \sigma$ + oscillators



- Strong coupling limit of planar gauge theory

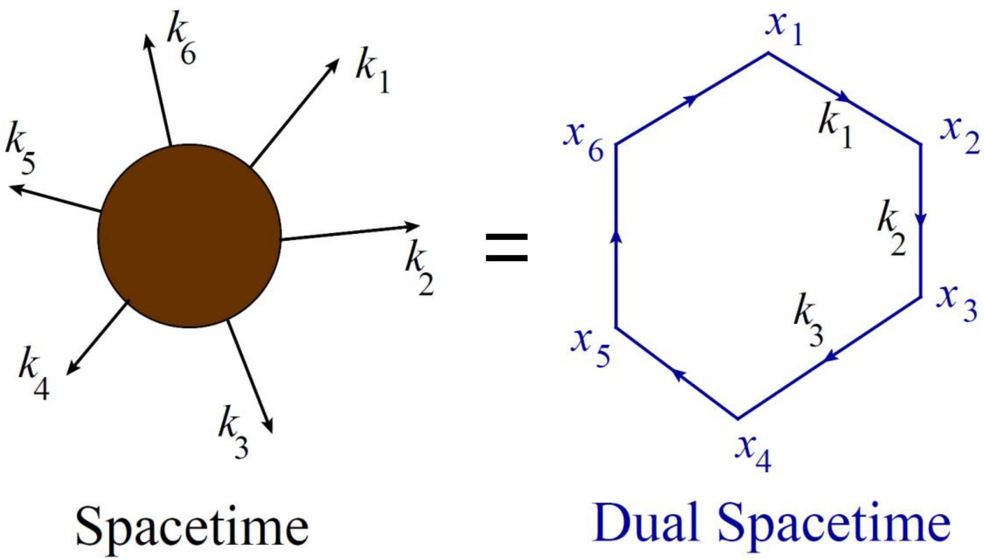
is semi-classical limit of string theory: world-sheet stretches tight around minimal area surface in AdS.

- Boundary determined by momenta of external states: light-like polygon with null edges = momenta k^μ



A. Sever

Amplitudes = Wilson loops



- Polygon vertices x_i are not positions but dual momenta, $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Duality verified to hold at weak coupling too

The [Dual] Conformal Group

SO(4,2) $\supset$ SO(3,1) [rotations+boosts] + translations+dilatations + special-conformal

15 = 3 + 3 + 4 + 1 + 4

- The nontrivial generators are special conformal K^μ
- Correspond to inversion · translation · inversion
- To obtain a [dual] conformally invariant function $f(x_{ij}^2)$ just have to check invariance under inversion,

$$x_i^\mu \to x_i^\mu / x_i^2$$

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$\boxed{u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}}$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for $n = 4,5$

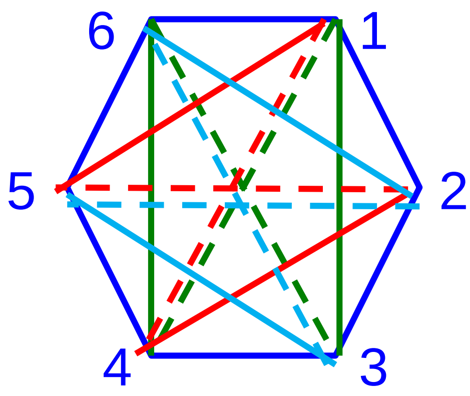
$n = 6$ → precisely 3 ratios:

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

$$v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

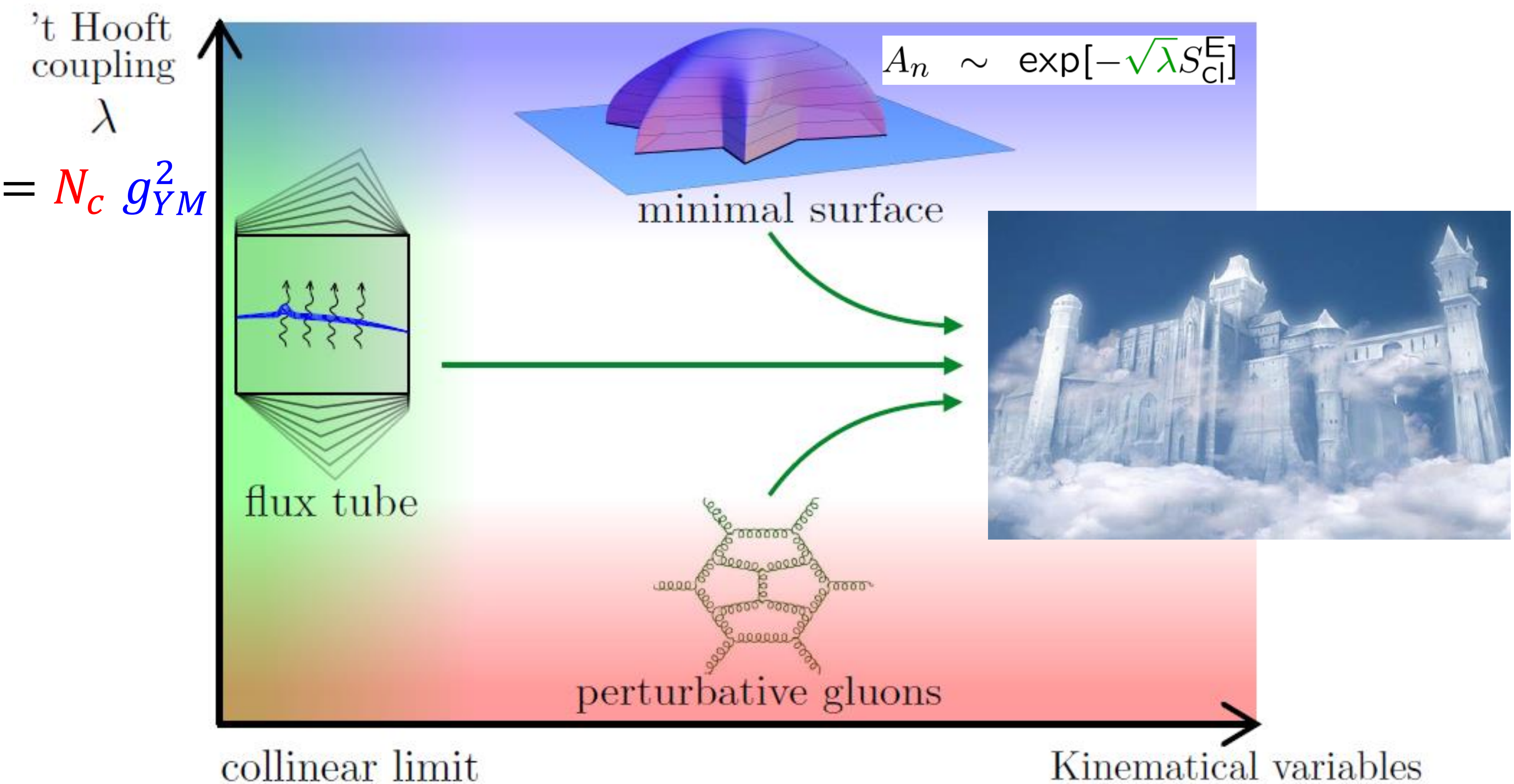
$$w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

In general, $3n$-15 ratios



Solving Planar N=4 SYM Scattering

Images: A. Sever, N. Arkani-Hamed



Integrand = the Amplituhedron

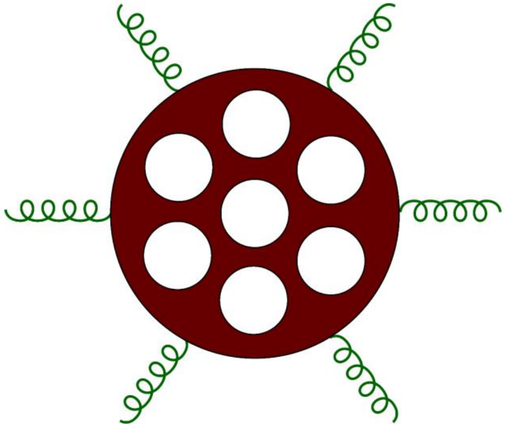
Arkani-Hamed, Trnka,
1312.2007, 1312.7878

Still have to do integrals over the loop momentum

Hexagon function bootstrap

LD, Drummond, Henn, 1108.4461, 1111.1704;
Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 190x.yyyyy and 190x.zzzzz

Use analytical properties of perturbative amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops

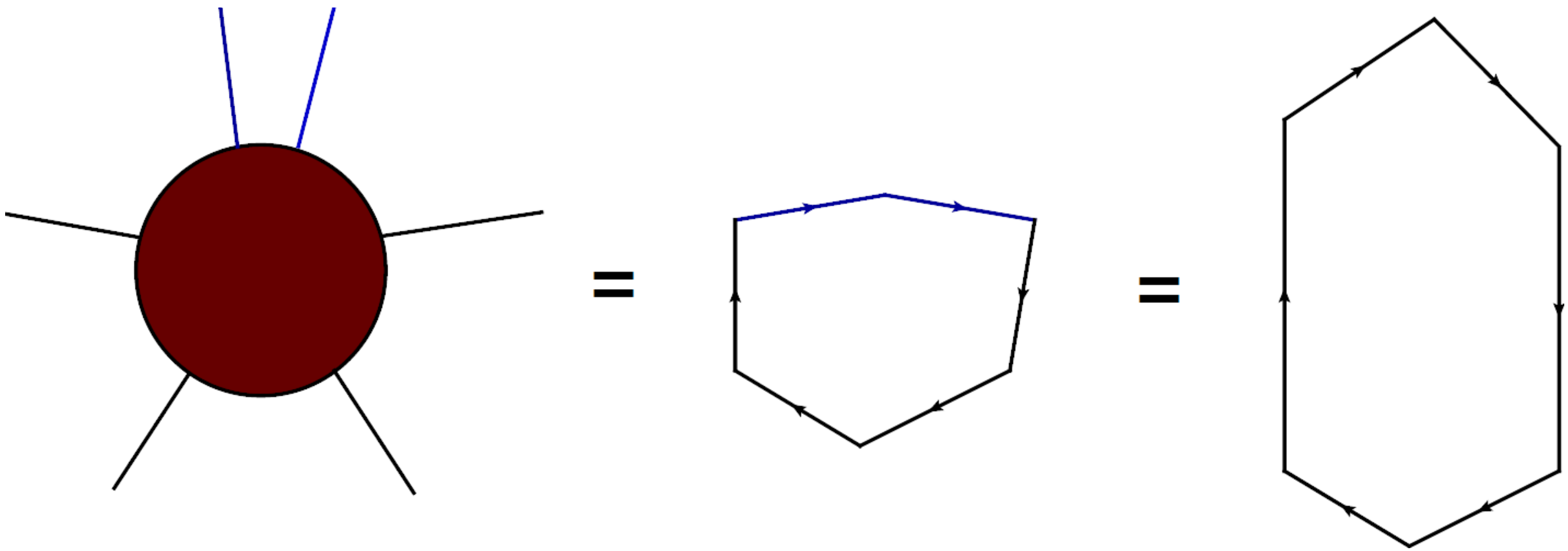


First step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

Rich theoretical “data” mine

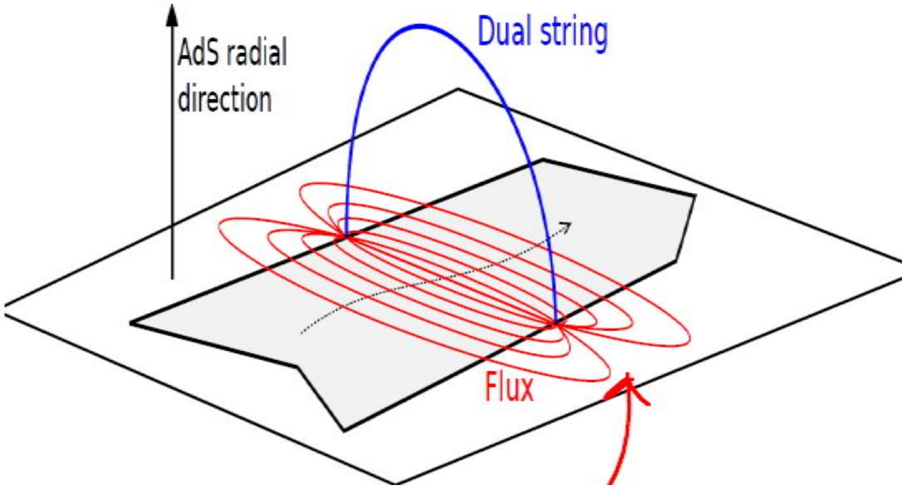
- Rare to have perturbative results to 6 or 7 loops
- Usually high loop order → single numbers
- Here we have analytic functions of 3 variables (6 variables in 7-point case)
- Many limits to study (and exploit)

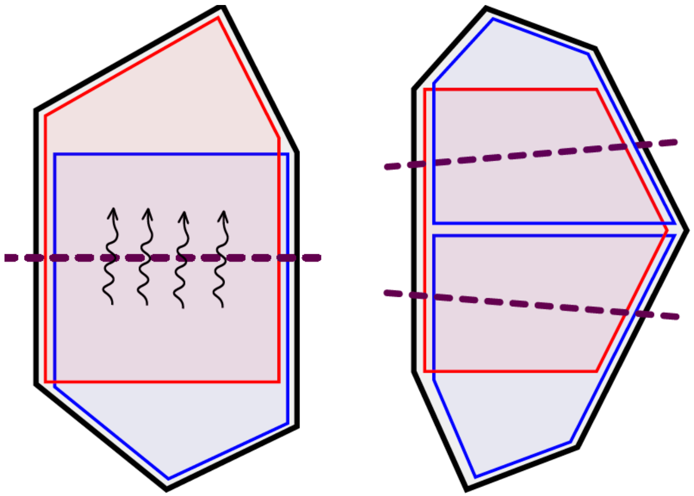
(Near) collinear (OPE) limit



Flux tubes at finite coupling

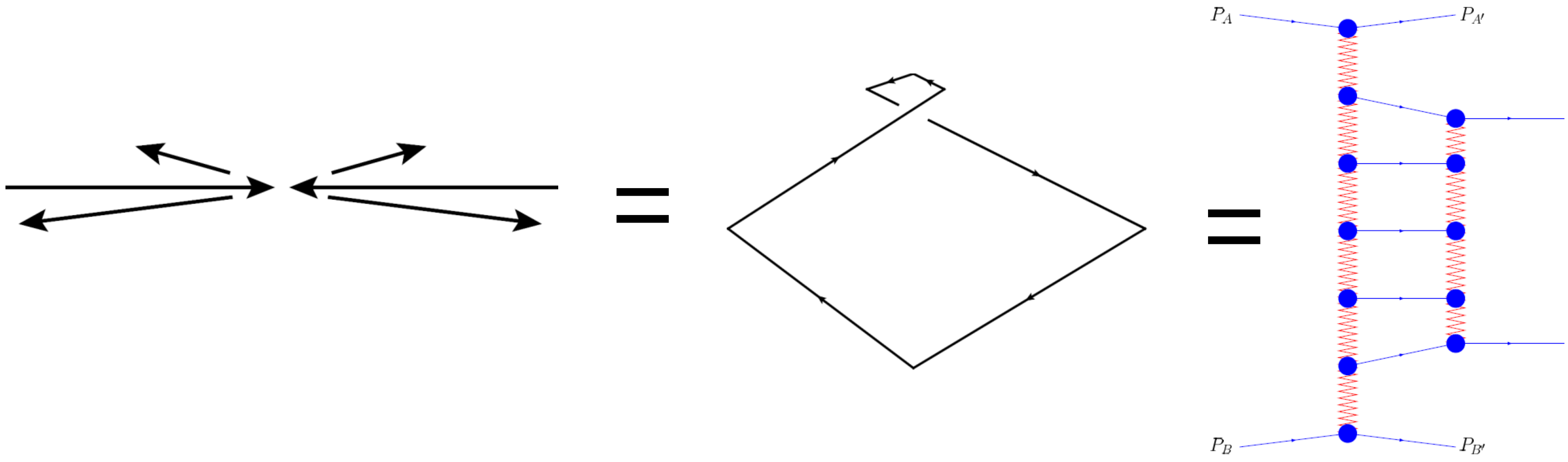
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in ’t Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

Multi-regge limit

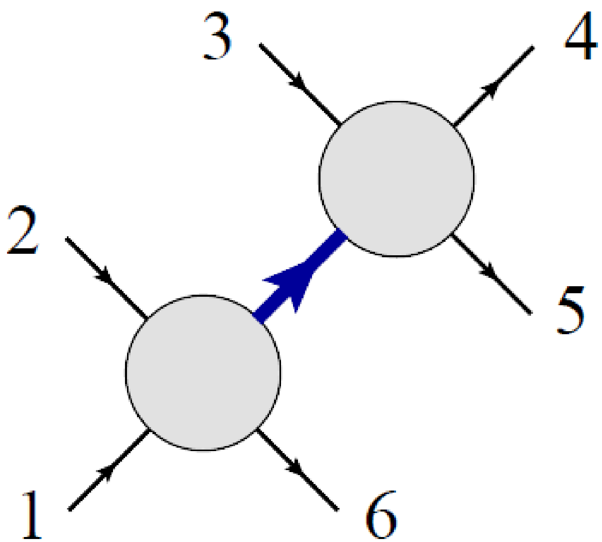


- Amplitude factorizes in Fourier-Mellin space

Bartels, Lipatov, Sabio Vera, 0802.2065, Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186; Pennington, 1209.5357;
Basso, Caron-Huot, Sever, 1407.3766 (analytic continuation from OPE limit);
Broedel, Sprenger, 1512.04963; Lipatov, Prygarin, Schnitzer, 1205.0186;
LD, von Hippel, 1408.1505; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek, 1606.08807

Factorization on multi-particle pole

Bern, Chalmers, hep-ph/9503236; LD, von Hippel, 1408.1505;
Basso, Sever, Vieira (Sever talk at Amplitudes 2015)



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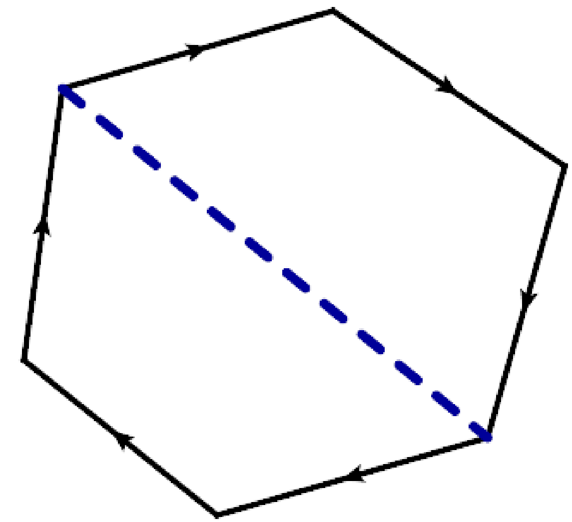


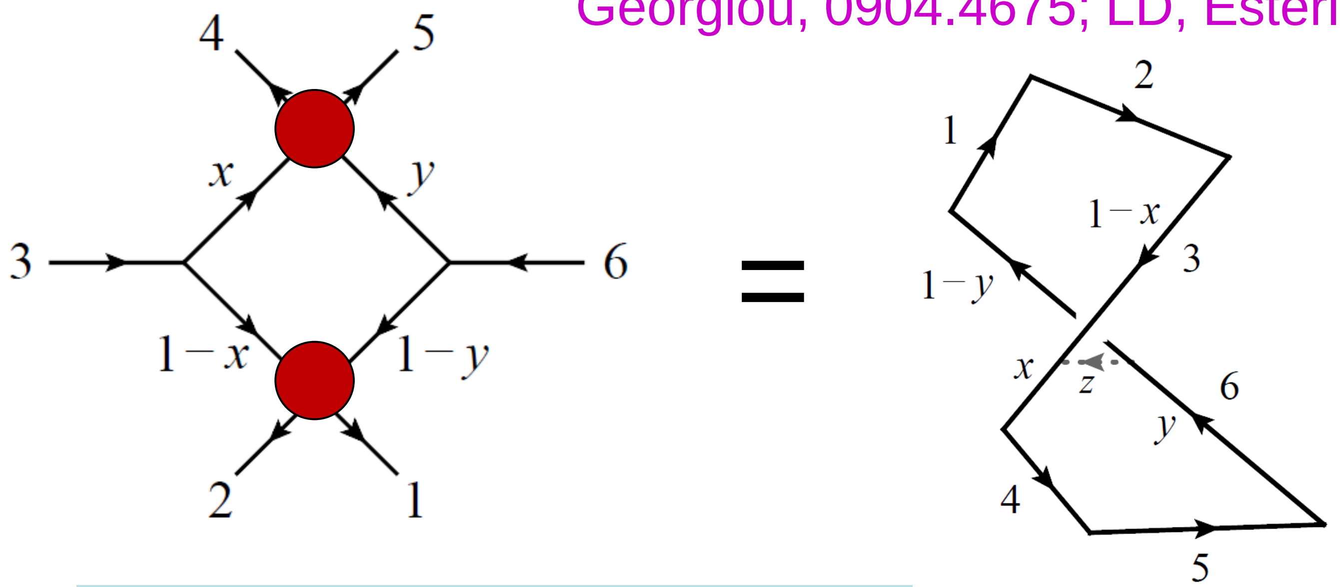
- Virtual Sudakov region, $A \sim \exp[-\ln^2\delta]$,

  $\delta \sim s_{345}$

- Can study to very high accuracy in planar N=4 SYM

Double-parton-scattering-like limit

Georgiou, 0904.4675; LD, Esterlis, 1602.02107

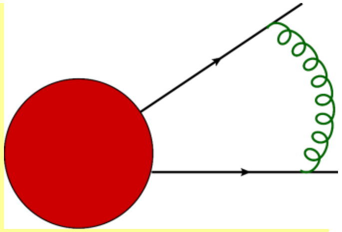


- Self-crossing limit of Wilson loop
- Overlaps MRK limit
- Another Sudakov region
- Singularities ~ Wilson line RGE

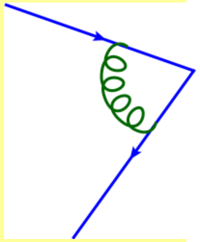
Korchemsky and Korchemskaya hep-ph/9409446

Removing Divergences

- On-shell amplitudes IR divergent
  due to long-range gluons



- Polygonal Wilson loops UV divergent
  at cusps, anomalous dimension γ_K
  planar N=4: Beisert, Eden, Staudacher, hep-th/0610251



- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205,
  Alday, Gaiotto, Maldacena, 0911.4708

- Normalized amplitude is finite,
  (dual) conformally invariant.
- BDS-like → also maintain important relation due to causality (Steinmann)

BDS-like ansatz

$$\frac{\mathcal{A}_6^{\rm BDS-like}}{\mathcal{A}_6^{\rm MHV\,(0)}} = \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon) \frac{1}{2} \hat{M}_6(L\epsilon) + C^{(L)} \right)\right]$$

where $f^{(L)}(\epsilon) = \frac{1}{4}\gamma_K^{(L)} + \epsilon\, \frac{L}{2}\mathcal{G}_0^{(L)} + \epsilon^2\, f_2^{(L)}$ are constants, and

$$\begin{aligned} \hat{M}_6(\epsilon) &= M_6^{\text{1-loop}}(\epsilon) + Y(u,v,w) \\ &= \sum_{i=1}^{6} \left[-\frac{1}{\epsilon^2}(1 - \epsilon \ln s_{i,i+1}) - \ln s_{i,i+1} \ln s_{i+1,i+2} + \frac{1}{2} \ln s_{i,i+1} \ln s_{i+3,i+4} \right] + 6\zeta_2 \end{aligned}$$

$$Y(u,v,w) = \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w) + \tfrac{1}{2}(\ln^2 u + \ln^2 v + \ln^2 w)$$

- BDS-like ansatz contains all IR poles, but **no 3-particle invariants**.
- BDS-like removes Y from BDS
- Y is dual conformally invariant part of one-loop amplitude $M_6^{\text{1-loop}}$ containing all 3-particle invariants

6-point BDS-like normalized amplitude

Define

$$\boxed{\frac{\mathcal{A}_6^{\mathrm{MHV}}}{\mathcal{A}_6^{\mathrm{BDS-like}}} \equiv \mathcal{E}(u, v, w)}$$

No 3-particle invariants in denominator of $\mathcal{E}$
→ Necessary for Steinmann constraints to hold
→ A unique choice (up to constant)

Key “initial” condition

- Two-loop 6-gluon result first computed numerically from both amplitude and Wilson loop sides of duality   Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1466; Drummond, Henn, Korchemsky, Sokatchev, 0803.1466
- Wilson loop side then evaluated analytically $\rightarrow$ 17 pages of [Goncharov] polylogarithms. Del Duca, Duhr, Smirnov, 0911.5332, 1003.1702
- Simplified to a few lines in term of classical polylogs $\mathrm{Li}_n(x)$, demonstrating power of symbol Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Basic bootstrap assumption

- MHV: $\mathcal{E}^{(L)}(u,v,w)$ is a linear combination of weight $2L$ hexagon functions at any loop order L
- NMHV: BDS-like normalized super-amplitude

$$\hat{\mathcal{P}}_{\rm NMHV} \equiv \frac{\mathcal{A}_{\rm NMHV}}{\mathcal{A}_{\rm MHV}^{\rm BDS-like}}$$

Drummond, Henn, Korchemsky, Sokatchev, 0807.1095; LD, von Hippel, McLeod, 1509.08127

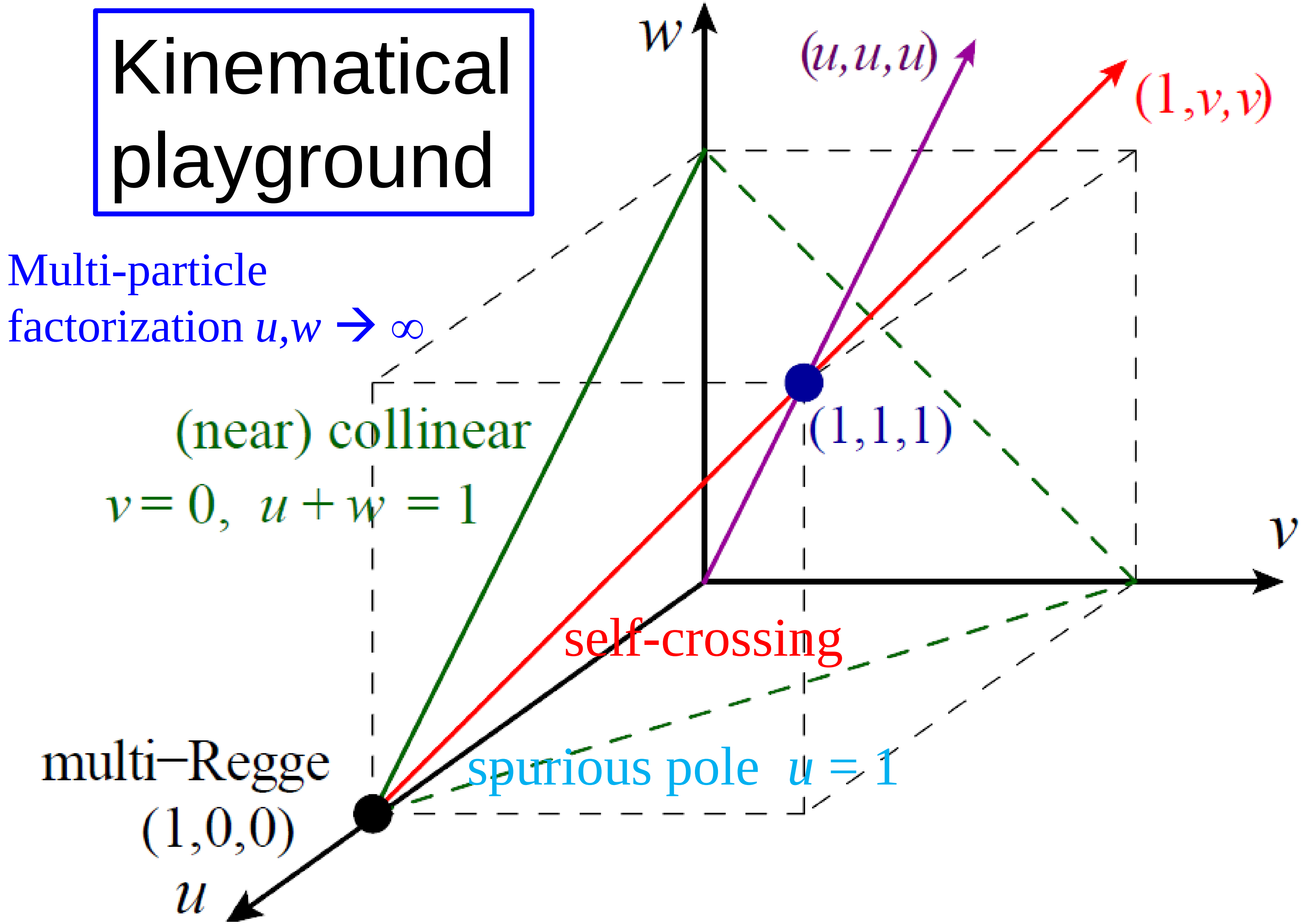
has expansion

$$\begin{aligned}\hat{\mathcal{P}}_{\rm NMHV} \;=\; \frac{1}{2}\Big[&[(1)+(4)]E(u,v,w)+[(2)+(5)]E(v,w,u)+[(3)+(6)]E(w,u,v)\\ &+[(1)-(4)]\tilde{E}(u,v,w)-[(2)-(5)]\tilde{E}(v,w,u)+[(3)-(6)]\tilde{E}(w,u,v)\Big]\end{aligned}$$

Grassmann-containing dual superconformal invariants, (a) = [bcdef]

$E, \tilde{E}$ = hexagon functions

Kinematical playground



Iterated integrals

Chen; Goncharov; Brown

- Generalized polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions f.
- Define by derivatives:

$$d\,f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d\ln s_k$$

$\mathcal{S}$ = finite set of rational expressions, "symbol letters", and

$f^{s_k} \equiv \{n-1,1\}$ component of a "coproduct" Δ

f^{s_k} are also pure functions, weight n-1

- Iterate: $d\,f^{s_k} \;\Rightarrow\; f^{s_j,s_k} \equiv \{n-2,1,1\}$ component
- Symbol = {1,1,…,1} component (maximally iterated)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example 1: Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Subsector of hexagon functions.
- Gen'lize classical polylogs: $\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(t) = -\ln(1-t)$
- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \qquad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Symbol letters: $\mathcal{S} = \{u, 1-u\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$: 2^{2L}

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\text{Li}_n(u) = \int_0^u \frac{dt}{t} \text{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\text{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.
  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$

Example 2: Single-valued harmonic polylogarithms of one complex variable

Brown, C. R. Acad. Sci. Paris, Ser. I 338 (2004) 527

- Also a subsector of hexagon functions, in the "multi-Regge limit"
- 4 symbol letters: $\mathcal{S} = \{z, 1-z, \bar{z}, 1-\bar{z}\}$
- But also require function to be real analytic in $(z, \bar{z}) \in \mathbb{C} - \{0, 1\}$
- Constrains the first entry of the symbol to be $z\bar{z} \leftrightarrow \ln|z|^2$ or $(1-z)(1-\bar{z}) \leftrightarrow \ln|1-z|^2$
- Brown: One SVHPL for each HPL
- Powerful constraint: $4^{2L} \rightarrow 2^{2L}$ functions

Symbol of $R_6^{(2)}(u,v,w)$

GSVV, 1006.5703

$$
\begin{aligned}
-8\,\mathcal{S}[R_6^{(2)}] \;=\; & u \otimes (1-u) \otimes \frac{u}{(1-u)^2} \otimes \frac{u}{1-u} \\
& + 2\,(u \otimes v + v \otimes u) \otimes \frac{w}{1-v} \otimes \frac{u}{1-u} \\
& + 2\,v \otimes \frac{w}{1-v} \otimes u \otimes \frac{u}{1-u} \\
& + u \otimes (1-u) \otimes y_u y_v y_w \otimes y_u y_v y_w \\
& - 2\,u \otimes v \otimes y_w \otimes y_u y_v y_w
\end{aligned}
$$

$$+ \text{ 5 permutations of } (u, v, w)$$

Nine hexagon symbol letters

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

- y_i not independent of u_i :

$$y_u \equiv \frac{u - z_+}{u - z_-} \;, \; \ldots$$ where $$z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right]$$ $$\Delta = (1 - u - v - w)^2 - 4uvw$$

- Suppose number of functions went like $\mathbf{5^{2L}}$
  Then $\mathbf{L=6 \rightarrow 5^{12} > 2 \cdot 10^8}$ **(!!!)**
- Fortunately it is way better than this:
  need only **3,692** weight 12 functions
- Why? Consistency of mixed partial derivatives, plus certain (mostly) physical constraints

Branch cut condition

- All massless particles → all branch cuts start at origin in

$$s_{i,i+1}, \quad s_{i,i+1,i+2}$$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad \text{or } v \text{ or } w$$

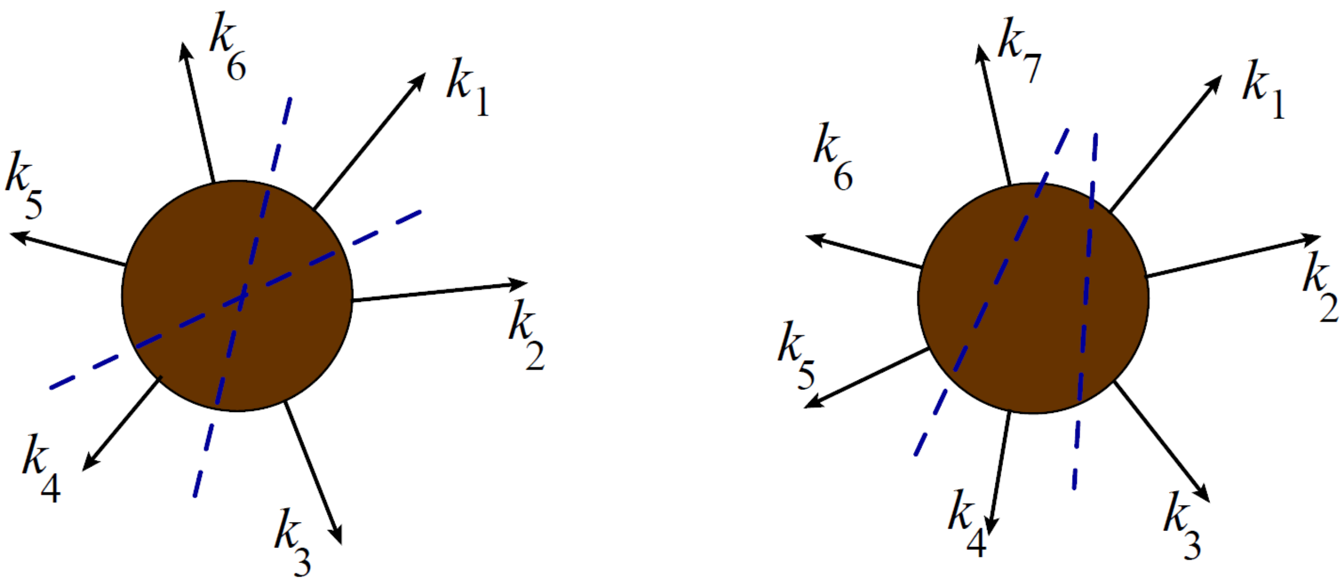
→ First symbol entry $\in \{u, v, w\}$ GMSV, 1102.0062

- Powerful constraint: At weight 8 (four loops) we would have 1,675,553 functions without it; exactly 6,916 with it.
- But almost all of the 6,916 functions are still unphysical.

Steinmann relations

Steinmann, Helv. Phys. Acta (1960)   Bartels, Lipatov, Sabio Vera, 0802.2065

- Amplitudes should not have overlapping branch cuts:



Not Allowed   Allowed

$$\boxed{\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0}$$

Steinmann relations (cont.)

S. Caron-Huot, LD, M. von Hippel, A. McLeod, 1609.00669

$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right]=0$$

+ cyclic conditions

$$u=\frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad v=\frac{s_{23}s_{56}}{s_{234}s_{123}} \qquad w=\frac{s_{61}s_{34}}{s_{345}s_{234}}$$

$\ln^2 u$ $\ln^2 \frac{uv}{w}$

NO **OK**

$$\frac{uv}{w}=\frac{s_{12}s_{23}s_{45}s_{56}}{s_{34}s_{61}s_{123}^2}$$

First two entries restricted to 6 out of 9:

$\mathrm{Li}_2(1-1/u)$ $\mathrm{Li}_2(1-1/v)$ $\mathrm{Li}_2(1-1/w)$

$\ln^2\frac{uv}{w}$ $\ln^2\frac{vw}{u}$ $\ln^2\frac{wu}{v}$

Analogous constraints for $n=7$

LD, J. Drummond, T. Harrington, A. McLeod, G. Papathanasiou, M. Spradlin, 1612.08976

Minimal Steinmann space

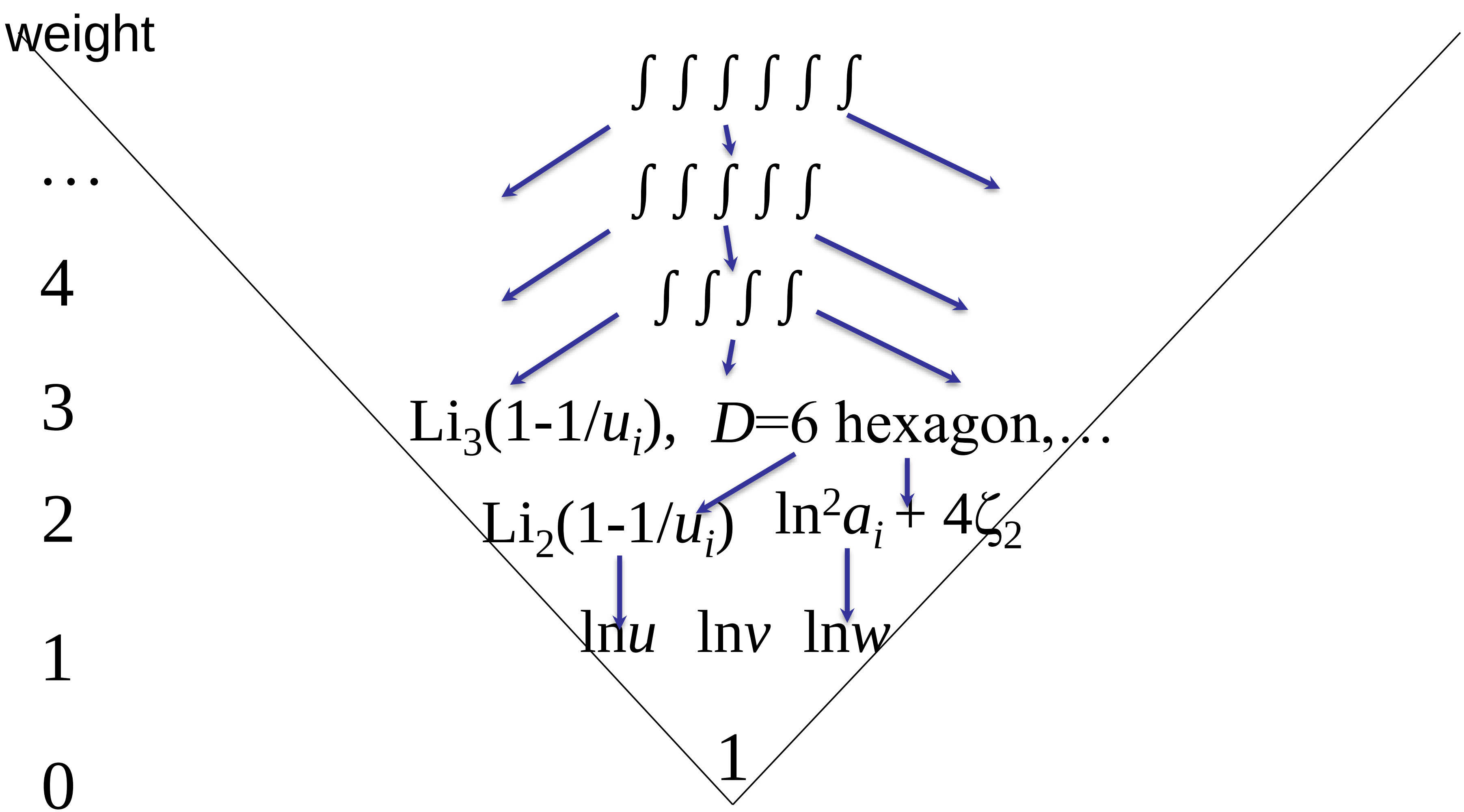
- At higher weights, we find that all zeta values are not independent elements of the basis, except $\zeta_4, \zeta_6, \zeta_8, \zeta_{10}, \cdots$
- That is,

$$\zeta_2, \zeta_3, \zeta_5, \zeta_2\,\zeta_3, \zeta_3^2, \zeta_7, \zeta_2\zeta_5, \zeta_3\zeta_4, \zeta_{5,3}, \zeta_3\zeta_5, \zeta_2\zeta_3^2, \cdots$$

are absorbable into other functions

- There are also additional Steinmann constraints, restricting pairs of adjacent entries, but deeper into the symbol than the first two entries.

Heuristic view of the space



Master Table

(MHV,NMHV): parameters left in $(\mathcal{E}^{(L)},\ E^{(L)} \& \tilde{E}^{(L)})$

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. All functions	(6,6)	(25,27)	(92,105)	(313,372)	(991,1214)	(2951,3692?)
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear limit	(0,0)	(0,0)	$(0^*, 0^*)$	$(0^*, 2^*)$	$(1^{*3}, 5^{*3})$	$(6^{*2}, 17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*, 0^*)$	$(1^{*2}, 2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*, 0^*)$	$(1^*, 0^*)$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. all MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. $T^2 F^2 \ln^4 T$ OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
12. all $T^2 F^2$ OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

(0,0) → amplitude uniquely determined

Also have MHV at $L = 7$

Properties of the Amplitude

- Having determined the 6-point amplitudes to 6 (7) loops for NMHV (MHV), we study their numerical and number-theoretic properties.
- What kinds of nested sum (MZV's,…) appear?
- Numerics are difficult for generic (u,v,w), feasible on "simple lines" like $(u,u,1)$, $(u,1,1)$, (u,u,u).
- Planar N=4 SYM should have finite radius of convergence of perturbative expansion (unlike QCD, QED, whose perturbative series are asymptotic).
- For BES solution to cusp anomalous dimension, using coupling $g^2 = \frac{\lambda}{16\,\pi^2}$, radius is $\frac{1}{16}$
- Ratio of successive coefficients $\gamma_K^{(L)} / \gamma_K^{(L-1)} \to -16$

At $(u,v,w) = (1,1,1)$, amplitude → MZVs

Allowed MZV's obey a Galois "co-action" principle, restricting the combinations that can appear

Brown, Panzer, Schnetz

MHV

$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] - 96\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

NMHV

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = -\frac{36271}{9}\,\zeta_8 - 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] + 132\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

6 loops at (1,1,1)

MHV

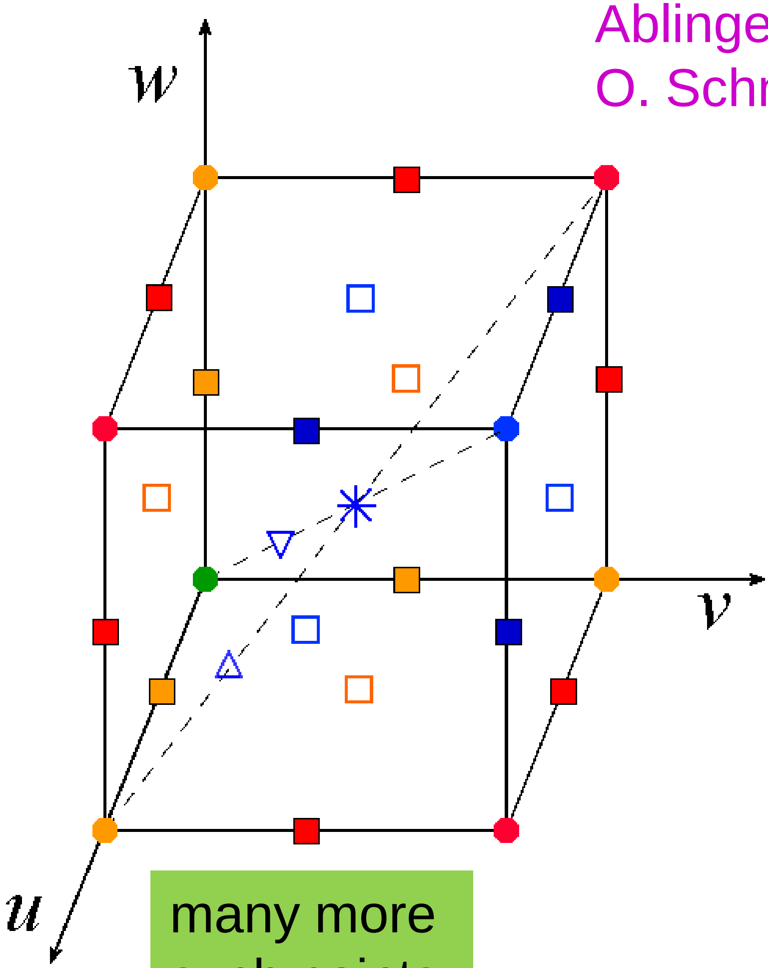
$$\begin{aligned}\mathcal{E}^{(6)}(1,1,1) = &-\frac{2273108143}{6219}\zeta_{12} + \frac{260}{3}\Big[140\zeta_5\zeta_7 - 56\zeta_2\zeta_3\zeta_7 - 10\zeta_2(\zeta_5)^2 - 60\zeta_4\zeta_3\zeta_5 + 49\zeta_6(\zeta_3)^2\Big]\\ &+ 384\Big[\zeta_2\zeta_{7,3} + 14\zeta_2\zeta_3\zeta_7 + 3\zeta_2(\zeta_5)^2 - 7\zeta_6(\zeta_3)^2\Big]\\ &+ 120\Big[4\zeta_4\zeta_{5,3} + 20\zeta_4\zeta_3\zeta_5 - 7\zeta_6(\zeta_3)^2\Big]\\ &+ \frac{5392}{3}\Big[\zeta_{9,3} + 27\zeta_3\zeta_9 + 20\zeta_5\zeta_7 - 2\zeta_2\zeta_3\zeta_7 - \zeta_2(\zeta_5)^2 - 6\zeta_4\zeta_3\zeta_5 - 5\zeta_6(\zeta_3)^2\Big]\end{aligned}$$

NMHV

$$\begin{aligned}E^{(6)}(1,1,1) = &\frac{5066300219}{6219}\zeta_{12} - \frac{344}{3}\Big[140\zeta_5\zeta_7 - 56\zeta_2\zeta_3\zeta_7 - 10\zeta_2(\zeta_5)^2 - 60\zeta_4\zeta_3\zeta_5 + 49\zeta_6(\zeta_3)^2\Big]\\ &- 528\Big[\zeta_2\zeta_{7,3} + 14\zeta_2\zeta_3\zeta_7 + 3\zeta_2(\zeta_5)^2 - 7\zeta_6(\zeta_3)^2\Big]\\ &+ 60\Big[4\zeta_4\zeta_{5,3} + 20\zeta_4\zeta_3\zeta_5 - 7\zeta_6(\zeta_3)^2\Big]\\ &- \frac{9952}{3}\Big[\zeta_{9,3} + 27\zeta_3\zeta_9 + 20\zeta_5\zeta_7 - 2\zeta_2\zeta_3\zeta_7 - \zeta_2(\zeta_5)^2 - 6\zeta_4\zeta_3\zeta_5 - 5\zeta_6(\zeta_3)^2\Big]\end{aligned}$$

Menagerie of "cyclotomic" polylogs at unity

Ablinger, Blumlein, Schneider, 1105.6063, 1310.5645;
O. Schnetz, **HyperlogProcedures**



- ● MZVs
- ■, □ Alternating sums
- ✳ 4th roots of unity
- ▽, △ 6th roots of unity

finite
1 variable singular
2 variables singular
3 variables singular

Galois co-action principle applies to entire function space at every point at which we have checked it!!

many more such points

e.g.

$$u = v = w, \qquad y_u = y_v = y_w = y,$$
$$u = \frac{y}{(1+y)^2} \qquad 1 - u = \frac{1+y+y^2}{(1+y)^2}$$

On the line $(u,u,1)$, everything collapses to HPLs of u.
In a linear representation,
and a very compressed notation,

$$H_1^u H_{2,1}^u = H_1^u H_{0,1,1}^u = 3H_{0,1,1,1}^u + H_{1,0,1,1}^u \to 3h_7^{[4]} + h_{11}^{[4]}$$

2 and 3 loop answers:

$$\begin{aligned}
R_6^{(2)}(u,u,1) &= h_1^{[4]} - h_3^{[4]} + h_9^{[4]} - h_{11}^{[4]} - \frac{5}{2}\zeta_4\,, \\
R_6^{(3)}(u,u,1) &= -3h_1^{[6]} + 5h_3^{[6]} + \frac{3}{2}h_5^{[6]} - \frac{9}{2}h_7^{[6]} - \frac{1}{2}h_9^{[6]} - \frac{3}{2}h_{11}^{[6]} - h_{13}^{[6]} - \frac{3}{2}h_{17}^{[6]} \\
&\quad + \frac{3}{2}h_{19}^{[6]} - \frac{1}{2}h_{21}^{[6]} - \frac{3}{2}h_{23}^{[6]} - 3h_{33}^{[6]} + 5h_{35}^{[6]} + \frac{3}{2}h_{37}^{[6]} - \frac{9}{2}h_{39}^{[6]} \\
&\quad - \frac{1}{2}h_{41}^{[6]} - \frac{3}{2}h_{43}^{[6]} - h_{45}^{[6]} - \frac{3}{2}h_{49}^{[6]} + \frac{3}{2}h_{51}^{[6]} - \frac{1}{2}h_{53}^{[6]} - \frac{3}{2}h_{55}^{[6]} \\
&\quad + \zeta_2\left[-h_1^{[4]} + 3h_3^{[4]} + 2h_5^{[4]} - h_9^{[4]} + 3h_{11}^{[4]} + 2h_{13}^{[4]}\right] \\
&\quad + \zeta_4\left[-2h_1^{[2]} - 2h_3^{[2]}\right] + \zeta_3^2 + \frac{413}{24}\zeta_6\,,
\end{aligned}$$

4 loop answer →

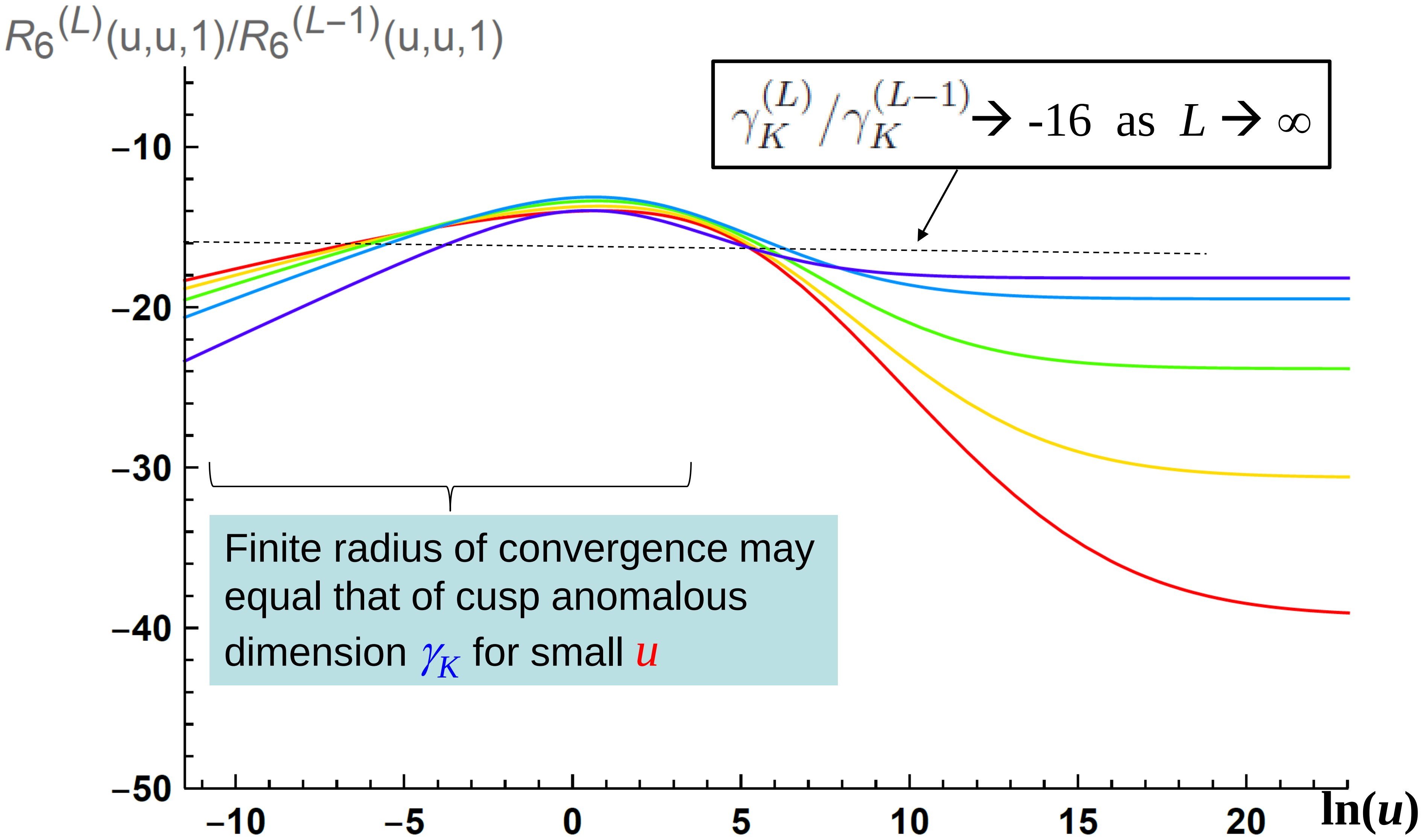
5 loop answer is several pages

6 loop answer is a novel!

7 loop answer has ~ 10,000 terms

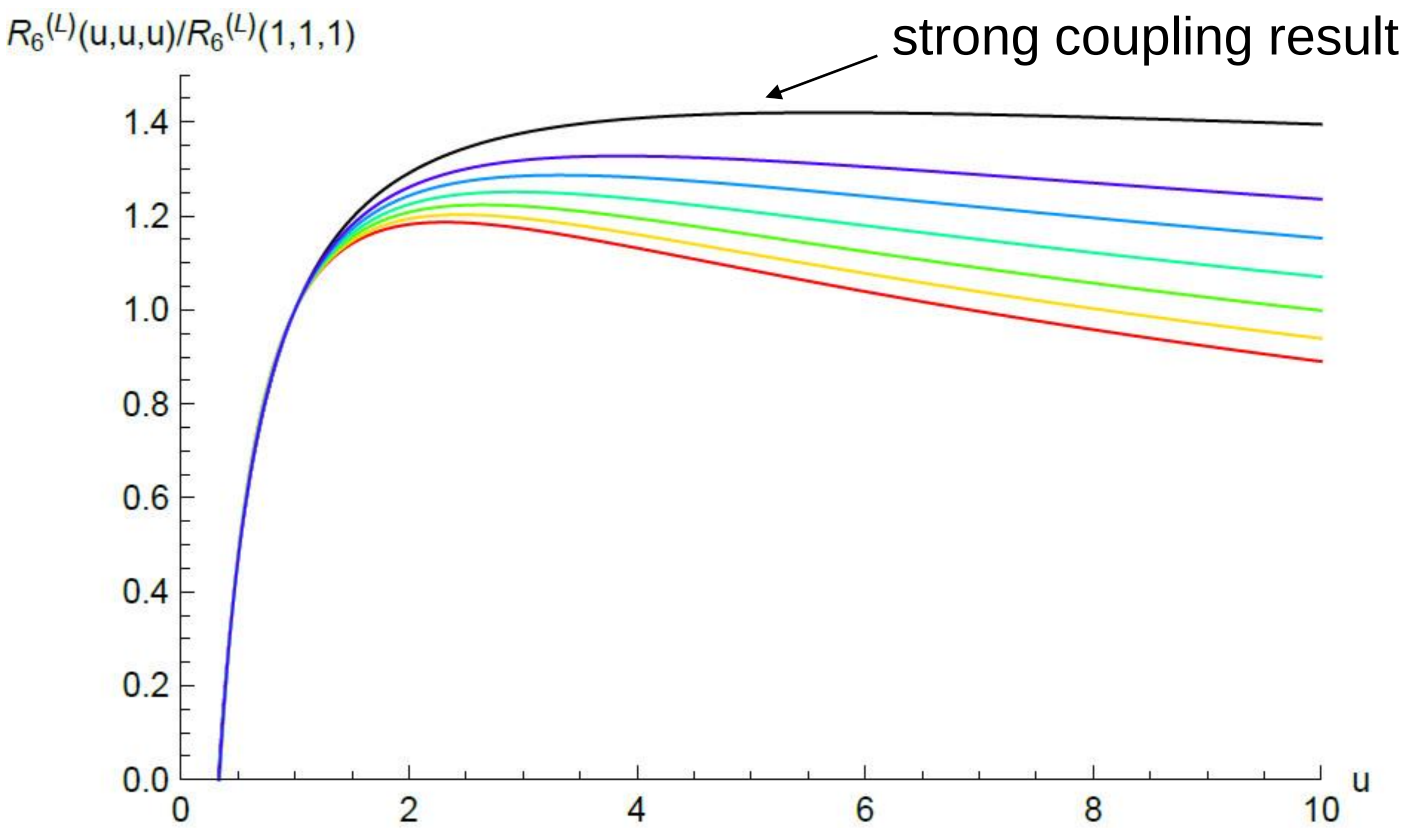
$$\begin{aligned}
R_6^{(4)}(u,u,1) = {} & 15h_1^{[8]} - 41h_3^{[8]} - \frac{31}{2}h_5^{[8]} + \frac{105}{2}h_7^{[8]} - \frac{7}{2}h_9^{[8]} + \frac{53}{2}h_{11}^{[8]} + 12h_{13}^{[8]} - 42h_{15}^{[8]} \\
& + \frac{5}{2}h_{17}^{[8]} + \frac{11}{2}h_{19}^{[8]} + \frac{9}{2}h_{21}^{[8]} - \frac{41}{2}h_{23}^{[8]} + h_{25}^{[8]} - 13h_{27}^{[8]} - 7h_{29}^{[8]} - 5h_{31}^{[8]} \\
& + 6h_{33}^{[8]} - 11h_{35}^{[8]} - 3h_{37}^{[8]} + 3h_{39}^{[8]} - 4h_{43}^{[8]} - 4h_{45}^{[8]} - 11h_{47}^{[8]} + \frac{3}{2}h_{49}^{[8]} - \frac{3}{2}h_{51}^{[8]} \\
& - 3h_{53}^{[8]} - 5h_{55}^{[8]} + \frac{3}{2}h_{57}^{[8]} - \frac{3}{2}h_{59}^{[8]} + 9h_{65}^{[8]} - 25h_{67}^{[8]} - 9h_{69}^{[8]} + 27h_{71}^{[8]} - 2h_{73}^{[8]} \\
& + 9h_{75}^{[8]} + 2h_{77}^{[8]} - 23h_{79}^{[8]} + 2h_{81}^{[8]} - h_{85}^{[8]} - 8h_{87}^{[8]} + 2h_{89}^{[8]} - 3h_{91}^{[8]} + \frac{5}{2}h_{97}^{[8]} \\
& - \frac{7}{2}h_{99}^{[8]} - \frac{1}{2}h_{101}^{[8]} + \frac{5}{2}h_{103}^{[8]} + \frac{1}{2}h_{105}^{[8]} + \frac{1}{2}h_{107}^{[8]} + \frac{1}{2}h_{109}^{[8]} - \frac{5}{2}h_{111}^{[8]} + 15h_{129}^{[8]} \\
& - 41h_{131}^{[8]} - \frac{31}{2}h_{133}^{[8]} + \frac{105}{2}h_{135}^{[8]} - \frac{7}{2}h_{137}^{[8]} + \frac{53}{2}h_{139}^{[8]} + 12h_{141}^{[8]} - 42h_{143}^{[8]} \\
& + \frac{5}{2}h_{145}^{[8]} + \frac{11}{2}h_{147}^{[8]} + \frac{9}{2}h_{149}^{[8]} - \frac{41}{2}h_{151}^{[8]} + h_{153}^{[8]} - 13h_{155}^{[8]} - 7h_{157}^{[8]} \\
& - 5h_{159}^{[8]} + 6h_{161}^{[8]} - 11h_{163}^{[8]} - 3h_{165}^{[8]} + 3h_{167}^{[8]} - 4h_{171}^{[8]} - 4h_{173}^{[8]} \\
& - 11h_{175}^{[8]} + \frac{3}{2}h_{177}^{[8]} - \frac{3}{2}h_{179}^{[8]} - 3h_{181}^{[8]} - 5h_{183}^{[8]} + \frac{3}{2}h_{185}^{[8]} - \frac{3}{2}h_{187}^{[8]} \\
& + 9h_{193}^{[8]} - 25h_{195}^{[8]} - 9h_{197}^{[8]} + 27h_{199}^{[8]} - 2h_{201}^{[8]} + 9h_{203}^{[8]} + 2h_{205}^{[8]} - 23h_{207}^{[8]} \\
& + 2h_{209}^{[8]} - h_{213}^{[8]} - 8h_{215}^{[8]} + 2h_{217}^{[8]} - 3h_{219}^{[8]} + \frac{5}{2}h_{225}^{[8]} - \frac{7}{2}h_{227}^{[8]} - \frac{1}{2}h_{229}^{[8]} \\
& + \frac{5}{2}h_{231}^{[8]} + \frac{1}{2}h_{233}^{[8]} + \frac{1}{2}h_{235}^{[8]} + \frac{1}{2}h_{237}^{[8]} - \frac{5}{2}h_{239}^{[8]} \\
& + \zeta_2\Big[2h_1^{[6]} - 14h_3^{[6]} - \frac{15}{2}h_5^{[6]} + \frac{37}{2}h_7^{[6]} - \frac{5}{2}h_9^{[6]} + \frac{25}{2}h_{11}^{[6]} + 7h_{13}^{[6]} - \frac{1}{2}h_{17}^{[6]} \\
& \qquad + \frac{5}{2}h_{19}^{[6]} + \frac{7}{2}h_{21}^{[6]} + \frac{9}{2}h_{23}^{[6]} - 3h_{25}^{[6]} + 3h_{27}^{[6]} + 2h_{33}^{[6]} - 14h_{35}^{[6]} - \frac{15}{2}h_{37}^{[6]} \\
& \qquad + \frac{37}{2}h_{39}^{[6]} - \frac{5}{2}h_{41}^{[6]} + \frac{25}{2}h_{43}^{[6]} + 7h_{45}^{[6]} - \frac{1}{2}h_{49}^{[6]} + \frac{5}{2}h_{51}^{[6]} + \frac{7}{2}h_{53}^{[6]} \\
& \qquad + \frac{9}{2}h_{55}^{[6]} - 3h_{57}^{[6]} + 3h_{59}^{[6]}\Big] \\
& + \zeta_4\left[\frac{15}{2}h_1^{[4]} - \frac{55}{2}h_3^{[4]} - \frac{41}{2}h_5^{[4]} + \frac{15}{2}h_9^{[4]} - \frac{55}{2}h_{11}^{[4]} - \frac{41}{2}h_{13}^{[4]}\right] \\
& + \left(\zeta_2\zeta_3 - \frac{5}{2}\zeta_5\right)\left[h_3^{[3]} + h_7^{[3]}\right] - \left(\zeta_3^2 - \frac{73}{4}\zeta_6\right)\left[h_1^{[2]} + h_3^{[2]}\right] \\
& - \frac{3}{2}\zeta_2\zeta_3^2 - \frac{5}{2}\zeta_3\zeta_5 - \frac{471}{4}\zeta_8 + \frac{3}{2}\zeta_{5,3}\,.
\end{aligned}$$

Numerical values on $(u,u,1)$



Numerical values on (u,u,u)

Alday, Gaiotto, Maldacena, 0911.4708



- Why such an **amazing proportionality** of **each** perturbative coefficient at small u, and also with the strong coupling result???

Beyond 6 gluons

- Cluster algebras provide strong clues to right polylogarithmic function space
  Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Foster, Gurdogan, 1710.10953
- Symbol of 3-loop MHV 7-point amplitude bootstrapped first. 42 letter alphabet. No need for OPE constraints
  Drummond, Papathanasiou, Spradlin 1412.3763
- With Steinmann relations, could go to 4-loop MHV and 3-loop non-MHV LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976
- And now (extended Steinmann) 4-loop NMHV
  Drummond, Gurdogan, Papathanasiou, 1812.04640
- Still need to go from symbols → functions!

Summary & Outlook

- Steinmann hexagon (heptagon) polylogarithmic functions provide solution space for planar N=4 SYM amplitudes/WLs over full kinematical phase space, for 6 or 7 gluons, both MHV and NMHV, to high loop orders.
- To determine answer uniquely, need very little additional information.
- ~ cusp finite radius of convergence at small u.
- Interesting number-theoretic restrictions.
- Can we go to finite coupling for generic kinematics? What are the right finite-coupling functions? Clues from OPE/integrability?
- How many of these lessons can we apply to QCD?

Extra Slides

Cosmic Galois Theory

Studies the symmetries of 'periods' (integrals of rational functions over domains given by rational inequalities)

- The space of functions appearing in the six-point amplitude is (conjecturally) stable under the coaction
- This property can be formulated as a 'coaction principle'

$$\Delta \mathcal{H}^{\text{hex}} \subset \mathcal{H}^{\text{hex}} \otimes \mathcal{H}^{\pi}$$

  which incorporates the branch cut condition, but also constrains the constants that can show up
- This can be alternately formulated in terms of the action of the 'cosmic Galois group' C which is dual to this coaction

$$C \times \mathcal{H}^{\text{hex}} \to \mathcal{H}^{\text{hex}}$$

Cosmic Galois Theory

- The Lie algebra of C includes a set of elements ∂_{2m+1} that act on the zeta values as

$$\partial_{2m+1}\zeta_{2n+1} = \delta_{m,n}$$

  and that satisfy the Leibniz rule. So, for example,

$$\partial_3(\zeta_7\zeta_3^2) = 2\zeta_7\zeta_3$$

- There is no ∂_2, because including even zeta values on both sides of the coaction leads to contradictions

- These operators also act nontrivially on multiple zeta values

  Brown, arXiv:1102.1310 [math.NT]

MZV restrictions

Weight	All MZVs	$\mathcal{H}^{\rm hex}(1,1,1)$	$\mathcal{H}^{\rm hex}$ indep.
0	1	1	1
1	—	—	—
2	ζ_2	ζ_2	—
3	ζ_3	—	—
4	ζ_4	ζ_4	ζ_4
5	$\zeta_5,\ \zeta_2\zeta_3$	$5\zeta_5 - 2\zeta_2\zeta_3$	—
6	$\zeta_6,\ (\zeta_3)^2$	ζ_6	ζ_6
7	$\zeta_7,\ \zeta_2\zeta_5,\ \zeta_3\zeta_4$	$7\zeta_7 - \zeta_2\zeta_5 - 3\zeta_3\zeta_4$	—
8	$\zeta_8,\ \zeta_{5,3},\ \zeta_3\zeta_5,\ \zeta_2(\zeta_3)^2$	$\zeta_8,\ \zeta_{5,3} + 5\zeta_3\zeta_5 - \zeta_2(\zeta_3)^2$	ζ_8

∂_3

∂_5

∂_3

BDS-like-prime

- To get the amplitudes into the minimal space requires, starting at 3 loops, one more redefinition of the BDS ansatz, by a multi-loop constant ρ :

$$\mathcal{A}_6^{\mathrm{BDS-like}'} = \mathcal{A}_6^{\mathrm{BDS-like}} \times \rho$$

$$\begin{aligned}\rho \;=\; & 1 + (\zeta_3)^2\, a^3 - 10\zeta_3\zeta_5\, a^4 \\ & + \left[-\zeta_4(\zeta_3)^2 + \frac{105}{2}\zeta_3\zeta_7 + \frac{57}{2}(\zeta_5)^2\right] a^5 \\ & + \left[\frac{25}{4}\zeta_6(\zeta_3)^2 + 7\zeta_4\zeta_3\zeta_5 - 294\zeta_3\zeta_9 - \frac{651}{2}\zeta_5\zeta_7\right] a^6 + \cdots\end{aligned}$$

$$a = \frac{N_c g^2}{8\pi^2} = \frac{\lambda}{8\pi^2}$$

- What is the meaning of ρ ?

7 gluons: 6 variables, 42 letters

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}\,, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle}\,,$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}\,, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}\,, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

- plus cyclic, $i \rightarrow i+1$ (mod 7), $a_{ji} \rightarrow a_{j,i+1}$ (6 x 7 = 42)

Number of (first 2 entry) Steinmann heptagon symbols

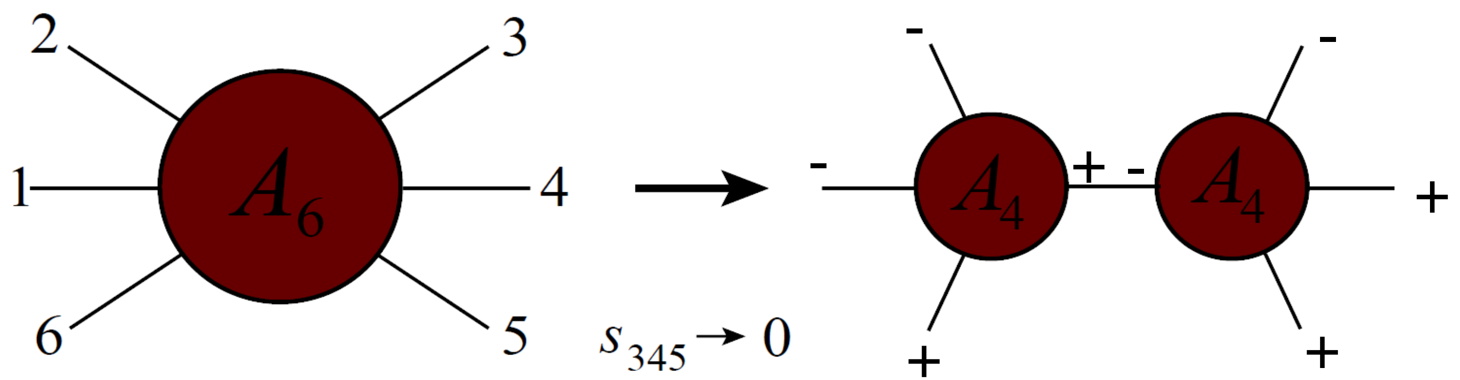
Weight $k =$	1	2	3	4	5	6	7	$7''$
parity +, flip +	4	16	48	154	467	1413	4163	3026
parity +, flip −	3	12	43	140	443	1359	4063	2946
parity −, flip +	0	0	3	14	60	210	672	668
parity −, flip −	0	0	3	14	60	210	672	669
Total	7	28	97	322	1030	3192	9570	7309

Table 1. Number of Steinmann heptagon symbols at weights 1 through 7, and those satisfying the MHV next-to-final entry condition at weight 7.

Enough to get symbols of 4 loop MHV & 3 loop NMHV amplitude.
Even less boundary data needed: just well-defined collinear limits.

NMHV Multi-Particle Factorization

Bern, Chalmers, hep-ph/9503236; LD, von Hippel, 1408.1505



$$A_6^{\rm NMHV}(k_i) \xrightarrow{s_{345}\to 0} A_4(k_6,k_1,k_2,K)\,\frac{F_6(K^2,s_{i,i+1})}{K^2}\,A_4(-K,k_3,k_4,k_5)$$

Look at for NMHV: MHV tree has no pole

$$\mathcal{A}_{\rm MHV}^{(0)} = i\,\frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \to \infty \qquad w = \frac{s_{61}s_{34}}{s_{345}s_{234}} \to \infty$$

u/w and $v = \dfrac{s_{23}s_{56}}{s_{234}s_{123}}$ fixed

Multi-Particle Factorization (cont.)

$(1) = (4) \to \infty, \text{ rest finite}$

→ look at $E(u,v,w)$

Or rather at $U(u,v,w) = \ln E(u,v,w)$

$$\frac{\mathcal{A}_{\mathrm{NMHV}}}{\mathcal{A}_{\mathrm{BDS-like}}} \approx e^{U}[(1) + (4)]$$

Factorization limit of U

$$U^{(1)}(u,v,w) = -\frac{1}{4}\ln^2(uw/v) - \zeta_2$$

$$U^{(2)}(u,v,w)|_{u,w\to\infty} \; = \; \frac{3}{4}\zeta_2 \ln^2(uw/v) - \frac{1}{2}\zeta_3 \ln(uw/v) + \frac{71}{8}\zeta_4$$

$$U^{(3)}(u,v,w)|_{u,w\to\infty} \; = \; \frac{1}{3}\zeta_3 \ln^3(uw/v) - \frac{75}{8}\zeta_4 \ln^2(uw/v) + (7\,\zeta_5 + 8\,\zeta_2\,\zeta_3)\ln(uw/v) - \frac{721}{8}\zeta_6 - 3\,(\zeta_3)^2$$

…

- Simple polynomial in $\ln(uw/v)$, form dictated by Steinmann relations

$$\boxed{\frac{uw}{v} = \frac{s_{12}s_{34}}{s_{56}} \cdot \frac{s_{45}s_{61}}{s_{23}} \cdot \frac{1}{s_{345}^2}}$$

- Sudakov logs due to on-shell intermediate state

- All orders form available via analytic continuation from the near-collinear (OPE) limit.

Basso, Sever, Vieira (Sever talk at Amplitudes 2015)

All hexagon letter are rational in terms of y_i

$$
\begin{aligned}
u &= \frac{y_u(1-y_v)(1-y_w)}{(1-y_u y_v)(1-y_u y_w)}, \quad v = \frac{y_v(1-y_w)(1-y_u)}{(1-y_v y_w)(1-y_v y_u)}, \quad w = \frac{y_w(1-y_u)(1-y_v)}{(1-y_w y_u)(1-y_w y_v)} \\
1-u &= \frac{(1-y_u)(1-y_u y_v y_w)}{(1-y_u y_v)(1-y_u y_w)}, \quad \text{etc.}, \quad \sqrt{\Delta} = \frac{(1-y_u)(1-y_v)(1-y_w)(1-y_u y_v y_w)}{(1-y_u y_v)(1-y_v y_w)(1-y_w y_u)}
\end{aligned}
$$

$$\mathcal{S} = \{y_i,\ 1-y_i,\ 1-y_i y_j,\ 1-y_u y_v y_w\}$$

"extra" 10th letter

MRK Master formulae

$$w = -z, \quad w^* = -\bar{z}$$

- MHV:

$$e^{R+i\pi\delta}|_{\rm MRK} = \cos\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\,\boxed{\Phi_{\rm Reg}(\nu,n)}$$

$$\times\left(-\frac{1}{1-u}\frac{|1+w|^2}{|w|}\right)^{\boxed{\omega(\nu,n)}}$$

NLL: Fadin, Lipatov, 1111.0782; Caron-Huot, 1309.6521

- NMHV:

$$\exp(R^{\rm NMHV}+i\pi\delta)|_{\rm MRK} = \boxed{\mathcal{P}}\exp(R^{\rm MHV}+i\pi\delta)$$

$$= \cos\pi\omega_{ab} - i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\boxed{\frac{d\nu}{(i\nu+\frac{n}{2})^2}}|w|^{2i\nu}$$

$$\times\,\Phi^{\boxed{\rm NMHV}}_{\rm Reg}(\nu,n)\left(-\frac{1}{1-u}\frac{|1+w|^2}{|w|}\right)^{\omega(\nu,n)}$$

LL: Lipatov, Prygarin, Schnitzer, 1205.0186

NMHV MRK limit

Like g, h for R_6:
Extract p, q from V, $\tilde{V}$
→ linear combinations of SVHPLs [Brown, 2004]

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \left[g_r^{(L)}(w,w^*) + 2\pi i\, h_r^{(L)}(w,w^*)\right]$$

$$\mathcal{P}_{\rm MRK}^{(L)} = (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \Big[\frac{1}{1+w^*} \big(p_r^{(L)}(w,w^*) + 2\pi i\, q_r^{(L)}(w,w^*)\big)$$
$$+ \frac{w^*}{1+w^*} \big(p_r^{(L)}(w,w^*) + 2\pi i\, q_r^{(L)}(w,w^*)\big)\Big|_{(w,w^*)\to(\frac{1}{w},\frac{1}{w^*})} \Big] + \mathcal{O}(1-u)$$

- Then match p, q to master formula for factorization in Fourier-Mellin space

MRK limits agree with all-orders predictions

Basso, Caron-Huot, Sever 1407.3766

- BFKL eigenvalue:

$$E^{(1)}(\nu,n),\ E^{(2)}(\nu,n),\ E^{(3)}(\nu,n)$$

LL, NLL, NNLL, NNNLL

- Impact factors:

$$\Phi_{\mathrm{Reg}}^{\mathrm{(N)MHV},(1)}(\nu,n),\ \Phi_{\mathrm{Reg}}^{\mathrm{(N)MHV},(2)}(\nu,n),\ \Phi_{\mathrm{Reg}}^{\mathrm{(N)MHV},(3)}(\nu,n),\ \Phi_{\mathrm{Reg}}^{\mathrm{(N)MHV},(4)}(\nu,n)$$

- All zeta-valued linear combinations of:

derivatives of $\ln\Gamma(1 \pm i\nu + \frac{n}{2})$ $\quad \frac{i\nu}{\nu^2+\frac{n^2}{4}},\ \frac{n}{\nu^2+\frac{n^2}{4}}$

L	$\gamma_K^{(L)}/\gamma_K^{(L-1)}$	$\bar{R}_6^{(L)}(1,1,1)$	$\overline{\ln \mathcal{W}}_{\rm hex}^{(L)}(\frac{3}{4},\frac{3}{4},\frac{3}{4})$
2	-1.6449340	∞	-2.7697175
3	-3.6188549	-7.0040885	-5.0036164
4	-4.9211827	-6.5880519	-5.8860842
5	-5.6547494	-6.7092373	-6.3453695
6	-6.0801089	-6.8736364	??
7	-6.3589220	—	—
8	-6.5608621	—	—
9	-6.7164600	—	—
10	-6.8410049	—	—
11	-6.9432839	—	—
12	-7.0288902	—	—
13	-7.1016320	—	—
14	[illegible]		

Iterative Construction of Steinmann hexagon functions

$\{n\text{-}1,1\}$ coproduct F^x characterizes first derivatives, defines F up to additive constant (a multiple zeta value).

$$\boxed{\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}}$$

$$\frac{\partial \ln y_u}{\partial u}$$

1. Insert general linear combinations for F^x
2. Apply "integrability" constraint that mixed-partial derivatives are equal (largest linear algebra computation)
3. Stay in space of functions with good branch cuts and obeying Steinmann by imposing a few more "zeta-valued" conditions in each iteration.

Iterative construction

$$\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}$$

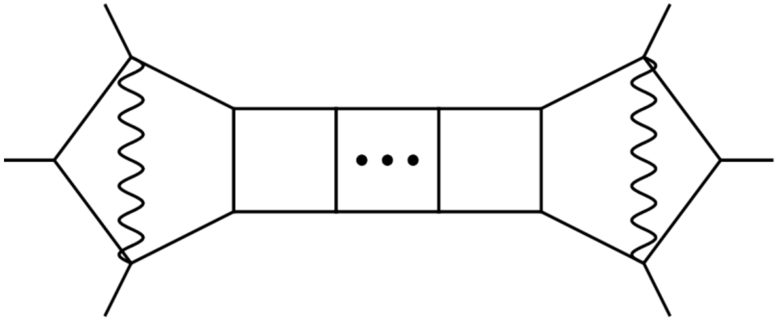
- F weight n, from F^x weight n-1 (already classified)
- Just need to impose: 1. mixed-partials: $\frac{\partial^2 F}{\partial u_i \partial u_j} = \frac{\partial^2 F}{\partial u_j \partial u_i}, \qquad i \neq j$

$$\begin{aligned}
F^{u,v} &= F^{v,u} - F^{y_u,y_v} + F^{y_v,y_u}\,, \\
F^{v,w} &= F^{w,v} - F^{y_v,y_w} + F^{y_w,y_v}\,, \\
F^{w,u} &= F^{u,w} - F^{y_w,y_u} + F^{y_u,y_w}\,, \\
F^{1-u,1-v} &= F^{1-v,1-u} + F^{y_u,y_v} - F^{y_u,y_w} - F^{y_v,y_u} + F^{y_v,y_w} + F^{y_w,y_u} - F^{y_w,y_v} \\
F^{1-v,1-w} &= F^{1-w,1-v} + F^{y_v,y_w} - F^{y_v,y_u} - F^{y_w,y_v} + F^{y_w,y_u} + F^{y_u,y_v} - F^{y_u,y_w} \\
F^{1-w,1-u} &= F^{1-u,1-w} + F^{y_w,y_u} - F^{y_w,y_v} - F^{y_u,y_w} + F^{y_u,y_v} + F^{y_v,y_w} - F^{y_v,y_u} \\
F^{u,1-v} &= F^{1-v,u} + F^{y_u,y_w} - F^{y_w,y_u}\,, \\
F^{v,1-w} &= F^{1-w,v} + F^{y_v,y_u} - F^{y_u,y_v}\,, \\
F^{w,1-u} &= F^{1-u,w} + F^{y_w,y_v} - F^{y_v,y_w}\,, \\
F^{u,1-w} &= F^{1-w,u} + F^{y_u,y_v} - F^{y_v,y_u}\,, \\
F^{v,1-u} &= F^{1-u,v} + F^{y_v,y_w} - F^{y_w,y_v}\,, \\
F^{w,1-v} &= F^{1-v,w} + F^{y_w,y_u} - F^{y_u,y_w}\,,
\end{aligned}$$

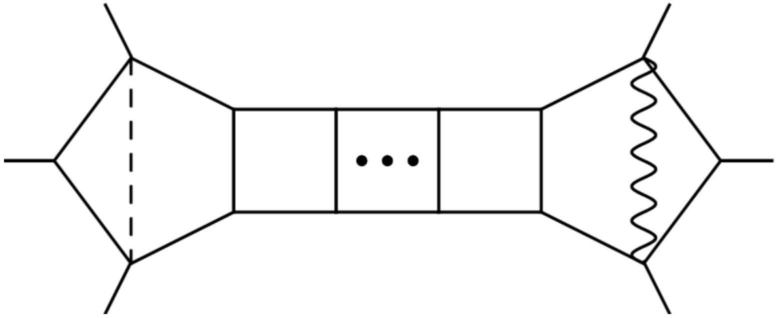
$$\begin{aligned}
F^{u,y_u} &= F^{y_u,u}\,, \\
F^{v,y_v} &= F^{y_v,v}\,, \\
F^{w,y_w} &= F^{y_w,w}\,, \\
F^{u,y_w} &= F^{w,y_u} - F^{y_u,w} + F^{y_w,u}\,, \\
F^{v,y_u} &= F^{u,y_v} - F^{y_v,u} + F^{y_u,v}\,, \\
F^{w,y_v} &= F^{v,y_w} - F^{y_w,v} + F^{y_v,w}\,, \\
F^{1-v,y_v} &= F^{y_v,1-v} - F^{y_u,1-u} + F^{1-u,y_u} + F^{y_u,w} - F^{w,y_u} - F^{y_w,v} + F^{v,y_w} \\
F^{1-w,y_w} &= F^{y_w,1-w} - F^{y_v,1-v} + F^{1-v,y_v} + F^{y_v,u} - F^{u,y_v} - F^{y_u,w} + F^{w,y_u} \\
F^{1-u,y_u} &= F^{y_u,1-u} - F^{y_w,1-w} + F^{1-w,y_w} + F^{y_w,v} - F^{v,y_w} - F^{y_v,u} + F^{u,y_v} \\
F^{1-u,y_v} &= F^{y_v,1-u} + F^{y_v,w} - F^{w,y_v}\,, \\
F^{1-v,y_w} &= F^{y_w,1-v} + F^{y_w,u} - F^{u,y_w}\,, \\
F^{1-w,y_u} &= F^{y_u,1-w} + F^{y_u,v} - F^{v,y_u}\,, \\
F^{1-u,y_w} &= F^{y_w,1-u} + F^{y_w,v} - F^{v,y_w}\,, \\
F^{1-v,y_u} &= F^{y_u,1-v} + F^{y_u,w} - F^{w,y_u}\,, \\
F^{1-w,y_v} &= F^{y_v,1-w} + F^{y_v,u} - F^{u,y_v}\,.
\end{aligned}$$

- 2. No bad branch cuts: $F^{1-u_i}(y_i = 1, y_j, y_k) \; = \; 0$

Infinite class of integrals



$\Omega^{(L)}$



$\tilde{\Omega}^{(L)}$

- Differential equations Drummond, Henn, Trnka, 1010.3679
  easy to solve in space of Steinmann hexagon functions

Caron-Huot, LD, von Hippel, McLeod, Papathanasiou, 1806.01361

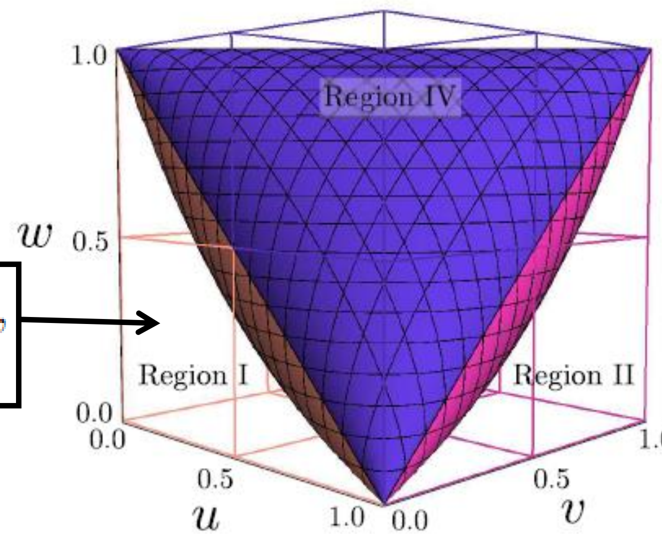
Menagerie of functions includes:

1. HPLs: One variable, symbol letters $\{u, 1-u\}$. Near-collinear limit, lines $(u,u,1)$, $(u,1,1)$
2. Cyclotomic Polylogarithms [Ablinger, Blumlein, Schneider, 1105.6063]: One variable, letters $\{y_u, 1+y_u, 1+y_u+y_u^2\}$. For line (u,u,u).
3. SVHPLs [F. Brown, 2004]: Two variables, letters $\{z, 1-z, \bar{z}, 1-\bar{z}\}$. First entry/single-valuedness constraint (real analytic function in z plane). Multi-Regge limit.
4. Full hexagon functions. Three variables, symbol letters $\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$, branch-cut condition(s)

Hexagon functions as generalized polylogarithms in y_i

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

$$\boxed{\text{Region I}: \quad \begin{cases} \Delta > 0\,, \quad 0 < u_i < 1\,, \quad \text{and} \quad u+v+w<1, \\ 0 < y_i < 1\,. \end{cases}}$$

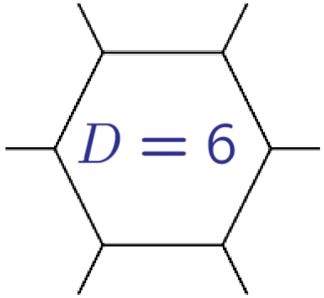


$$\mathcal{G} = \Big\{ G(\vec{w};y_u) | w_i \in \{0,1\} \Big\} \cup \Big\{ G(\vec{w};y_v) \Big| w_i \in \Big\{0,1,\frac{1}{y_u}\Big\} \Big\} \cup \Big\{ G(\vec{w};y_w) \Big| w_i \in \Big\{0,1,\frac{1}{y_u},\frac{1}{y_v},\frac{1}{y_u y_v}\Big\} \Big\}$$

- Useful for analytics and for numerics for $\Delta > 0$

  GiNaC implementation: Vollinga, Weinzierl, hep-th/0410259

First true (y-containing) hexagon function



$$\Rightarrow \quad \tilde{\Phi}_6(u,v,w)$$

A real integral so it must be Steinmann

- Weight 3, totally symmetric in $\{u,v,w\}$
- First parity odd function, so:

$$\tilde{\Phi}_6^u = \tilde{\Phi}_6^v = \tilde{\Phi}_6^w = \tilde{\Phi}_6^{1-u} = \tilde{\Phi}_6^{1-v} = \tilde{\Phi}_6^{1-w} = 0$$

- Only independent {2,1} coproduct:

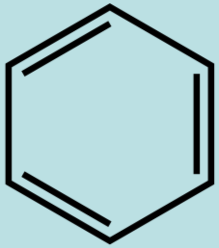
$$\tilde{\Phi}_6^{y_u} = -\Omega^{(1)}(v,w,u) = -H_2^u - H_2^v - H_2^w - \ln v \, \ln w + 2\,\zeta_2$$

$$H_2^u = \mathrm{Li}_2(1-u)$$

- Encapsulates first order differential equation found earlier

LD, Drummond, Henn, 1104.2787

Simple all-loop constraints on $\mathcal{E}$



- S_3 permutation **symmetry** in $\{u, v, w\}$
- Even under "**parity**".
- "Remainder function" $\boldsymbol{R_6}$ vanishes in **collinear** limit ($\boldsymbol{R_6 \rightarrow R_5 = 0}$)

$v \rightarrow 0 \qquad u + w \rightarrow 1$

$$\frac{\mathcal{A}_6^{\mathrm{MHV}}}{\mathcal{A}_6^{\mathrm{BDS-like}}} \equiv \mathcal{E}(u,v,w) = \exp\left[R_6 - \frac{\gamma_K(a)}{8} Y\right] \qquad \gamma_K(a) = \text{cusp anom. dim.}$$

$$Y(u,v,w) \equiv \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w) + \frac{1}{2}\left(\ln^2 u + \ln^2 v + \ln^2 w\right)$$

$\bar{Q}$ equation for MHV

- First derivative of $\mathcal{E}$ constrained by dual superconformal invariance.
- In terms of final entry of symbol, restricts to 6 of 9 possible letters:

$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w\right\}$$

- In terms of $\{n\text{-}1,1\}$ coproducts, equivalent to:

$$\boxed{\mathcal{E}^{u} + \mathcal{E}^{1-u} = \mathcal{E}^{v} + \mathcal{E}^{1-v} = \mathcal{E}^{w} + \mathcal{E}^{1-w} = 0}$$

- Similar (but more intricate) constraints for NMHV
  [Caron-Huot], LD, von Hippel McLeod, 1509.08127

$\bar{\mathrm{Q}}$ equation for NMHV

Caron-Huot, He, 1112.1060; S. Caron-Huot (2015);
LD, von Hippel, McLeod, 1509.08127

$$\bar{Q}\hat{\mathcal{R}}_{6,1} = \frac{\gamma_K}{8}\int d^{2|3}\mathcal{Z}_7[\mathcal{R}_{7,2} - \hat{\mathcal{R}}_{6,1}\mathcal{R}_{7,1}^{\text{tree}}] + \text{ cyclic}$$

$$\bar{Q}_a^A = \sum_{i=1}^{n}\chi_i^A\frac{\partial}{\partial Z_i^a} \qquad \hat{\mathcal{R}}_{6,1} \equiv \frac{\mathcal{A}_{\text{NMHV}}}{\mathcal{A}_{\text{BDS-like}}}$$

prevents second (simpler) term
from generating new “final entries”

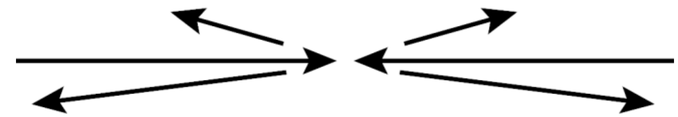
→ Only 18 out of 5 x 9 = 45 possible R-invariants x final entries:

$$(1)\, d\ln(uw/v), \quad (1)\, d\ln\left(\frac{(1-w)u}{w(1-u)y_v}\right),$$

$$[(2)+(5)+(3)+(6)]\, d\ln\left(\frac{v}{1-v}\right) + (1)\, d\ln\left(\frac{w}{y_u(1-w)}\right) + (4)\, d\ln\left(\frac{u}{y_w(1-u)}\right)$$

+ cyclic

2→4 multi-Regge limit



- Euclidean MRK limit vanishes
- To get nonzero result for physical region, first let

$u \to u\, e^{-2\pi i}$, then u → 1, v, w → 0

$$\frac{v}{1-u} \to \frac{1}{|1-z|^2} \qquad \frac{w}{1-u} \to \frac{|z|^2}{|1-z|^2}$$

$$\boxed{R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u)\, [g_r^{(L)}(z,\bar{z}) + 2\pi i\, h_r^{(L)}(z,\bar{z})]}$$

$g_r^{(L)}$ and $h_r^{(L)}$
all well understood by now;
all SVHPLs (Brown, 2004);
also NMHV behavior

Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186;
Pennington, 1209.5357; Basso, Caron-Huot, Sever, 1407.3766; Broedel, Sprenger, 1512.04963

Lipatov, Prygarin, Schnitzer, 1205.0186;
LD, von Hippel, 1408.1505

A minimal function space

- Want to describe, not only $(\mathcal{E}^{(L)},\ E^{(L)} \& \tilde{E}^{(L)})$ to a given loop order, but also derivatives ($\{n\text{-}k,1,1,\ldots,1\}$ coproducts) of even higher loop answers.
- How many functions do we need?
- We take multiple derivatives/coproducts of amplitudes we know, and ask how much more of the Steinmann space we can remove.

Minimal function space (cont.)

- First surprise already at weight 2
- The many, many {2,1,1,…,1} coproducts of the weight 12 functions $(\mathcal{E}^{(6)}, E^{(6)} \& \tilde{E}^{(6)})$ span only a 6 dimensional subspace of the 7 dimensional Steinmann space, with basis:

$\mathrm{Li}_2(1-1/u)$ $\ln^2\frac{uv}{w}+4\zeta_2$ plus cyclic

ζ_2 is not an independent element!

Minimal Steinmann space (cont.)

Weight	1	2	3	4	5	6	7	8	9	10	11
P-even OLD Steinmann	3	7	16	37	80	174	365	758	1543	3105	????
P-even NEW Steinmann	3	6	12	25	48	92	170	313	559	991	1718
P-odd OLD Steinmann	0	0	1	2	7	16	38	81	167	329	???
P-odd NEW Steinmann	0	0	1	2	6	13	30	59	120	223	418

Factor of 3 smaller at high weights.

Through at least weight 7, the fully trimmed Steinmann hexagon space is "just right" for the problem of 6 point scattering in planar N=4 super-Yang-Mills theory!

Numerical values on $(u,1,1)$