Physics 451 – Problem Set # 1

(due Tuesday, February 4)

1. Prove the Fierz identities:

$$
\begin{aligned}
\eta_\alpha \xi^\dagger_\beta &= -\frac{1}{2}(\sigma^k)_{\alpha\beta}(\xi^\dagger \overline{\sigma}_k \eta) \\
\eta_\alpha \xi^T_\beta &= \frac{1}{2} c_{\alpha\beta}(\xi^T c \eta) - \frac{1}{2}(\sigma^{k\ell} c)_{\alpha\beta}(\xi^T c \sigma_{k\ell} \eta) \\
\eta^*_\alpha \xi^\dagger_\beta &= \frac{1}{2} c_{\alpha\beta}(\xi^\dagger c \eta^*) - \frac{1}{2}(c\overline{\sigma}^{k\ell})_{\alpha\beta}(\xi^\dagger \overline{\sigma}_{k\ell} c \eta^*)
\end{aligned} \tag{1}
$$

   for η, ξ constant anticommuting Weyl spinors. Prove also the useful identity

$$
(\chi_1^\dagger \overline{\sigma}^m \chi_2)(\chi_3^\dagger \overline{\sigma}_m \chi_4) = (\chi_1^\dagger \overline{\sigma}^m \chi_4)(\chi_3^\dagger \overline{\sigma}_m \chi_2) \tag{2}
$$

   for the χ_1 anticommuting Weyl spinors.

2. Prove that, if (ϕ_k, ψ_k, F_k) are chiral supermultiplets and $W(\phi)$ is an analytic function, then

$$
\mathcal{L}_F = F_k \frac{\partial W}{\partial \phi_k} - \frac{1}{2}\psi_k^T c \psi_\ell \frac{\partial^2 W}{\partial \phi_k \partial \phi_\ell} \tag{3}
$$

   satisfies $\delta_\xi \mathcal{L}_F = 0$ up to integration by parts.

3. For λ^a the Weyl fermion in a vector supermultiplet, show explicitly that

$$
[\delta_\xi, \delta_\eta]\lambda^a = 2i(\xi^\dagger \overline{\sigma}^m \eta - \eta^\dagger \overline{\sigma}^m \xi)\mathcal{D}_m \lambda^a \ . \tag{4}
$$

   Some Fierzing is necessary.

4. Let (ϕ, ψ, F) be a chiral supermultiplet transforming under a representation r of a gauge group G, let t^a be the representation of the generators of G in r, and let (A^a_m, λ^a, D^a) be a vector supermultiplet in the adjoint representation of G. Show, using the transformation laws of the vector multiplet and the transformations

$$
\begin{aligned}
\delta_\xi \phi &= \sqrt{2}\xi^T c \psi \\
\delta_\xi \psi &= \sqrt{2} i \sigma^n c \xi^* \mathcal{D}_n \phi + \sqrt{2} F \xi \\
\delta_\xi F &= -\sqrt{2} i \xi^\dagger \overline{\sigma}^m \mathcal{D}_m \psi - 2g \xi^\dagger c \lambda^{a*} t^a \phi \ ,
\end{aligned} \tag{5}
$$

   for the chiral fields, that the following Lagrangian is invariant to supersymmetry (up to integration by parts):

$$
\begin{aligned}
\mathcal{L} &= \mathcal{D}_m \phi^* \mathcal{D}^m \phi + \psi^\dagger i \overline{\sigma}^m \mathcal{D}_m \psi + F^* F \\
&\quad -\sqrt{2} g(\phi^* \lambda^{Ta} t^a c \psi - \psi^\dagger c \lambda^{*a} t^a \phi) + g D^a \phi^* t^a \phi
\end{aligned} \tag{6}
$$

   Yes, I would like you to work this out in the non-Abelian case.