8. Supersymmetry in higher dimensions

Up to this point in the course, we have been studying $N=1$ SUSY in 4 dimensions. This symmetry has four fermionic charges

$$Q_\alpha \qquad Q^\dagger_\alpha \qquad \alpha = 1,2$$

which is the minimal number in 4-dimensions. In general, the number of SUSY charges is given by the size of the minimal spin-$\frac{1}{2}$ representation of the Lorentz group. In $d > 4$, these representations are larger, and the corresponding supersymmetric theories have a larger number of charges. I would now like to study some of these theories. They are relevant for two reasons. First, unifying models of the fundamental interactions, e.g. string theory, actually have more than 4 dimensions. String theory has up to 11 dimensions. So these theories might be relevant to the real world. But even if not, they provide a route to the construction of 4-d theories with $N > 1$ SUSY, by dimensional reduction of the higher dimensional theory.

First of all, how big is a spinor representation in d-dimensions? In $d=4$, we used a special trick to find all of the representations of the Lorentz group. In

$d > 4$, it is again possible to find all finite-dimensional representations of the Lorentz group, but this is quite complicated. Instead, I will construct the spin-$\frac{1}{2}$ representations by a trick due to Dirac, this trick, plus judicious separation of the pieces of reducible representations, gives the minimal representation with spin-$\frac{1}{2}$ in any dimension.

Our task is to find a set of finite-dimensional matrices that satisfy the algebra

$$[M^{ab}, M^{cd}] = M^{ad}\eta^{bc} - M^{ac}\eta^{bd} - M^{bd}\eta^{ac} + M^{bc}\eta^{ad}$$

Dirac suggested that we look for matrices that satisfy the anticommutation relations ("Clifford algebra" or "Dirac algebra")

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}$$

then

$$\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$$

satisfies the Lorentz algebra: Proof:

$$\tfrac{1}{2}\gamma^a\gamma^b \tfrac{1}{2}\gamma^c\gamma^d = \tfrac{1}{16}\{\gamma^a\gamma^d\, 2\eta^{bc} - 2\eta^{ac}\gamma^b\gamma^d + 2\eta^{bd}\gamma^c\gamma^a - 2\eta^{ad}\gamma^c\gamma^b + \gamma^c\gamma^d\gamma^a\gamma^b\}$$

or

$$[\tfrac{1}{2}\gamma^a\gamma^b, \tfrac{1}{2}\gamma^c\gamma^d] = \tfrac{1}{8}\gamma^a\gamma^d\eta^{bc} - \tfrac{1}{8}\gamma^b\gamma^d\eta^{ac} - (-\tfrac{1}{8}\gamma^c\gamma^a)\eta^{bd} + (-\tfrac{1}{8}\gamma^c\gamma^b)\eta^{ad}$$

antisymmetrizing on (ab) and then on (cd) gives the desired result. In $d=2,3$ the Dirac algebra has a simple representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

In $d=4$, there is no 2×2 matrix representation of the Dirac algebra. The smallest representation is 4×4 and has the form (up to unitary equivalence)

$$\gamma^a = \left(\begin{array}{c|c} & \sigma^a \\ \hline \bar\sigma^a & \end{array}\right) = \left(\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & \sigma^i \\ \hline -\sigma^i & 0 \end{array}\right) \right)$$

Then

$$\Sigma^{ab} = \tfrac{1}{4}[\gamma^a,\gamma^b] = \left(\begin{array}{c|c} \frac{1}{4}(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a) & \\ \hline & \frac{1}{4}(\bar\sigma^a\sigma^b - \bar\sigma^b\sigma^a) \end{array}\right)$$

$$= \left(\begin{array}{c|c} \sigma^{ab} & \\ \hline & \bar\sigma^{ab} \end{array}\right)$$

So the Dirac representation of the Lorentz group ("Dirac spinor") is a reducible representation that splits into

$$(\tfrac{1}{2},0) \oplus (0,\tfrac{1}{2})$$

To study spinors in higher dimensions, I would like first to construct the Dirac spinor representation in any dimension and then to investigate how this representation might split into irreducible components.

Let's first construct the Dirac spinor in 5-dimensions.

This is not so difficult: Notice that, in 4-dimensions

$$\hat{\Gamma} = \gamma^0\gamma^1\gamma^2\gamma^3$$

satisfies $\{\hat{\Gamma}, \gamma^a\} = 0$ since γ^a commutes w. itself, anticommutes w. all other γ^b's.

$$\hat{\Gamma}^2 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3$$
$$= (-1)^3(-1)^2(-1)(\gamma^0)^2(\gamma^1)^2(\gamma^2)^2(\gamma^3)^2 = (-1)^6(-1)^3 = -1$$

so setting

$$\gamma^4 = (\gamma^0\gamma^1\gamma^2\gamma^3)$$

we have a representation of the Dirac algebra in 5 dimensions.

Actually, it is not hard to show that, if (γ^a) is a representation of the Dirac algebra in d dimensions, with d even, there is a representation in $(d+1)$ dimensions of the same size.

Define again:

$$\hat{\Gamma} = \gamma^0\gamma^1 \cdots \gamma^{d-1}$$

Again $\{\hat{\Gamma}, \gamma^a\} = 0$. The square of $\hat{\Gamma}$ is:

$$\hat{\Gamma}^2 = \gamma^0\gamma^1 \cdots \gamma^{d-1}\gamma^0\gamma^1 \cdots \gamma^{d-1}$$
$$= (-1)^{d-1}(-1)^{d-2}\cdots(-1)\cdot(-1)^{d+1} = (-1)^{\frac{d(d-1)}{2}}(-1)^{d-1} = (-1)^{\left(\frac{d+2}{2}\right)(d-1)}$$

$$= \begin{cases} 1 & \text{in } d = 2, 6, 10, \ldots \\ -1 & \text{in } d = 4, 8, 12, \ldots \end{cases}$$

in fact

$$\Gamma \equiv \begin{cases} \gamma^0\gamma^1 \cdots \gamma^{d-1} & d = 2, 6, 10, \ldots \\ i\gamma^0\gamma^1 \cdots \gamma^{d-1} & d = 4, 8, 12, \ldots \end{cases}$$

satisfies $\{\Gamma, \gamma^a\} = 0 \qquad \Gamma^\dagger = \Gamma \qquad \Gamma^2 = 1$. In any even dimensionality d, the matrices

$$\gamma^0, \gamma^1, \cdots, \gamma^{d-1}, -i\Gamma$$

satisfy the Dirac algebra in $(d+1)$ dimensions.

Using Γ, we can see ~~that~~ the Dirac spinor representation in $d=4$ is reducible, and ~~that~~ this is true in any even dimension. Since $\{\Gamma, \gamma^a\} = 0$, $[\Gamma, \Sigma^{ab}] = 0$. So the eigenspaces of Γ are independent representations of the Lorentz group. Now

$$\operatorname{tr} \Gamma = 0$$

(to see this, note that $\operatorname{tr}(\gamma^0\gamma^1 \cdots \gamma^{d-1}) = (-1)^{d-1} \operatorname{tr}[\gamma^{d-1}\gamma^0\gamma^1 \cdots \gamma^{d-2}]$
$= (-1) \cdot \operatorname{tr}[\gamma^0\gamma^1 \cdots \gamma^{d-1}]$.)

so Γ has equal #s of $+1$ and -1 eigenvalues. Γ then splits the Dirac spinor into two representations of half the size.
In $d=4$

$$\Gamma = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

so these two representations are just the $(\tfrac{1}{2}, 0)$ and $(0, \tfrac{1}{2})$.

Now that we see how to get from $d = 2N$ to $d = 2N+1$, how do we get to $d = 2N+2$? It turns out that the only way is to double the size of the Dirac spinor. Given a rep. of the Dirac algebra in odd dimensionality d, here is representation in even $(d+1)$!

$$\gamma^0_{(d+1)} = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right) \qquad \gamma^j_{(d+1)} = \left(\begin{array}{c|c} 0 & i\gamma^j_{(d)} \\ \hline -i\gamma^j_{(d)} & 0 \end{array}\right) \qquad \gamma^d_{(d+1)} = \left(\begin{array}{c|c} 0 & \gamma^0_{(d)} \\ \hline -\gamma^0_{(d)} & 0 \end{array}\right)$$

This is the representation of minimal size. Since this representation is of the form

$$\gamma^a = \left(\begin{array}{c|c} & \sigma^a \\ \hline \bar{\sigma}^a & \end{array}\right)$$

we find
$$\Sigma^{ab} = \frac{i}{4}\left(\begin{array}{c|c} \sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a & 0 \\ \hline 0 & \bar{\sigma}^a\sigma^b - \sigma^b\bar{\sigma}^a \end{array}\right)$$

which manifestly splits. The split representations with definite $\Gamma = +1$ or -1 are "Weyl spinors".

Then,

for d even, the Dirac matrices are $2^{d/2} \times 2^{d/2}$
and there are Weyl spinors with dimension $2^{d/2-1}$

for d odd, the Dirac matrices are $2^{(d-1)/2} \times 2^{(d-1)/2}$
but Dirac spinors do not split into Weyl spinors.

If it were true that the minimal spinor in d-dimensions were the complex Weyl or Dirac spinor, we could easily count the number of supercharges in the minimal SUSY algebra in any dimension.

$d = 4$	2 complex	= 4 charges	$(N=1)$
$d = 5, 6$	4 complex	= 8 charges	$(N=2)$
$d = 7, 8$	8 complex	= 16 charges	$(N=4)$
$d = 9, 10$	16 complex	= 32 charges	$(N=8)$
$d = 11, 12$	32 complex	= 64 charges	$(N=16)$

$\vdots$

However, there is another possible way to split a Dirac or even a Weyl spinor. If we can impose a reality condition, we can make the spinor smaller by another factor of 2.

A reality condition on a Dirac spinor is called a Majorana condition; the ~~reality~~ real spinor is a Majorana spinor. A Majorana spinor is unitarily equivalent to its conjugate:

$$\bar{\psi} = \psi^T C \quad \text{or} \quad \psi = (C^{-1})^T \bar{\psi}^T$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ and C is unitary. It is possible to restrict C to have definite symmetry: $C^T = aC$, $a = \pm 1$.

The Majorana reduction is only possible if ψ and $\bar{\psi}$ transform consistently under Lorentz transformations. Since $\bar{\psi}\psi$ is a scalar:

$$\psi \to (1 + \tfrac{1}{2}\omega_{ab}\Sigma^{ab})\psi \qquad \bar{\psi} \to \bar{\psi}(1 - \tfrac{1}{2}\omega_{ab}\Sigma^{ab})$$

The second relation implies

$$\psi^T = \bar{\psi}C^{-1} \to \bar{\psi}C^{-1}\, C(1 - \tfrac{1}{2}\omega_{ab}\Sigma^{ab})C^{-1}$$

$$= \psi^T(1 + \tfrac{1}{2}\omega_{ab}(-C\Sigma^{ab}C^{-1}))$$

so $(\Sigma^{ab})^T = -C\Sigma^{ab}C^{-1}$. This is satisfied if

$$C\gamma^a C^{-1} = b(\gamma^a)^T \qquad \text{for } b = \pm 1$$

The same condition insures that C preserves the Dirac algebra:

$$C\{\gamma^a, \gamma^b\}C^{-1} = b^2\{\gamma^a, \gamma^b\} = b^2 \cdot 2\eta^{ab} \Rightarrow b^2 = 1$$

There are two criteria ~~that~~ connect the phases a and b. First, we can insist ~~that~~ the Dirac Lagrangian is self-consistent:

$$\bar{\psi} i\gamma^a\partial_a\psi = \psi^T C\, i\gamma^a\, \partial_a (C^{-1})^T\bar{\psi}^T$$

$$= a\, \psi^T i(C\gamma^a C^{-1})\partial_a\bar{\psi}^T$$

$$= ab\, \psi^T i(\gamma^a)^T\partial_a\bar{\psi}^T$$

$$= -ab\,\partial_a\psi^T\, i(\gamma^a)^T\bar{\psi}^T$$

$$= +ab\, \bar{\psi} i\gamma^a\partial_a\psi$$

the last (-1) is from fermion interchange.

so $ab = +1$. Alternatively, we can insist that the supersymmetry algebra is self-consistent. A form consistent with Lorentz invariance is:

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\,\gamma^a_{\alpha\beta} P_a$$

If the Q_α are Majorana, this is

$$(C^{-1})^T \{\bar{Q}^T, Q^T\} C = 2\gamma^a P_a$$

$$a\, C^{-1} \{\bar{Q}^T, Q^T\} C = 2\gamma^a P_a$$

$$\{\bar{Q}^T, Q^T\} = 2a\,(C\gamma^a C^{-1}) P_a$$

$$= 2ab\,(\gamma^a)^T P_a$$

this again implies $ab = +1$. So we have two choices:

C with $C^T = -C$ $\quad C\gamma^a C^{-1} = -(\gamma^a)^T$

$\hat{C}$ with $\hat{C}^T = +\hat{C}$ $\quad \hat{C}\gamma^a\hat{C}^{-1} = +(\gamma^a)^T$

Let's first work out the consequences of the existence of C.

$$C = -C^T$$

$$C\gamma^a = C\gamma^a C^{-1} C = -(\gamma^a)^T(-C^T) = +(C\gamma^a)^T$$

$$C\gamma^a\gamma^b = C\gamma^a\gamma^b C^{-1} C = +(\gamma^a)^T(\gamma^b)^T(-C^T) = -(C\gamma^b\gamma^a)^T$$

in general

$$C\gamma^{a_1}\cdots\gamma^{a_n} = (-1)^{n-1}(C\gamma^{a_n}\cdots\gamma^{a_1})^T$$

In 4-dimensions, we know ~~that~~ it is possible to construct a basis in the 16-dimensional space of Dirac matrices by writing antisymmetric products of Dirac matrices

$$1, \ \gamma^a, \ \gamma^{ab} = \tfrac{1}{2}[\gamma^a, \gamma^b], \ \gamma^{abc} = \gamma^{[a}\gamma^b\gamma^{c]}, \ \ldots$$

The numbers of these matrices in 4-dimensions are

$$1 \quad \gamma^a \quad \gamma^{ab} \quad \gamma^{abc} \quad \gamma^{abcd} = \epsilon^{abcd}(-iI)$$

$$1 + 4 + 6 + 4 + 1 = 16$$

The same result holds in any even dimensionality

$$1 \quad \gamma^a \quad \gamma^{ab} \quad \gamma^{abc} \quad \ldots \quad \gamma^{a_1 \ldots a_{d-1}} \quad \gamma^{a_1 \ldots a_d}$$

$$1 + d + \binom{d}{2} + \binom{d}{3} + \ldots + \underset{\substack{\shortparallel \\ d}}{\binom{d}{d-1}} + \underset{\substack{\shortparallel \\ 1}}{\binom{d}{d}}$$

$$= (1+1)^d = 2^d = [2^{d/2}] \times [2^{d/2}] \ \checkmark$$

If there exists a matrix C, each such antisymmetric product has definite symmetry!

$$C \cdot 1 = -C^T$$

$$C\gamma^a = +(C\gamma^a)^T$$

$$C\gamma^{ab} = -(C\gamma^{ba})^T = +(C\gamma^{ab})^T$$

$$C\gamma^{abc} = +(C\gamma^{cba})^T = -(C\gamma^{abc})^T$$

$$C\gamma^{a_1 \ldots a_N} = (-1)^{N-1}(C\gamma^{a_N \ldots a_1})^T = (-1)^{N-1}(-1)^{\frac{N(N-1)}{2}}(C\gamma^{a_1 \ldots a_N})^T$$

This give a consistency condition that C must satisfy. The space of $2^{d/2} \times 2^{d/2}$ matrices is spanned by

$$\frac{2^{d/2}(2^{d/2}+1)}{2} \text{ symmetric matrices and } \frac{2^{d/2}(2^{d/2}-1)}{2} \text{ antisymmetric matrices.}$$

so C can exist only if it predicts the right number of symmetric (S) and antisymmetric (A) matrices. Try this in d=4

	S	A
C		+1
$(C\gamma^a)$	4	
$(C\gamma^{ab})$	6	
$(C\gamma^{abc})$		4
$(C\gamma^{abcd})$		1
	10	6

consistent!

on the other hand, if we did this analysis with $\hat{C}$:

$$\hat{C} = +\hat{C}^T$$

$$\hat{C}\gamma^a = +(\hat{C}\gamma^a)^T$$

$$\hat{C}\gamma^{ab} = +(\hat{C}\gamma^{ba})^T = -(\hat{C}\gamma^{ab})^T$$

$$(\hat{C}\gamma^{abc}) = +(\hat{C}\gamma^{cba})^T = -(\hat{C}\gamma^{abc})^T$$

etc.

	$\underline{S}$	$\underline{A}$
$\hat{C}$	1	
$\hat{C}\gamma^a$	4	
$\hat{C}\gamma^{ab}$		6
$\hat{C}\gamma^{abc}$		4
$\hat{C}\gamma^{abcd}$	1	
	6	10

which is inconsistent, so $\hat{C}$ cannot exist in $d=4$. It is not hard to find C explicitly in $d=4$; in the basis

$$\gamma^a = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & \sigma^j \\ \hline -\sigma^j & 0 \end{array}\right) = (\sigma^1 \otimes 1\,,\ i\sigma^2 \otimes \sigma^j)$$

we have

$$C = (\sigma^3 \otimes -i\sigma^2) = \left(\begin{array}{c|c} -i\sigma^2 & 0 \\ \hline 0 & i\sigma^2 \end{array}\right).$$

Repeat this test in 6, 8, 10 dimensions!

$\underline{d=6}$

	$\underline{S}$	$\underline{A}$
C		1
$C\gamma^a$	6	
$C\gamma^{ab}$	15	
$C\gamma^{abc}$		20
$C\gamma^{abcd}$		15
$C\gamma^{abcde}$	6	
$C\gamma^{abcdef}$	1	
	28	36

	$\underline{S}$	$\underline{A}$
$\hat{C}$	1	
$\hat{C}\gamma^a$	6	
$\hat{C}\gamma^{ab}$		15
$\hat{C}\gamma^{abc}$		20
$\hat{C}\gamma^{abcd}$	15	
$\hat{C}\gamma^{abcde}$	6	
$\hat{C}\gamma^{abcdef}$		1
	28	36

Neither C nor $\hat{C}$ exists!

$d=8$

	$\underline{S}$	$\underline{A}$
C		1
$C\gamma^a$	8	
$2\text{-}\gamma$	28	
$3\text{-}\gamma$		56
$4\text{-}\gamma$		70
$5\text{-}\gamma$	56	
$6\text{-}\gamma$	28	
$7\text{-}\gamma$		8
$8\text{-}\gamma$		1
	120	136

inconsistent!

	$\underline{S}$	$\underline{A}$
$\hat{C}$	1	
$\hat{C}\gamma^a$	8	
$2\text{-}\gamma$		28
$3\text{-}\gamma$		56
$4\text{-}\gamma$	70	
$5\text{-}\gamma$	56	
$6\text{-}\gamma$		28
$7\text{-}\gamma$		8
$8\text{-}\gamma$	1	
	136	120 $\leftarrow \frac{16\cdot 15}{2}$

consistent!

$d=10$

	$\underline{S}$	$\underline{A}$
C		1
$C\gamma^a$	10	
$2\text{-}\gamma$	45	
$3\text{-}\gamma$		120
$4\text{-}\gamma$		210
$5\text{-}\gamma$	252	
$6\text{-}\gamma$	210	
$7\text{-}\gamma$		120
$8\text{-}\gamma$		45
$9\text{-}\gamma$	10	
$10\text{-}\gamma$	1	
	528	496 $\leftarrow \frac{32\cdot 31}{2}$

consistent!

	$\underline{S}$	$\underline{A}$
$\hat{C}$	1	
$\hat{C}\gamma^a$	10	
$2\text{-}\gamma$		45
$3\text{-}\gamma$		120
$4\text{-}\gamma$	210	
$5\text{-}\gamma$	252	
$6\text{-}\gamma$		210
$7\text{-}\gamma$		120
$8\text{-}\gamma$	45	
$9\text{-}\gamma$	10	
$10\text{-}\gamma$		1
	528	496

consistent!

The general pattern follows d mod 8:

C can be realized in $d = 2 \quad 4 \quad 10 \quad 12 \quad 18 \quad 20 \quad \ldots$

$\hat{C}$ can be realized in $d = 2, 8, 10, 16, 18, \ldots$

If C or $\hat{C}$ exists, we can impose the Majorana condition and decrease the size of a spinor representation by a factor of 2.

Of course, in all of these even dimensions, we can already make a Weyl reduction. So, we must ask, can we impose the Weyl and Majorana conditions simultaneously, or is it one or the other? The Weyl condition is (say)

$$\Gamma \Psi = + \Psi .$$

then $$\bar{\Psi}\Gamma = \Psi^\dagger \gamma^0 \Gamma = - \Psi^\dagger \Gamma \gamma^0 = - \Psi^\dagger \gamma^0 = - \bar{\Psi}$$

If Ψ is a Majorana spinor

$$\bar{\Psi}\Gamma = \Psi^T C \Gamma = \Psi^T (C \Gamma C^{-1}) C$$

so for compatibility we need: $C \Gamma C^{-1} = - \Gamma^T$

Now compute

$$C \Gamma C^{-1} = C \cdot \left(\tfrac{1}{i}\right) \gamma^0 \gamma^1 \cdots \gamma^{d-1} C^{-1}$$

$$= \left(\tfrac{1}{i}\right) (-1)^d \left[\gamma^{d-1} \gamma^{d-2} \cdots \gamma^1 \gamma^0\right]^T$$

$$= \left(\tfrac{1}{i}\right) (-1)^d (-1)^{(d-1)d/2} \left[\gamma^0 \gamma^1 \cdots \gamma^{d-1}\right]^T$$

$$= \begin{cases} -\Gamma^T & \text{in } d = 2, 6, 10, 14 \ldots \\ +\Gamma^T & \text{in } d = 4, 8, 12 \ldots \end{cases}$$

so the Majorana condition with C is compatible with the Weyl condition in

$$d = 2,\ 10,\ 18,\ \cdots \text{ only}$$

Similarly

$$\hat{C}\, \Gamma\, \hat{C}^{-1} = \begin{cases} -\Gamma^T & d = 2, 6, 10, \ldots \\ +\Gamma^T & d = 4, 8, 12, \ldots \end{cases}$$

So this is also compatible only in $d = 2, 10, 18, .$ Is it odd that these are the dimensions where both C and $\hat{C}$ exist? No, given C, we can construct $\hat{C} = C\Gamma$ in $d = 2, 10, 18, \ldots$

Finally, what about odd dimensions? If d is even, we construct the Dirac algebra in $(d+1)$ by adjoining $(-i\Gamma)$. If there exists a C in d-dim, this already satisfies

$$C = -C^T \qquad C\gamma^a C^{-1} = -(\gamma^a)^T \quad \text{for } a = 0, \ldots (d-1)$$

The last condition that must be satisfied is

$$C(\gamma^{d}) C^{-1} = -(\gamma^{d})^T$$

or

$$C(-i\Gamma) C^{-1} = -(-i\Gamma)^T$$

This works if $d = 2, \cancel{6}, 10, \cancel{14}, 18$ (cross out dims where C does not exist)

Similarly, $\hat{C}\, \gamma^d\, \hat{C}^{-1} = +(\gamma^d)^T$

in $d = \cancel{4}, 8, \cancel{12}, 16$ (cross out dims where $\hat{C}$ does not exist)

Putting together all of this information, we obtain the following table giving the minimal number of fermionic charges in d dimensions:

d =	2	1	M and W
	3	2	M with C
	4	4 (N=1)	W or M with C
	5	8 (N=2)	
	6	8	W
	7	16 (N=4)	
	8	16	W or M with $\hat{C}$
	9	16	M with $\hat{C}$
	10	16	W and M
	11	32 (N=8)	M with C
	12	64 (N=16)	W or M with C
	⋮	⋮	

By our previous analysis, we can have SUSY theories with all spins ≤ 1 in $d \leq 10$. We can have supergravity with all spins ≤ 2 in $d \leq 11$.

I should ~~note that~~ this table depends on the space-time signature: $SO(8)$ and $SO(10,2)$ also have Majorana-Weyl spinors.

It is disappointing that there are no Majorana spinors in 5 and 6 dimensions. Often it is convenient to discuss SUSY in these dimensions using a variant of Majorana spinors constructed as follows: Consider a C with the properties

$$C^T = -C \qquad C\gamma^a C^{-1} = +(\gamma^a)^T$$

Such a C exists in 4 and 6 dimensions

	S	A
C		1
$C\gamma^a$		4
2-γ	6	
3-γ	4	
4-γ		1
	10	6

✓

	S	A
C		1
$C\gamma^a$		6
2-γ	15	
3-γ	20	
4-γ		15
5γ		6
6γ	1	
	36	28

✓

Also, in 4-d, $C\Gamma C^{-1} = +\Gamma^T$, so C exists in 5 dimensions.

We showed earlier that such a C is incompatible with the Dirac Lagrangian. To make it consistent, we need to add to the spinor an extra index $i = 1,2$ and an extra antisymmetry:

$$\bar{\psi}^j = (\psi^i)^T C\, c_{ij} \qquad \text{where } c_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{ij} \text{ as earlier.}$$

$$\psi^i = c_{ij} (C^{-1})^T \bar{\psi}^{Tj}$$

Spinors satisfying this property are called symplectic-Majorana (sM) spinors. Such spinors can be defined in 4, 5, 6 dimensions. Further, since in $d=6$ $\quad C\Gamma C = -\Gamma^T$,

$$\Gamma\psi^i = +\psi^i \Rightarrow \quad \bar{\Psi}^j\Gamma = \psi^{iT} C c_{ij}\Gamma = \psi^{iT}(-\Gamma)^T C c_{ij}$$

$$\Rightarrow \quad \bar{\Psi}^j\Gamma = -\bar{\Psi}^j$$

so this Majorana condition is compatible with the Weyl condition and we may define symplectic-Majorana-Weyl spinors in 6 dimensions. The number of degrees of freedom is still

$d=5$	8 (N=2)	sM
6	8	sM and W

and the new description is equivalent to the one on p.16.

It will be important to us ~~that~~, with any of the three types of Majorana fermions we have discussed

$$\bar{\eta}\gamma^a\xi = -\bar{\xi}\gamma^a\eta$$

The proof follows that of the consistency of the Dirac Lagrangian.

As an application of this theory of spinors, I would like to construct supersymmetric Yang-Mills theory in various dimensions. The content of this theory (on shell) is a massless gauge boson and a massless Majorana fermions both in the adjoint rep. of the gauge group.

Let's first count on-shell particles to see when the degrees of freedom match between fermions and bosons. A massless vector boson in d dimensions has $(d-2)$ polarization states (transverse polarizations). An N-component Majorana fermion in d dimensions leads to N/2 massless particle states, which are their own antiparticles. So we have,

$$\text{in } d=4 \qquad A: \ (d-2)=2 \qquad \lambda: \ N=4 \to 2 \text{ particles}$$

$$d=6 \qquad A: \ (d-2)=4 \qquad \lambda: \ N=8 \to 4$$

(Weyl or SMW)

$$d=10 \qquad A: \ (d-2)=8 \qquad \lambda: \ \underset{(MW)}{N=16} \to 8$$

The $d=4$ theory is equivalent to the theory of a vector supermultiplet that we have already constructed. But the $d=6$ and $d=10$ theories are new. They are the simplest theories with $N=2$ and $N=4$ SUSY.

For all three cases, the Lagrangian and the transformation laws can be written:

$$\mathcal{L} = -\frac{1}{4}(F_{mn}^a)^2 + \frac{1}{2}\bar{\lambda}^a i \gamma^m \mathcal{D}_m \lambda^a$$

$$F_{mn}^a = \partial_m A_n^a - \partial_n A_m^a + g f^{abc} A_m^b A_n^c$$
$$\mathcal{D}_m \lambda^a = \partial_m \lambda^a + g f^{abc} A_m^b \lambda^c$$

with
$$\delta_\xi A_m^a = i\,\bar{\xi}\gamma_m \lambda^a$$
$$\delta_\xi \lambda^a = \frac{1}{2}\gamma^{pq} F_{pq}\,\xi$$

where, in $d=4,6,10$, ξ and λ^a are M, SMW, MW, respectively. Note that the transformation of A can be written in a manifestly real form as
$$\delta_\xi A_m^a = \frac{i}{2}(\bar{\xi}\gamma_m\lambda^a - \bar{\lambda}^a\gamma_m\xi)$$

and the transformation law for λ^a implies $\delta_\xi \bar{\lambda}^a = -\bar{\xi}(\frac{1}{2}\gamma^{pq}F_{pq})$

To verify SUSY, we need to show

(1) $$[\delta_\xi, \delta_\eta] = 2i\,\bar{\xi}\gamma^m\eta\,\partial_m + \text{(gauge)}$$

(2) $$\delta_\xi \mathcal{L} = 0$$

Let's start with (2):

Using $$\delta\left[-\frac{1}{4}(F_{mn})^2\right] = -\frac{1}{2}F_{mn}^a(\mathcal{D}^m\delta A^{na} - \mathcal{D}^n\delta A^{ma})$$
$$= (\mathcal{D}^m F_{mn}^a)\,\delta A^{na}$$

and $\delta(\frac{1}{2}\bar{\lambda}^a i\gamma^m \mathcal{D}_m \lambda)$

$$= \frac{1}{2}[\delta\bar{\lambda}^a\, i\gamma^m \mathcal{D}_m \lambda + \bar{\lambda}^a i\gamma^m \mathcal{D}_m \delta\lambda^a + \bar{\lambda}^a i\gamma^m\, g f^{abc}\delta A^b_m\, \lambda^c\,]$$

$$= \bar{\lambda}^a i\gamma^m \mathcal{D}_m \delta\lambda^a + \frac{i}{2}\bar{\lambda}^a\gamma^m\lambda^c g f^{abc}\,\delta A^b_m$$

we have

$$\delta_\xi \mathcal{L} = \mathcal{D}^m F_{mn}^{\ a}\,(-i\bar{\lambda}^a\gamma^n\xi)$$
$$+ \bar{\lambda}^a i\gamma^m \mathcal{D}_m(\tfrac{1}{2}\gamma^{pq}F_{pq}^{\ a}\,\xi)$$
$$+ \frac{i}{2} g f^{abc}(\bar{\lambda}^a\gamma^m\lambda^c)(i\bar{\xi}\gamma_m\lambda^b)$$

To deal with the 2^{nd} line, we can rewrite $\gamma^m\gamma^{pq}$ in terms of antisymmetric products of γ matrices

$$\gamma^m\gamma^p\gamma^q = \frac{1}{3}[\,\gamma^m\gamma^p\gamma^q + (2\eta^{mp}\gamma^q - \gamma^p\gamma^m\gamma^q) + (2\eta^{mp}\gamma^q - 2\eta^{mq}\gamma^p + \gamma^p\gamma^q\gamma^m)]$$

antisymmetrizing on $[pq]$

$$\gamma^m\gamma^{pq} = \gamma^{mpq} + \frac{1}{2}[(\tfrac{4}{3}\eta^{mp}\gamma^q - \tfrac{2}{3}\eta^{mq}\gamma^p) - (p\leftrightarrow q)]$$

$$= \gamma^{mpq} + \eta^{mp}\gamma^q - \eta^{mq}\gamma^p$$

So $\bar{\lambda}^a i\gamma^m\gamma^{pq}\frac{1}{2}\mathcal{D}_m F_{pq}^{\ a}\,\xi$

$$= \frac{i}{2}\bar{\lambda}^a\gamma^{mpq}\mathcal{D}_m F_{pq}^{\ a}\,\xi + i\bar{\lambda}^a\gamma^q\,\mathcal{D}^m F_{mq}^{\ a}\,\xi$$

The second term cancels the first line of $\delta_\xi \mathcal{L}$.
The first term vanishes by the Bianchi identity

$$D_{[m} F^a_{pq]} = 0$$

All that remains is

$$\delta_\xi \mathcal{L} = -\frac{i}{2} g f^{abc} \bar{\lambda}^a \gamma^m \lambda^b \, \bar{\xi} \gamma_m \lambda^c$$

Now, I claim that, for all three types of Majorana fermions, the following magic Fierz identity is valid

$$(*) \qquad \bar{\psi} \gamma^m \xi \, \bar{\eta} \gamma_m \chi - (\xi \leftrightarrow \eta)$$
$$= \bar{\eta} \gamma^m \xi \, \bar{\psi} \gamma_m \chi$$

Assuming $(*)$, we can rewrite this last term as

$$\delta_\xi \mathcal{L} = +\frac{i}{2} g f^{abc} \bar{\lambda}^a \gamma^m \lambda^b \, \bar{\lambda}^c \gamma_m \xi$$

with

$$f^{abc} \bar{\lambda}^a \gamma^m \lambda^b \, \bar{\lambda}^c \gamma_m \xi$$
$$= \frac{1}{2} f^{abc} \left(\bar{\lambda}^a \gamma^m \lambda^b \, \bar{\lambda}^c \gamma_m \xi - \bar{\lambda}^a \gamma^m \lambda^c \, \bar{\lambda}^b \gamma_m \xi \right)$$
$$= \frac{1}{2} f^{abc} \bar{\lambda}^c \gamma^m \lambda^b \, \bar{\lambda}^a \gamma_m \xi \qquad \text{by } (*)$$
$$= -\frac{1}{2} f^{abc} \bar{\lambda}^a \gamma^m \lambda^b \, \bar{\lambda}^c \gamma_m \xi$$
$$= 0 \; !$$

Now check (1):

$$[\delta_\xi,\delta_\eta]\, A_m^a = \delta_\xi(\, i\bar\eta\gamma_m\lambda^a) - \delta_\eta(\, i\bar\xi\gamma_m\lambda^a)$$

$$= i\bar\eta\,\gamma_m\,\tfrac{1}{2}\gamma^{pq}F_{pq}\,\xi - i\,\bar\xi\,\gamma_m\,\tfrac{1}{2}\gamma^{pq}F^a_{pq}\,\eta$$

now, for all three Majorana fermions:

$$\bar\eta\gamma^m\gamma^p\gamma^q\xi = -\,\bar\xi\gamma^q\gamma^p\gamma^m\eta$$

$$\bar\eta\gamma^m\gamma^{pq}\xi = +\,\bar\xi\gamma^{pq}\gamma^m\eta$$

so

$$[\delta_\xi,\delta_\eta]\, A_m^a = -\frac{i}{2}\,\bar\xi\,(\gamma^m\gamma^{pq} - \gamma^{pq}\gamma^m)\,\eta\, F_{pq}$$

$$= -\frac{i}{2}\,\bar\xi\,(2\eta^{mp}\gamma^q - 2\eta^{mq}\gamma^p)\,\eta\, F_{pq}$$

$$= -2i\,\bar\xi\gamma^q\eta\; F^a_{mq}$$

$$= 2i\,\bar\xi\gamma^q\eta\,(\partial_q A_m^a - \partial_m A_q^a - gf^{abc}A_m^b A_q^c)$$

The ~~quantity~~ on the RHS is the desired result, plus the same gauge piece that we found for the 4-d case. The corresponding CR for λ^a is

$$[\delta_\xi,\delta_\eta]\,\lambda^a = 2i\,\bar\xi\gamma^m\eta\, D_m\lambda^a$$

Is it correct?

$$[\delta_\xi, \delta_\eta]\, \lambda^a = \delta_\xi \left(\tfrac{1}{2} \gamma^{pq} F_{pq} \eta \right) - (\xi \leftrightarrow \eta)$$

$$= \gamma^{pq} \mathcal{D}_p \left(i \bar{\xi} \gamma_q \lambda^a \right) \eta - (\xi \leftrightarrow \eta)$$

$$= i \gamma^{pq} \eta \;\; \bar{\xi} \gamma_q \left(\mathcal{D}_p \lambda^a \right) - (\xi \leftrightarrow \eta)$$

This is as far as we can get off-shell. But if we are willing to use the equation of motion for λ:

$$i \gamma^p \mathcal{D}_p \lambda^a = 0$$

we can rewrite $\gamma^{pq} \cdot \gamma_q \mathcal{D}_p \lambda^a$

$$= \tfrac{1}{2} [\gamma^p \gamma^q - \gamma^q \gamma^p] \cdot \gamma_q \mathcal{D}_p \lambda^a$$

$$= (\gamma^p \gamma^q - \eta^{pq}) \quad \gamma_q \mathcal{D}_p \lambda^a$$

$$= \gamma^p \gamma^q \cdot \gamma_q \mathcal{D}_p \lambda^a - 1 \cdot \underbrace{\gamma^p \mathcal{D}_p \lambda^a}_{= 0 \text{ by eq of motion}}$$

then, <u>on-shell</u>

$$[\delta_\xi, \delta_\eta]\, \lambda^a = i \gamma^p \gamma^q \eta \;\; \bar{\xi} \gamma_q \mathcal{D}_p \lambda^a - (\eta \leftrightarrow \xi)$$

again using (*)

$$= i \bar{\xi} \gamma^m \eta \cdot \gamma^p \gamma_m \mathcal{D}_p \lambda^a$$

$$= i \bar{\xi} \gamma^m \eta \cdot (2\eta^{pm} - \gamma^m \gamma^p) \mathcal{D}_p \lambda^a$$

$$= 2i \bar{\xi} \gamma^m \eta \;\; \mathcal{D}_m \lambda^a \qquad \text{again using } \underline{\gamma^p \mathcal{D}_p \lambda^a = 0}$$

so the Lagrangian on p. 20 is supersymmetric in $d = 4, 6, 10$ as long as the Fierz identity (+) is correct. To complete the proof, I'll now prove this identity in the three cases.

Begin with Majorana fermions in $d=4$. Because the antisymmetric products of γ's form a basis in the space of 4×4 matrices, we can always rewrite:

$$\bar\psi\gamma^m\xi\;\bar\eta\gamma_m\chi = \sum_{AB} c_{AB}\;\bar\eta\Gamma^A\xi\;\;\bar\psi\Gamma^B\chi$$

where $\Gamma^A \in \{1, \gamma^a, \gamma^{ab}, \gamma^{abc}, \gamma^{abcd}\}$

To compute c_{AB}, replace $\xi\bar\eta \to \Gamma^C$ in this equation:

$$\bar\psi\gamma^m\Gamma^C\gamma_m\chi = \sum_{AB} c_{AB}\,(-\mathrm{tr}[\Gamma^C\Gamma^A])\,\bar\psi\Gamma^B\chi$$

Now $\mathrm{tr}\,\Gamma^C\Gamma^A = (\text{const})\cdot\delta^{AC}$, and also $\gamma^m\Gamma^C\gamma_m = (\text{const})\,\Gamma^C$, so c_{AB} is actually diagonal: $c_{AB}\propto\delta_{AB}$:

$$\bar\psi\gamma^m\xi\;\;\bar\eta\gamma_m\chi = \sum_A c_A\;\bar\eta\Gamma^A\xi\;\;\bar\psi\Gamma^A\chi$$

Now antisymmetrize on $(\xi\leftrightarrow\eta)$. On the right-hand side, we need keep only those terms antisymmetric in $(\xi\leftrightarrow\eta)$

with C in $d=4$

$$\bar\eta\xi = \eta^T C(-C^{-1})\bar\xi^T = -\eta^T\bar\xi^T = +\bar\xi\eta$$
$$\bar\eta\gamma^m\xi = \eta^T C\gamma^m(-C^{-1})\bar\xi^T = +\eta^T(\gamma^m)^T\bar\xi^T = -\bar\xi\gamma^m\eta$$
$$\bar\eta\gamma^{mn}\xi = +\bar\xi\gamma^{nm}\eta = -\bar\xi\gamma^{mn}\eta$$
$$\bar\eta\gamma^{mnp}\xi = -\bar\xi\gamma^{pnm}\eta = +\bar\xi\gamma^{mnp}\eta$$
$$\bar\eta\gamma^{mnpq}\xi = +\bar\xi\gamma^{qpnm}\eta = +\bar\xi\gamma^{mnpq}\eta$$

so only $\bar\eta\gamma^m\xi$, $\bar\eta\gamma^{mn}\xi$ need be included. Further, when we compute C_A for γ^{mn}, we encounter

$$\bar\Psi\gamma^m\gamma^{pq}\gamma_m X = C_{(pq)}[-\mathrm{tr}(\gamma^{pq}\gamma^{kl})]\bar\Psi\gamma_{kl}X$$

but

$$\gamma^m\gamma^p\gamma^q\gamma_m = 2\eta^{mp}\gamma^q\gamma_m - \gamma^p 2\eta^{mq}\gamma_m + \gamma^p\gamma^q\gamma^m\gamma_m$$
$$= 2\gamma^q\gamma^p - 2\gamma^p\gamma^q + \gamma^p\gamma^q\cdot 4$$

antisymmetrizing on $[pq]$

$$\gamma^m\gamma^{pq}\gamma_m = \gamma^{pq}[4-2-2] = 0$$

so $C_{(pq)} = 0$

It is worth noting a more general result, proved the same way:

$$\gamma^m\gamma^{a_1 - a_N}\gamma_m = \gamma^{a_1 - a_N}(d-2N)\cdot(-1)^N$$

anyway, we have reduced our Fierz identity to

$$\bar{\Psi}\gamma^m\xi\ \bar{\eta}\gamma_m\chi - (\xi\leftrightarrow\eta)$$

$$= 2\sum_p c_p\ \bar{\eta}\gamma^p\xi\ \bar{\Psi}\gamma_p\chi$$

where: $(\xi\bar{\eta}\to\gamma^q)$

$$\bar{\Psi}\gamma^m\gamma^q\gamma_m\chi = c_p\left[-\mathrm{tr}(\gamma^q\gamma^p)\right](\bar{\Psi}\gamma_p\chi)$$

Now $\gamma^m\gamma^q\gamma_m = -2\gamma^q$

$$-\mathrm{tr}\left[\gamma^q\gamma^p\right] = -\tfrac{1}{2}\mathrm{tr}\ \{\gamma^q,\gamma^p\} = -4\eta^{pq}$$

so $C_p = \frac{1}{2}$ and we indeed find

$$\bar{\Psi}\gamma^m\xi\ \bar{\eta}\gamma_m\chi - (\xi\leftrightarrow\eta) = \bar{\eta}\gamma^p\xi\ \bar{\Psi}\gamma_p\chi$$

The next easiest case is that of 10-d MW fermions.

For Weyl fermions $\quad \Gamma\xi = +\xi \qquad \bar{\xi}\Gamma = -\bar{\xi}$, so

$$\bar{\eta}\xi = \bar{\eta}\Gamma\xi = -\bar{\eta}\xi = 0$$

similarly $\quad \bar{\eta}\gamma^{a_1 - a_N}\xi = 0 \quad$ for N <u>even</u>

In addition, since $\quad \Gamma = \gamma^0\gamma^1 - \gamma^9$, we can replace

$$\gamma^{a_1 - a_N}\xi = \gamma^{a_1 - a_N}\Gamma\xi = \gamma^{b_1 - b_M}\xi$$

where the indices $b_1 - b_M$ are those <u>not</u> appearing in $a_1 - a_N$.

so we do not have to consider all Γ^A in the Fierz identity, only those with ≤ 5 indices.

Now impose the condition of antisymmetry in $\eta \leftrightarrow \xi$.

Up to 5 indices, only

$$\bar{\xi}\gamma^m\eta \qquad \bar{\xi}\gamma^{mn}\eta \qquad \bar{\xi}\gamma^{mnpqr}\eta$$

are antisymmetric, as we found on p. 26, and of these $\bar{\xi}\gamma^{mn}\eta = 0$

Further, when we compute the C_A coefficient for γ^{mnpqr}, this is proportional to

$$\gamma^m \gamma^{npqrs}\gamma_m = -(10 - 2\cdot 5)\gamma^{npqrs}$$

$$= 0 \quad !$$

So

$$\bar{\Psi}\gamma^m\xi \;\; \bar{\eta}\gamma_m\chi \;-\; (\xi\leftrightarrow\eta)$$

$$= 2\sum_p c_p \;\; \bar{\eta}\gamma^p\chi \;\; \bar{\Psi}\gamma_p\chi$$

to compute c_p, replace $\xi\bar{\eta} \to \gamma^q\left(\frac{1-\Gamma}{2}\right)$ (respecting the Weyl condition). Then

$$\bar{\Psi}\gamma^m\gamma^q\left(\tfrac{1-\Gamma}{2}\right)\gamma_m\chi = c_p\left[\tfrac{1}{?}\operatorname{tr}\gamma^q\left(\tfrac{1-\Gamma}{2}\right)\gamma^p\right]\bar{\Psi}\gamma_p\chi$$

$$\frac{1-\Gamma}{2}\gamma^m\chi = \gamma^m\left(\frac{1+\Gamma}{2}\right)\chi = \gamma^m\chi$$

$$\gamma^m\gamma^q\gamma_m = -(10-2)\gamma^q = -8\gamma^q$$

and $\quad -\mathrm{tr}\left[\gamma^q\left(\frac{1-\Gamma}{2}\right)\gamma^p\right] = -\frac{1}{2}\eta^{pq}\,\mathrm{tr}[1]$

$$= -\tfrac{1}{2}\eta^{pq}\cdot 32 = -16\,\eta^{pq}$$

so $\quad \bar\Psi(-8\gamma^q)\chi = c_q(-16)\,\bar\Psi\gamma^q\chi \Rightarrow c_q = +\frac{1}{2}$

and we find again

$$\bar\Psi\gamma^m\xi\;\bar\eta\gamma_m\chi - (\xi\leftrightarrow\eta) = \bar\eta\gamma^p\xi\;\bar\Psi\gamma_p\chi$$

Finally, consider the case of symplectic MW fermions in 6 dimensions. To begin, let's write the Fierz identity for Weyl fermions in 6-dimensions. We need to keep terms with an odd number of γ's only, and only up to 3 indices.

$$\bar\Psi\gamma^q\xi\;\bar\eta\gamma_q\chi = \sum_A c_A\;\bar\eta\Gamma^A\xi\;\bar\Psi\Gamma^A\chi$$

where $\Gamma^A = \gamma^p$ and $\Gamma^A = \gamma^{pqr}$ must be included. But

$$\gamma^m\gamma^{pqr}\gamma_m = -(6-3\cdot 2)\gamma^{pqr} = 0$$

so the c_A for γ^{pqr} are zero. We can compute c_A for γ^p

$$\bar\Psi\gamma^m\gamma^q\left(\tfrac{1-\Gamma}{2}\right)\gamma_m\chi = c_p\left[-\mathrm{tr}\left(\gamma^q\left(\tfrac{1-\Gamma}{2}\right)\gamma^p\right)\right]\bar\Psi\gamma_p\chi$$

$$\bar\Psi\gamma^q\chi\cdot[-(6-2)] = c_p(-\tfrac{1}{2})\cdot 8\;\bar\Psi\gamma^q\chi$$

$$-4 = c_p(-4) \Rightarrow c_p = +1$$

so

$$\bar{\Psi}\gamma^q\xi \quad \bar{\eta}\gamma_q\chi \quad = + \bar{\eta}\gamma^q\xi \quad \bar{\Psi}\gamma_q\chi$$

for Weyl fermions in 6 dimensions. Now add the symplectic index and use the Majorana condition:

$$(\bar{\Psi}\gamma^q\xi)\,(\bar{\eta}\gamma_q\chi)$$

$$= (\bar{\Psi}^i\gamma^q\xi^i)\,(\bar{\eta}^j\gamma_q\chi^j) \;=\; \bar{\eta}^j\gamma^q\xi^i \;\bar{\Psi}^i\gamma_q\chi^j$$

$$(\bar{\Psi}^i\gamma^q\xi^i)(\bar{\eta}^j\gamma_q\chi^j) - (\bar{\Psi}^i\gamma^q\eta^i)(\bar{\xi}^j\gamma_q\chi^j)$$

$$= (\bar{\eta}^j\gamma^q\xi^i - \bar{\xi}^j\gamma^q\eta^i)(\bar{\Psi}^i\gamma_q\chi^j)$$

now $+\bar{\xi}^j\gamma^q\eta^i = \xi^{kT} C_1 c_{kj}\,\gamma^q\, c_{il}(-C^{-1})\bar{\eta}^{Tl}$

$$= -\xi^{kT}(+\gamma^q)^T c_{kj}c_{il}\;\bar{\eta}^{Tl}$$

$$= +\bar{\eta}^l\gamma^q\xi^k\;c_{kj}c_{il}$$

Now $\quad c_{kj}c_{il} = \delta_{ki}\delta_{jl} - \delta_{kl}\delta_{ij}$

so $\quad = \bar{\eta}^j\gamma^q\xi^i - \bar{\eta}^k\gamma^q\xi^k\,\delta_{ij}$

The first term here cancels the $\bar{\eta}\gamma^q\xi$ term above and we are left with: $\quad (\bar{\Psi}\gamma^q\xi)(\eta\gamma_q\chi) - (\bar{\Psi}\gamma^q\eta)(\bar{\xi}\gamma_q\chi)$

$$= +(\bar{\eta}\gamma^q\xi)(\bar{\Psi}\gamma_q\chi) \quad \text{as desired!}$$