7. Applications of Superspace

Now that we have developed the superspace formalism, which gives actions and other expressions that are manifestly $N=1$ supersymmetric, I would like to describe some applications.

① Supergraph perturbation theory.

It is possible to develop Feynman perturbation theory directly from the superspace action. I will not describe in detail how to do this (there is an extensive discussion in Wess + Bagger) but I would like to quote some results. The propagators in this formalism are

$$\overset{\Phi}{\longleftarrow} = \langle \Phi \Phi^* \rangle = i\,\frac{\frac{1}{16}\bar{D}^2 D^2}{p^2} = i \times \begin{pmatrix}\text{projector onto a} \\ \text{chiral superfield}\end{pmatrix}$$

$$\overset{V}{\sim\!\sim} = \langle V V \rangle = i\frac{1}{p^2} \quad \text{(in a Feynman gauge)}$$

The vertices are localized in superspace. Note that, while the chiral field propagator has components that are fairly conventional, the vector field propagator leads to compound Feynman rules such as

$$\langle \phi\, D \rangle = \frac{-i}{p^2} \qquad \langle \psi_\alpha\, \lambda_\beta \rangle = \frac{-i\,C_{\alpha\beta}}{p^2}$$

so the whole field content of the vector superfield becomes involved in the perturbation theory.

On the other hand, the results of this perturbation theory must be manifestly supersymmetric and so must exhibit the SUSY nonrenormalization theorems. Here are some examples:

a) All vacuum diagrams (e.g. ) are proportional to $\int d^4\theta \; [\theta\text{-indep}] = 0$. Thus, <u>all</u> <u>perturbative contributions to the vacuum energy vanish</u> diagram by diagram.

b) All contributions to the effective Lagrangian for chiral superfields are of the form

 $= \int d^4\theta \; (\text{function of } \Phi, \Phi^*)$

Notice that all such interactions, when reduced to components, have derivatives. The structure

$$\int d^2\theta \; (\text{function of } \Phi)$$

is not generated by supergraphs. So, there are no <u>perturbative corrections to the superpotential</u>.

c.) From this argument, you might think that it is also not possible to generate the structure

$$\int d^2\theta\ W^2$$

If we rescale $gV \to V$ so that g is removed from all couplings, the kinetic term of the gauge field is

$$\int d^2\theta\ \frac{1}{4g^2} W^2 + h.c.$$

So the radiative corrections of the form $\int d^2\theta \cdot \Delta \cdot W^2$ generate coupling constant renormalization. In a fixed gauge, this structure can be rewritten as

$$\int d^4\theta\ \Delta \cdot V(p)\, p^2\, V(-p)$$

and can be generated by supergraphs. However, in a well-chosen regulator, this contribution appears only at 1-loop order. Later in this lecture, I will make this ~~more~~ clear by connecting this result to the axial vector anomaly.

d.) In 1-loop order, corrections to the Fayet Iliopoulos term

$$\int d^4x\ \kappa D = \int d^4x\, d^4\theta\ \kappa V$$

are proportional to the sum of U(1) charges of the matter fields:

 $\propto \mathrm{tr}\, Q$

So if $\mathrm{tr}\, Q = 0$, this correction vanishes to 1-loop order. It is possible to show that, in that case, the correction to K vanishes to all orders.

(2) Soft SUSY-breaking perturbations

In a thy with exact SUSY, there are no quadratically divergent corrections to parameters in the Lagrangian. Now, the real world is not supersymmetric, so it is worth asking whether we can add perturbations to $\mathcal{L}$ while preserving this property. From ordinary quantum field thy, we might expect that perturbations to $\mathcal{L}$ do not disturb the symmetry structure if the terms we add have dimension < 4. However, it is not so hard to find an operator with dimension <4 that leads to quadratic divergences. Consider

$$\delta\mathcal{L} = \int d^4x \; g\phi^*\phi^2 + h.c.$$

Then $\displaystyle\int \frac{d^4k}{k^2} \rightarrow$  leads to $\displaystyle\int d^4x \, (g\Lambda^2)\,\phi$

that is, we induce a correction to the potential that is quadratically divergent and can shift $\langle\phi\rangle$ by a divergent amount. We would like to know if this is a generic situation, or if there are specific operators we can add that avoid this difficulty.

Grisaru and Girardello analyzed this problem using superspace. Imagine that we write a supersymmetric

Lagrangian that contains some extra chiral field Φ_0 or vector field V_0. Perturbation theory with this Lagrangian will never generate quadratically divergent corrections to $\mathcal{L}$.

Now replace

$$\Phi_0 \to \theta^2 F \quad \text{or} \quad V_0 \to \theta^2\bar{\theta}^2 d$$

These replacements break SUSY. But the terms that result still cannot generate quadratic divergences.

Let's apply this to various cases.

$$\int d^2\theta \,(\Phi_0\Phi,\ \Phi_0\Phi^2,\ \Phi_0\Phi^3) \to (f\phi,\ f\phi^2,\ f\phi^3)$$

$$\int d^4\theta \ \Phi^\dagger\Phi V_0 \to d\phi^*\phi$$

so the perturbations of these forms are "soft" in the SUSY sense that they do not generate quadratic divergences.

Also, the Lagrangian

$$\int d^2\theta \ \Phi_0 W^2$$

is supersymmetric specifically if Φ_0 is a chiral superfield; this generates

$$f\,\lambda^{Ta} C \lambda^a$$

a gaugino mass term. So, a $\delta\mathcal{L}$ of the form

$\delta\mathcal{L}$ = [(holomorphic polynomial in Φ_k) + h.c.]

+

(scalar mass term $\quad m_{k\ell}\,\phi_k^*\phi_\ell$)

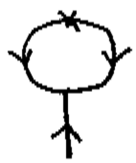
+

(gaugino mass term $\quad m_a\,\lambda^{Ta} c\,\lambda^a$)

is soft in the sense that we require. Of possible SUSY-breaking perturbations with dimension ≤ 3, the only terms that do not appear on the list above are

$$g\,\phi^*\phi^2 + \text{h.c.} \qquad m\,\psi^T c\,\psi$$

We have already seen that the first of these interactions generates quadratic divergences. For the matter fermion mass, if we link it to a superpotential vertex:



$$\sim m\lambda\left(\int\frac{d^4k}{k^2}\right)\phi \sim m\lambda\Lambda^2\phi$$

we can also generate a divergence. So the list at the top of this page is complete.

③ N=1 SUSY nonlinear sigma models

Up to now, we have only considered the simplest kinetic term for the chiral superfield

$$\int d^4\theta \; \Phi^* \Phi$$

From superspace, it is clear that the most general 2-derivative Lagrangian for a set of chiral superfields is

$$\int d^4\theta \;\; K(\Phi_k, \Phi_k^*)$$

Let's now investigate the physics of this more general interaction.

We need to compute $[K(\Phi,\Phi^*)]_{\theta^2\bar\theta^2}$, so start expanding:

$$\mathcal{K} = K(\phi,\phi^*) + \frac{\partial K}{\partial \phi_k}(\sqrt{2}\,\theta^T c\,\psi_k + \cdots)$$
$$+ \frac{1}{2}\frac{\partial^2 K}{\partial\phi_k \partial\phi_\ell}(\sqrt{2}\,\theta^T c\,\psi_k + \cdots)(\sqrt{2}\,\theta^T c\,\psi_\ell + \cdots)$$
$$+ \cdots$$

Abbreviate:

$$\frac{\partial K}{\partial \phi_k} = K_{,k} \qquad \frac{\partial K}{\partial \phi_k^*} = K_{,\bar{k}} \quad \text{etc.}$$

the terms involving two derivatives on ϕ or ϕ^* are

$$K_{,k}\left(-\tfrac{1}{4}\theta^2\bar\theta^2\partial^2\phi_k\right) + K_{,\bar k}\left(-\tfrac{1}{4}\theta^2\bar\theta^2\partial^2\phi_k^*\right)$$
$$+\tfrac{1}{2}K_{,kl}\left(i\bar\theta\bar\sigma^a\theta\,\partial_a\phi_k\right)\left(i\bar\theta\bar\sigma^b\theta\,\partial_b\phi_l\right) + h.c.$$
$$+ K_{,k\bar l}\left(-i\bar\theta\sigma^a\theta\,\partial_a\phi_l^*\right)\left(i\bar\theta\bar\sigma^b\theta\,\partial_b\phi_k\right)$$

$$= \theta^2\bar\theta^2\Big\{-\tfrac{1}{4}K_{,k}\,\partial^2\phi_k - \tfrac{1}{4}K_{,\bar k}\,\partial^2\phi_k^*$$
$$\div \tfrac{1}{4}K_{,kl}\,\partial_a\phi_k\partial^a\phi_l - \tfrac{1}{4}K_{,\bar k\bar l}\,\partial_a\phi_k^*\partial^a\phi_l^*$$
$$+ \tfrac{1}{2}K_{,k\bar l}\,\partial_a\phi_l^*\partial^a\phi_k\Big\}$$

Note that

$$\int d^4x\,\left(-\tfrac{1}{4}K_{,k}\,\partial^2\phi_k\right) = \int d^4x\,\tfrac{1}{4}\partial_a(K_{,k})\,\partial^a\phi_k$$
$$= \int d^4x\left(\tfrac{1}{4}K_{,kl}\,\partial_a\phi_l\partial^a\phi_k + \tfrac{1}{4}K_{,k\bar l}\,\partial_a\phi_l^*\partial^a\phi_k\right)$$

in which the first term cancels the 2nd line above and the second term adds to the third line to give finally:

$$\theta^2\bar\theta^2\quad K_{,k\bar l}\,\partial_a\phi_l^*\partial^a\phi_k$$

It is not surprising that the terms with $\partial_k\phi\,\partial_l\phi$ cancel. If K were a function of Φ but not Φ^*, we would have

$$\int d^4\theta\;F(\Phi) = \int d^2\theta\left(-\tfrac{1}{4}\bar D^2\right)F(\Phi) = 0$$

so any nonzero contributions to the kinetic term must involve both ϕ^* and ϕ.

Following our analysis of $\int d^4\theta\, \Phi^\dagger \Phi$, we can find very quickly the term

$$\int d^4\theta\, K(\Phi,\Phi^*) = K_{,\bar{k}\ell}\left(\partial_a\phi_k^*\partial^a\phi_\ell + \tfrac{1}{2}\psi_k^\dagger i\bar{\sigma}\cdot\partial\psi_\ell - \tfrac{1}{2}\partial_a\psi_k^\dagger\bar{\sigma}^a\psi_\ell + F_k^* F_\ell\right)$$

$$+ (\text{terms at least } \underline{\text{cubic}} \text{ in } \partial\phi, \psi, F.)$$

Let's now gather these nonlinear terms:

$$K_{,\bar{k}\ell m}(-\sqrt{2}\psi_k^* c\bar{\theta})(i\bar{\theta}\bar{\sigma}^a\theta\,\partial_a\phi_\ell)(\sqrt{2}\theta^T c\psi_m)$$

$$= K_{,\bar{k}\ell m}(-2i)(-\tfrac{1}{2}\bar{\theta}^2)(\tfrac{1}{2}\theta^2)\,\psi_k^* c c\bar{\sigma}^a c c\,\partial_a\phi_\ell\psi_m$$

$$= K_{,\bar{k}\ell m}\,\frac{i}{2}\,\theta^2\bar{\theta}^2\,\psi_k^*(\bar{\sigma}^a\partial_a\phi_\ell)\psi_m$$

$$K_{,\bar{k}\ell m}\,(\bar{\theta}^2 F_k^*)\tfrac{1}{2}(\sqrt{2}\,\theta^T c\psi_\ell)(\sqrt{2}\theta^T c\,\psi_m)$$

$$= K_{,\bar{k}\ell m}\,\bar{\theta}^2 F_k^*\,\theta^2\,\tfrac{1}{2}\,\psi_\ell^T ccc\psi_m$$

$$= K_{,\bar{k}\ell m}\,(-\tfrac{1}{2})\,\theta^2\bar{\theta}^2 F_k^*\psi_\ell^T c\psi_m$$

the conjugates of these terms also appear, as well as:

$$K_{,\bar{k}\bar{\ell}mn}\,\tfrac{1}{2}(-\sqrt{2}\psi_k^\dagger c\bar{\theta})(-\sqrt{2}\bar{\theta}^T c\psi_\ell^*)\,\tfrac{1}{2}(\sqrt{2}\psi_m^T c\theta)(\sqrt{2}\theta c\psi_n)$$

$$= K_{,\bar{k}\bar{\ell}mn}(-\tfrac{1}{2}\bar{\theta}^2)(\tfrac{1}{2}\theta^2)\,\psi_k^\dagger ccc\psi_\ell^*\;\psi_m^T ccc\psi_n$$

$$= K_{,\bar{k}\bar{\ell}mn}\,(-\tfrac{1}{4})\,\psi_k^\dagger c\psi_\ell^*\,\psi_m^T c\psi_n$$

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$$\int d^4\theta\, K(\Phi,\bar\Phi) = K_{,\bar k l}\left\{ \partial_a\phi_k^* \partial^a\phi_l + \tfrac{i}{2}\left(\psi_k^\dagger \bar\sigma\cdot\partial\psi_l - \partial_a\psi_k^\dagger\bar\sigma^a\psi_l\right) + F_k^* F_l \right\}$$

$$+ \tfrac{i}{2} K_{,\bar k l m}\, \psi_k^\dagger \bar\sigma^\rho \partial_\rho\phi_l \psi_m - \tfrac{i}{2} K_{,\bar l \bar m k}\, \psi_l^\dagger \partial_a\phi_m^* \bar\sigma^a \psi_k$$

$$- \tfrac{1}{2} K_{\bar k l m}\, F_k^* \psi_l^T c \psi_m + \tfrac{1}{2} K_{,\bar k \bar l m}\, \psi_k^\dagger c \psi_l^* F_m$$

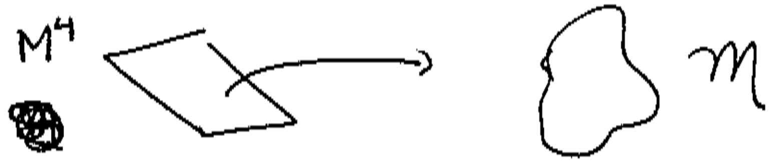
$$- \tfrac{1}{4} K_{\bar k \bar l m n}\, \psi_k^\dagger c \psi_l^* \psi_m^T c \psi_n$$

Now, what is the significance of this structure?

In ordinary quantum field thy, the most general 2-derivative action on a multiplet of scalar fields is a nonlinear sigma model:

$$\int d^4x \quad g_{kl}\, \partial_a\phi^k \partial^a\phi^l$$

This has the interpretation of being a quantum field thy in which the field is a coordinate on a manifold $\mathcal{M}$ w. metric g_{kl}



The SUSY version above is a nonlinear sigma model on some sort of complex manifold.

A physicist's notion of a complex manifold is quite particular. We image a manifold parametrized by complex coordinates z^k $\bar{z}^k$

with Riemannian geometry and a Hermitian metric

$$ds^2 = g_{\bar{k}l}\, d\overset{*}{z}{}^k dz^l$$

Mathematicians have a much more general notion of a complex manifold. First, the manifold should have an even number of dimensions (2N). Next, we need a way to sort coordinates into holomorphic and antiholomorphic. This can be done by postulating a complex structure, a tensor $J_i{}^j$ s.t.

$$(J^2)_i{}^k \quad J_i{}^j J_j{}^k = -\delta_i{}^k$$

with equal numbers of $+i$ and $-i$ eigenvalues. If we choose coordinates $z, \bar{z}$ in the eigenspaces of J, then

$$J_i{}^j = +i\,\delta_i^j \qquad J_{\bar{i}}{}^{\bar{j}} = -i\,\delta_{\bar{i}}^{\bar{j}} \qquad i,j = 1\text{—}N$$

However, this refers only to one coordinate patch. To be a complex structure, J must be defined globally (i.e. the patches must fit together)

Now we have a complex manifold, but the metric could still have the general form

$$d^2s = g_{\bar{i}j}\, d\overset{*}{z}{}^i dz^j + g_{ij}\, dz^i dz^j + g_{\bar{i}\bar{j}}\, dz^{\bar{i}} dz^{\bar{j}}$$

and the geometry might not be Riemannian. If we postulate also a Hermitian metric and Γ^{l}_{km} a Riemannian connection for which J is covariantly constant, we obtain a Kähler manifold.

Given a Kähler manifold, let $J_{i\bar{k}} = J_i{}^j g_{j\bar{k}}$, and define the Kähler form

$$k = J_{i\bar{k}}\, dz^i \wedge d\bar{z}^k$$

$$= i\, g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k \quad \text{in a coordinate patch}$$

if J is covariantly constant $\nabla_m J_i{}^j = 0$, with a symmetric connection, this implies,

$$\partial k = \bar{\partial} k = 0$$

with $\partial = dz^i \frac{\partial}{\partial z^i}$ $\quad \bar{\partial} = d\bar{z}^j \frac{\partial}{\partial \bar{z}^j}$

so k is a doubly closed form. Locally, then, k can be integrated to

$$k = \partial \bar{\partial} K(z, \bar{z})$$

where K is a function called the Kähler potential.

Note that the "Kähler transformation"

$$K(z,\bar{z}) \to K(z,\bar{z}) + F(z) + \bar{F}(\bar{z})$$

yields the same Kähler form and ultimately, the same metric and geometry.

We now see that, for a Kähler manifold

$$g_{i\bar{j}} = K_{,i\bar{j}}$$

Let's also compute the connection and curvature:

$$\Gamma_{i\ k}^{\ \bar{j}} = \tfrac{1}{2} g^{\bar{j}l}[\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}] = 0$$

$$\Gamma_{\bar{i}\ k}^{\ \bar{j}} = \tfrac{1}{2} g^{\bar{j}l}[\partial_{\bar{i}} g_{kl} + \partial_k g_{\bar{i}l} - \partial_l g_{\bar{i}k}] = 0$$

with $g_{\bar{i}l} = K_{\bar{i}l}$ etc.

so i indices are not transformed to $\bar{i}$ indices by parallel transport. ✓

$$\Gamma_{i\ k}^{\ j} = \tfrac{1}{2} g^{j\bar{l}}[\partial_i g_{k\bar{l}} + \partial_k g_{i\bar{l}} - \partial_{\bar{l}} g_{ik}]$$

$$= g^{j\bar{l}} K_{,ik\bar{l}}$$

$$\Gamma_{\bar{i}\ \bar{k}}^{\ \bar{j}} = g^{\bar{j}l} K_{\bar{i}\bar{k}l}$$

the nonvanishing elements of the curvature tensor are:

$$R_{i\bar{j}\ \bar{l}}^{\ \ \bar{k}} = \partial_i \Gamma_{\bar{j}\ \bar{l}}^{\ \bar{k}} - \partial_{\bar{j}} \Gamma_{i\ \bar{l}}^{\ \bar{k}} + \Gamma_{i\ m}^{\ \bar{k}} \Gamma_{\bar{j}\ \bar{l}}^{\ \bar{m}} - \Gamma_{\bar{j}\ \bar{m}}^{\ \bar{k}} \Gamma_{i\ \bar{l}}^{\ \bar{m}}$$

$$= \partial_i \Gamma_{\bar{j}\ \bar{l}}^{\ \bar{k}} = g^{\bar{k}m} K_{i\bar{j}m\bar{l}} - g^{\bar{k}m} \partial_i g_{m\bar{n}} g^{\bar{n}p} K_{\bar{j}\bar{l}p}$$

or

$$R_{i\bar{j}k\bar{l}} = K_{i\bar{j}k\bar{l}} - K_{ik\bar{n}} g^{\bar{n}p} K_{\bar{j}\bar{l}p}$$

note that is expression has the symmetry of the Riemann term $(i\bar{j}) \leftrightarrow (k\bar{l})$.

Now let's put this back into the Lagrangian on p.10. We now recognize that the $\partial_m\phi^* \partial^m\phi$ piece is a nonlinear sigma model on a Kähler manifold with metric

$$g_{\bar{k}l} = K_{,\bar{k}l}$$

and Kähler potential $K(\phi,\phi^*)$. Note that the metric is not affected by a Kähler transformation

$$K(\phi,\phi^*) \to K(\phi,\phi^*) + F(\phi) + [F(\phi)]^*$$

In fact, the whole SUSY action is not affected, since $\int d^4\theta \; F(\Phi) = 0$ if $F(\Phi)$ is a chiral superfield. Using also

$$\Gamma^n_{lm} = g^{n\bar{k}} K_{,\bar{k}lm} \qquad g^{ab} = \underline{\text{inverse}} \text{ of } g_{ab}$$

the Lagrangian on p. 10 becomes

$$\int d^4\theta \; K(\Phi,\Phi^*)$$

$$= g_{\bar{k}l}\Big\{ \partial_a\phi^{*k}\partial^a\phi^l + \tfrac{i}{2}\psi^{k+}\bar{\sigma}^a(\partial_a\psi^l + \Gamma^l_{mn}\partial_a\phi^m\psi^n)$$

$$-\tfrac{i}{2}\,(\partial_a\psi^{k+} + \Gamma^{\bar{k}}_{\bar{m}\bar{n}}\psi^{+m}\partial_a\phi^{*n})\,\bar{\sigma}^a\psi^l + F^{*k}F^l\Big\}$$

$$-\tfrac{1}{2}K_{\bar{k}lm}F^{*k}\psi^{Tl}c\,\psi^m + \tfrac{1}{2}K_{\bar{k}\bar{l}m}\psi^{+k}c\,\psi^{*l}F^m$$

$$-\tfrac{1}{4}K_{\bar{k}\bar{l}mn}\,\psi^{k+}_c\psi^{l*}\,\psi^{mT}_c\psi^n$$

The derivative on the ψ field is

$$\mathcal{D}_a \psi^l = \partial_a \psi^l + \Gamma^l_{mn}\, \partial_a \phi^m \psi^n$$

Interpreting ψ^l as a vector in the tangent space of $\mathcal{M}$, this covariant derivative keeps track of the parallel transport of ψ relative to the coordinate directions $\partial_a \phi^m$. We can integrate $\mathcal{D}_a$ by parts:

$$\int d^4x \;\; g_{\bar{k}l} (-\mathcal{D}_a \psi^{\bar{k}\dagger}) \psi^l = \int d^4x \;\psi^{\bar{k}\dagger}\, \mathcal{D}_a \psi^l$$

Finally, eliminating F, we find a 4-fermion term of the structure

$$-\tfrac{1}{4}\left(K_{\bar{k}\bar{l}mn} - K_{\bar{k}\bar{l}\bar{p}}\, g^{p\bar{q}}\, K_{\bar{q}mn}\right) \psi^{k\dagger}_c \psi^{l*} \psi^{mT}_c \psi^n$$

which is just the Riemann curvature of $\mathcal{M}$. So, finally, the most general kinetic energy term for a chiral superfield is a nonlinear sigma model on a Kähler manifold

$$\mathcal{L} = \int d^4\theta \; K(\Phi, \Phi^*)$$

$$= g_{\bar{k}l} \left\{ \partial_a \phi^{*k} \partial^a \phi^l + \psi^{\dagger k}\, i\bar{\sigma}^a \mathcal{D}_a \psi^l \right\}$$

$$- \tfrac{1}{4} R_{\bar{i}\bar{j}kl}\; \psi^{\dagger i}_c \psi^{*j} \psi^{kT}_c \psi^l$$

In particular, if a supersymmetric model has a manifold of degenerate vacuum states with $N=1$ SUSY, those states form not only a complex manifold but a Kähler manifold.

(4) Energy-momentum in $N=1$ SUSY

As a final application of superspace, I would like to explain what the equation of energy-momentum conservation looks like in $N=1$ SUSY.

Let's begin with an example, a model with a ~~chiral~~ superfield with superpotential $W(\Phi)$:

$$\int d^4\theta \ \Phi^* \Phi \quad + \quad \int d^2\theta \ W(\Phi) \quad + \quad h.c.$$

It is not so clear how to find the superspace equation of motion for Φ. Note that we cannot simply vary $\int d^4\theta \, \Phi^*\Phi$ freely with respect to Φ because Φ is a constrained field. However, we can rewrite the action as

$$\int d^2\theta \left\{ \left(-\tfrac{1}{4}\bar{D}^2\right)(\Phi^*\Phi) + W(\Phi) \right\} \quad + \quad \int d^2\bar{\theta} \, (W(\Phi))^*$$

and vary Φ freely under the integral $\int d^2\theta$. Notice that, in the first term, $\bar{D}^2(\Phi^*\Phi) = (\bar{D}^2\Phi^*)\Phi$, since $\bar{D}_{\dot{a}}\Phi = 0$.

Then we can read off the equation of motion

$$-\frac{1}{4}\bar{D}^2\Phi^* + \frac{\partial W}{\partial \Phi} = 0$$

Now let

$$J_{\alpha\beta} = +i(\bar{\sigma}\cdot\partial)_{\alpha\beta}\Phi^* \; \Phi - i\,\Phi^*(\bar{\sigma}\cdot\partial)_{\alpha\beta}\Phi$$
$$+ \frac{1}{2}\bar{D}_\alpha\Phi^* \, D_\beta\Phi$$

This object is reminiscent of the energy-momentum tensor in being a quadratic form in the components of the chiral field. Using the equation of motion above, we can work out the equation of motion for $J_{\alpha\beta}$:

$$(\bar{D}J)_\beta \equiv - c_{\gamma\alpha}\bar{D}_\gamma J_{\alpha\beta}$$

$$= -(c_{\gamma\alpha}\bar{D}_\gamma (i\bar{\sigma}\cdot\partial)_{\alpha\beta}\Phi^*)\;\Phi + i c_{\gamma\alpha}\bar{D}_\gamma\Phi^*(\bar{\sigma}\cdot\partial)_{\alpha\beta}\Phi$$

$$-\frac{1}{2}(c_{\gamma\alpha}\bar{D}_\gamma\bar{D}_\alpha)\Phi^* \, D_\beta\Phi + \frac{1}{2}\bar{D}_\alpha\Phi^* c_{\gamma\alpha}\bar{D}_\gamma D_\beta\Phi$$

using $\{\bar{D}_\alpha, D_\beta\} = 2i(\sigma\cdot\partial)_{\alpha\beta}$

$$= \frac{1}{4}[(-c_{\gamma\alpha}\bar{D}_\gamma\bar{D}_\alpha), D_\beta]\Phi^* \; \Phi + c_{\gamma\alpha}\bar{D}_\gamma\Phi^*\{\bar{D}_\alpha, D_\beta\}\Phi$$

$$+\frac{1}{2}\bar{D}^2\Phi^* \, D_\beta\Phi + c_{\gamma\alpha}\bar{D}_\alpha\Phi^*\{\bar{D}_\gamma, D_\beta\}\Phi$$

The two terms with $\{\bar{D}_\alpha, D_\beta\}\Phi$ cancel by the antisymmetry of $c_{\gamma\alpha}$. (since $\bar{D}_\gamma\Phi = 0$)

also $\quad [\bar{D}^2, D_\beta]\, \Phi^* = -D_\beta \bar{D}^2 \Phi^*$

so

$$(\bar{D} J)_\beta = 2 \frac{\partial W}{\partial \Phi} D_\beta \Phi - D_\beta \left(\frac{\partial W}{\partial \Phi} \right) \Phi$$

$$= D_\beta \left(3 W[\Phi] - \Phi \frac{\partial W}{\partial \Phi} \right)$$

so, we find ~~that~~

$$(\bar{D} J)_\beta = D_\beta S$$

where $J_{\alpha\beta}$ is a Hermitian, general superfield and S is a chiral superfield. Ferrara and Zumino recognized ~~that~~ this is the general form of the equation of energy-momentum conservation in superspace. They provided several more examples, here is a simple one: The equation of motion of an Abelian gauge field is

$$-D \bar{D}^2 D V = 0 \quad \text{or} \quad D_\alpha W_\alpha = 0$$

This implies that

$$J_{\alpha\beta} = (cW)^*_\alpha (W)_\beta$$

satisfies $\quad (\bar{D} J)_\beta = - c_{\gamma\alpha} \bar{D}_\gamma (cW)^*_\alpha (W)_\beta$

$$= - \bar{D}_\gamma W^*_\gamma (W)_\beta = 0$$

The Ferrara - Zumino equation is satisfied with the trivial RHS $S = 0$.

Actually, we can have $S = 0$ also in the case of the chiral superfield if the superpotential is

$$W(\Phi) = \lambda \Phi^3$$

This is the superpotential that gives a classically scale invariant theory, since λ is a dimensionless coupling constant. I would now like to show in some generality that

$$(\bar{D} J)_\beta = D_\beta S \quad \Rightarrow \text{ energy-momentum conservation}$$

$$S = 0 \quad \Rightarrow \text{ scale invariance.}$$

Introduce a component-field expansion of $J_{\alpha\beta}$ or of J_a defined by $J_{\alpha\beta} = \bar{\sigma}^a_{\alpha\beta} J_a$:

$$J_a = C_a(x) + \theta^T c \chi_a(x) + \theta^2 M_a(x) - \bar{\theta}^T c \chi^*_a(x) + \bar{\theta}^2 M^*_a(x)$$
$$+ \bar{\theta} \bar{\sigma}^b \theta \, v_{ab}(x) + \bar{\theta}^2 \theta^T c \rho_a(x) - \theta^2 \bar{\theta}^T c \rho^*_a(x)$$
$$+ \theta^2 \bar{\theta}^2 d_a(x)$$

Since we will differentiate with respect to $\bar{D} = -\frac{\partial}{\partial \bar{\theta}} + i \sigma \cdot \partial \theta$, it will be useful to rewrite this replacing x by

$$y^a = x^a + i \bar{\theta} \sigma^a \theta \qquad \text{s.t.} \qquad \bar{D} y^a = 0$$

$$J_a = C_a(y) + \theta^T c \chi_a(y) + \theta^2 M_a(y) - \bar\theta^T c \chi_a^* + \bar\theta^2 M_a^*$$
$$+ \bar\theta \bar\sigma^b \theta \,(v_{ab} - i\, \partial_b C_a)$$
$$+ \bar\theta^2 \theta^T c \,(\rho_a + \tfrac{i}{2} c (\bar\sigma^b)^T \partial_b \chi_a^*) - \theta^2 \bar\theta^T c\,(\rho_a^* + \tfrac{i}{2} c \bar\sigma^b \partial_b \chi_a)$$
$$+ \theta^2 \bar\theta^2 \left(d_a - \tfrac{i}{2} \partial_b v_{ab} - \tfrac{1}{4} \partial^2 C_a \right)$$

Then

$$-\bar D_\gamma c_{\gamma\alpha} J_{\alpha\beta} = -(c\bar\sigma^a)_{\gamma\beta} \left(-\frac{\partial}{\partial \bar\theta_\gamma}\right) J_a(y)$$

$$= (c\sigma^a)_{\beta\gamma} \Big\{ (c\chi_a^*)_\gamma + 2(c\bar\theta)_\gamma M_a^* - (\bar\sigma^b\theta)_\gamma (v_{ab} - i\,\partial_b C_a)$$
$$+ 2(c\bar\theta)_\gamma \,\theta^T c\,(\rho_a + \tfrac{i}{2} c(\bar\sigma^b)^T \partial_b \chi_a^*)$$
$$+ \theta^2 \left[c(\rho_a^* + \tfrac{i}{2} c \bar\sigma^b \partial_b \chi_a)\right]_\gamma$$
$$+ 2(c\bar\theta)_\gamma \,\theta^2 \left(d_a - \tfrac{i}{2}\partial_b v_{ab} - \tfrac{1}{4} \partial^2 C_a \right) \Big\}$$

This should be compared with $D_\beta S$, with

$$S = s(y) + \theta^T c \psi(y) + \theta^2 f(y)$$

Using $D_\beta y^a = -2i(\bar\theta\bar\sigma^a)_\beta$,

$$D_\beta S = (c\psi)_\beta + 2(c\theta)_\beta f - 2i(\bar\theta\bar\sigma^a)_\beta \partial_a s$$
$$- 2i(\bar\theta\bar\sigma^a\partial_a)_\beta \,\theta^T c \psi - 2i(\bar\theta\bar\sigma^a\partial_a)_\beta \,\theta^2 f$$

Now we can compare terms in $(\bar D J)_\beta = D_\beta S$.

$$\underline{1}: \quad -(\bar\sigma^a)^T \chi_a^* = c\,\psi \quad \rightarrow \quad \psi = c(\bar\sigma^a)^T \chi_a^*$$

$$\underline{\bar\theta}: \quad -2((\bar\sigma^a)^T\bar\theta)_\beta\, m_a^* = -2i\,((\bar\sigma^a)^T\bar\theta)_\beta\, \partial_a S$$

$$\rightarrow \qquad \partial_a S = -i\, m_a^*$$

$$\underline{\theta}: \quad -(c\,\sigma^a\bar\sigma^b\theta)_\beta\,(v_{ab} - i\,\partial_b c_a) = 2(c\theta)_\beta\, f$$

$$\rightarrow \qquad f = -\tfrac{1}{2}(v_{aa} - i\,\partial_a c_a)$$

We have now identified the components of S in terms of components of J. Next, $D_\beta S$ has no $\underline{\theta}^2$ term; this implies

$$\rho_a^* = -\tfrac{i}{2}\, c\,\bar\sigma^b\partial_b\chi_a$$

In the $\bar\theta\theta$ term, replace $\qquad \bar\theta_\alpha\theta_\beta = \tfrac{1}{2}(\sigma^c)^T_{\alpha\beta}\,\bar\theta\bar\sigma_c\theta$

$$2\, c\sigma^a c\, \tfrac{1}{2}(\sigma^c)^T c\, \left(\rho_a + \tfrac{i}{2}\, c(\bar\sigma^b)^T\partial_b\chi_a^*\right)\ (\bar\theta\bar\sigma_c\theta)$$

$$= -2i\,(\bar\sigma^a)^T(\tfrac{1}{2}\sigma^c)^T c\ \partial_a\psi\ (\bar\theta\bar\sigma_c\theta)$$

so

$$0 = \bar\theta\bar\sigma_c\theta\ \Big\{-(\bar\sigma^a)^T(\sigma^c)^T c\ \left(+\tfrac{i}{2}c(\bar\sigma^b)^T\partial_b\chi_a^* + \tfrac{i}{2}c(\bar\sigma^b)^T\partial_b\chi_a^*\right)$$
$$+\, i\,(\bar\sigma^a)^T(\sigma^c)^T c\, c\ \bar\sigma^b\,\partial_a\chi_b^*\Big\}$$

$$0 = \bar\theta\bar\sigma_c\theta\ \{+i\}\,(\partial_b\chi_a^\dagger - \partial_a\chi_b^\dagger)(\bar\sigma^b\sigma^c\bar\sigma^a)$$

$$\rightarrow \qquad \bar\sigma^a\sigma^c\bar\sigma^b(\partial_b\chi_a - \partial_a\chi_b) = 0$$

multiply by (σ_c) :

$$\rightarrow (-2\sigma^a\bar{\sigma}^b)(\partial_b \chi_a - \partial_a \chi_b) = 0$$

$$(-2)\{(2\eta^{ab} - \sigma^b\bar{\sigma}^a)\partial_b \chi_a - \sigma^a\bar{\sigma}^b \partial_a \chi_b\} = 0$$

or $\partial_a \mathcal{J}^a = 0$ where $\mathcal{J}^a = (\chi^a - \sigma^a\bar{\sigma}^b\chi_b)$

Finally, the $\theta^2\bar{\theta}$ term gives:

$$\theta^2 [-2(\bar{\sigma}^a)^T\bar{\theta}]_\beta (d_a - \tfrac{i}{2}\partial_b v_{ab} - \tfrac{1}{4}\partial^2 c_a)$$

$$= -2i\theta^2 [(\bar{\sigma}^a)^T\bar{\theta}]_\beta \, \partial_a f$$

so

$$d_a - \tfrac{i}{2}\partial_b v_{ab} - \tfrac{1}{4}\partial^2 c_a + i\tfrac{1}{2}\partial_a(v_{bb} - i\partial_b c_b) = 0$$

The imaginary part is $\partial_b(v_{ab} - \eta_{ab} v_{cc}) = 0$

So if

$$t_{ab} = (v_{ab} - \eta_{ab} v_{cc})$$

$$\mathcal{J}_b = (\chi_b - \sigma_b \bar{\sigma}\cdot\chi)$$

the relation $(\bar{D}J)_\beta = D_\beta S$ implies $\partial_b \mathcal{J}^b = \partial_b t^{ab} = 0$

t_{ab} should be the energy-momentum tensor. The spinor conserved current should be the supersymmetry current!

Now, what if $S = 0$? In that case, $\psi = 0$ and $f = 0$. These relations imply

$$\bar{\sigma}^a \mathcal{J}_a = 0 \qquad t_a^{\;a} = 0 \qquad \partial_a C^a = 0$$

The second of these relation is the equation for scale invariance. If D^a is the dilatation current, $D^a = (t^{ab} x_b)$

$$\partial_a (D^a) = \partial_a (t^{ab} x_b) = t^a_{\;a} = 0$$

Scale invariance plus SUSY implies an additional fermionic symmetry called super<u>con</u>formal in<u>variance</u>, and this leads to $\bar{\sigma}^a \mathcal{J}_a = 0$. Finally, there is a new <u>current</u> that is conserved. This is the current of "R-symmetry", the $U(1)$ symmetry on superfields:

$$\Phi(x, \theta) \to e^{i \frac{2}{3}\alpha} \Phi(x, e^{-i\alpha}\theta)$$

Notice that this is a symmetry of scale invariant superpotentials satisfying

$$\left(\frac{\partial W}{\partial \Phi_k}\right) \Phi_k = 3W$$

i.e. superpotentials <u>cubic</u> in superfields. Under R-symmetry

$$\phi_k \to e^{i\frac{2}{3}\alpha} \phi_k \qquad \psi_k \to e^{-i\frac{1}{3}\alpha} \psi_k$$

The vector superfield transforms as $V(x,\theta,\bar{\theta}) \to V(x, e^{-i\alpha}\theta, e^{+i\alpha}\bar{\theta})$

which implies

$$\lambda^a \to e^{+i\alpha} \lambda^a$$

Then, R symmetry is also a symmetry of the supersymmetric gauge interactions eg.

$$\sqrt{2} g \, \phi^* \lambda^{aT} c \, \psi$$

Although Yang-Mills theory is classically scale invariant, this invariance is violated quantum-mechanically by coupling constant renormalization. This effect is described by the equation

$$t_a{}^a = \frac{1}{2g} \beta(g) (F_{mn}^a)^2$$

In SUSY, we recognize this equation as ~~being~~ part of a superfield relation that also implies nonconservation of the superconformal current and the R-current. I would like to claim that these results are summarized in the equation

$$S = \frac{\beta(g)}{g} \cdot \left(\frac{1}{2} W^{Ta} c W^a \right)$$

In particular, the Θ^2 component of this equation is

$$\{ v_{aa} - i \partial_a C^a) = \left(\frac{2\beta(g)}{3g}\right)\left[\frac{1}{4} (F_{mn}^a)^2 - \frac{i}{8} \epsilon^{mnpq} F_{mn}^a F_{pq}^a + \cdots \right]$$

Now, if $J_{\alpha\beta} = (cW^*)_\alpha (Wc)_\beta$, we can show that

$$C^a = \frac{1}{2} \lambda^\dagger \bar{\sigma}^a \lambda = \frac{1}{2} J_R^a \quad \text{w. the normalization of}$$

R-charges above, and $\quad v_{ab} \sim F^{ac} F^b{}_c + \cdots$

which is the usual normalization of the energy-momentum tensor.

Since

$$t_{ab} = (v_{ab} - \eta_{ab} v_c{}^c)\,, \qquad -3 v_{aa} = t_{aa}$$

Thus, the above equation does include (up to signs)

$$t_{aa} = \frac{\beta(g)}{2g}\left(F_{mn}^a\right)^2$$

as required.

In a non-SUSY gauge theory

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\Big[\frac{11}{3}C_2(G) - \sum_{\substack{\text{chiral}\\ \text{matter}\\ \text{fermions}}}\frac{2}{3}C(r_k) - \sum_{\substack{\text{complex}\\ \text{matter}\\ \text{scalars}}}\frac{1}{3}C(r_k)\Big]$$

where

$$C(r) = \operatorname{tr}[t^a t^b]_r \qquad C_2(G) = C(\text{adjoint}) = N \text{ for } SU(N)$$

With SUSY, we should include the gauginos with the gauge fields:

$$\frac{11}{3}C_2(G) - \frac{2}{3}C_2(G) = 3\,C_2(G)$$

and add the contributions of chiral fermions and scalars in a chiral supermultiplet. Then

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\Big[3C_2(G) - \sum_{\substack{\text{chiral}\\ \text{supermult.}}} C(r_k)\Big]$$

We now predict

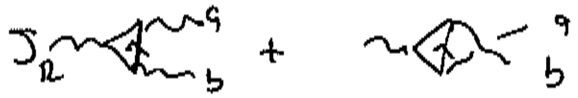
$$\partial_a J_R^a = 2\,\partial_a C^a = 2\cdot\frac{\beta(g)}{12g}\,\epsilon^{mnpq}F_{mn}^a F_{pq}^a$$

or

$$\partial_a J_R^a = \frac{g^2}{32\pi^2}\left[C_2(G) - \sum_{\substack{\text{chiral}\\ \text{s.f.'s.}}} \tfrac{1}{3}\, C(r_k)\right] \epsilon^{mnpq} F^a_{mn} F^a_{pq}$$

Now J_R is a chiral current, and so it should have a gauge anomaly!

$$\partial_a J_R^a = \frac{g^2}{32\pi^2}\, \mathrm{tr}\,[R\, t^a t^b]\, \epsilon^{mnpq} F^a_{mn} F^b_{pq}$$

from 

The trace is over chiral fermion representations, with their R charges.

$$\mathrm{tr}\, R t^a t^b = (+1)\, C_2(G)\, \delta^{ab} + \sum_k (-\tfrac{1}{3})\, C(r_k)$$

So the equation above is exactly right!

You might recall ~~that~~, with a proper choice of regulator, the chiral anomaly is present in 1-loop diagrams only. The analogous regulator for a SUSY theory gives only a 1-loop contribution to the β function. However, the β-function is conventionally defined with a different scheme, and so the conventional β-function receives contributions of all orders. Relating the two schemes, Shifman, Vainshtein, and Zakharov found the exact β-function of N=1 SUSY gauge theories. This result is nicely explained in Arkani-Hamed & Murayama, hep-th/9707133.