6. Superspace

Our discussion of the physics of the vector supermultiplet makes it clear that we need a more manifestly supersymmetric description of N=1 SUSY. Salam and Strathdee found a very nice way to represent N=1 SUSY transformations as translations in a "superspace" with fermionic coordinates.

So consider a space with coordinates $(x^m, \theta_\alpha, \theta^\dagger_\alpha)$ where θ_α is a Weyl fermion. I would like to show that functions on this space $\Phi(x,\theta,\theta^\dagger)$ naturally give representations of the SUSY algebra.

A first idea is that supersymmetry should be a translation in a fermionic coordinate

$$\theta \to \theta + \xi$$

But this is not sophisticated enough, because it leads to $\{Q, Q^\dagger\} = 0$, which as we saw gives a trivial quantum theory. We need to mix θ and x translations in order to form representations of the SUSY CR's.

To begin, let me say a little about the formalism of derivatives on superspace. Note that, if ξ is a constant spinor:

$$\left(-\frac{\partial}{\partial\theta}\xi\right)\theta = \xi \qquad \left(\xi^* \frac{\partial}{\partial\theta^*}\right)\theta^* = \xi^*$$

these relations should be related by complex conjugation, that requires

$$\left(-\frac{\partial}{\partial\theta}\right)^* = \frac{\partial}{\partial\theta^*} \quad \text{or} \quad \left(\frac{\partial}{\partial\theta}\right)^* = -\frac{\partial}{\partial\theta^*}$$

Now I would like to represent

$$\delta_\xi \Phi = \mathcal{Q}_\xi \Phi$$

where $\mathcal{Q}_\xi$ is a differential operator on superspace:

$$\mathcal{Q}_\xi = \left(-\frac{\partial}{\partial\theta} - i\theta^*\bar\sigma^m\partial_m\right)\xi + \xi^\dagger\left(\frac{\partial}{\partial\theta^*} + i\bar\sigma^m\theta\,\partial_m\right)$$

This equation represents the SUSY CR's by virtue of the fact that $\mathcal{Q}_\xi$ obeys the CR: (proof below)

$$[\mathcal{Q}_\xi, \mathcal{Q}_\eta] = -2i\left(\xi^\dagger\bar\sigma^m\eta - \eta^\dagger\bar\sigma^m\xi\right)\partial_m$$

The difference of (-1) on the RHS from the CR of $[\delta_\xi, \delta_\eta]$ is slightly delicate and will be discussed later. $\mathcal{Q}_\xi$ is also constructed to obey

$$(\mathcal{Q}_\xi\Phi)^* = \mathcal{Q}_\xi\Phi^*$$

in accord with the property of $\delta_\xi\Phi$. This relation uses $\left(-\frac{\partial}{\partial\theta}\right)^* = \frac{\partial}{\partial\theta^*}$.

Let me now prove the CR of $\mathcal{Q}_\xi$:

$$[\mathcal{Q}_\xi, \mathcal{Q}_\eta] = (\text{terms w. } \tfrac{\partial}{\partial\theta}, \tfrac{\partial}{\partial\theta^*} \text{ in } \mathcal{Q}_\xi) \text{ acting on } (\text{terms w } \theta, \theta^* \text{ in } \mathcal{Q}_\eta)$$

$$- \quad (\xi \leftrightarrow \eta)$$

$$= \left(-\tfrac{\partial}{\partial\theta}\xi\right)\left(\eta^\dagger i\bar\sigma^m \theta \partial_m\right) + \left(\xi^* \tfrac{\partial}{\partial\theta^*}\right)\left(-i\,\theta^* \bar\sigma^m \partial_m \eta\right)$$

$$- (\xi \leftrightarrow \eta)$$

$$= \left(i\,\eta^\dagger \bar\sigma^m \xi\, \partial_m - i\,\xi^\dagger \bar\sigma^m \eta\, \partial_m\right) - (\xi \leftrightarrow \eta)$$

which is what I claimed. Note that this CR closes completely generally, and off-shell. So this method provides an algorithm for constructing representations of SUSY.

If $\Phi(x,\theta,\theta^*)$ is a representation of SUSY, how is this connected to our fermion representations? To study this, note first that there are limits to how much Φ can vary with θ and θ^*. For example, θ_α has only two components, and $(\theta_1)^2 = (\theta_2)^2 = 0$. So the most general function of x, θ has the form:

$$f(x,\theta) = \phi(x) + \theta_1 \hat\psi_1(x) + \theta_2 \hat\psi_2(x) + \theta_1\theta_2 \mathcal{F}(x)$$

where $\phi, \mathcal{F}$ are commuting functions of x^m and $\hat\psi_1\, \hat\psi_2$ are anticommuting functions of x. It is most convenient to organize the expansion slightly differently as:

$$f(x,\theta) = \phi(x) + \sqrt{2}\,\theta^T c\,\psi(x) + \theta^T c\,\theta\,F(x)$$

where $\psi_\alpha(x)$ now transforms as a Weyl spinor. For a general function $\Phi(x,\theta,\theta^+)$, we would have 16 coefficient functions, 8 commuting and 8 anticommuting.

Unfortunately, we do not know any representation of $N=1$ with 8 Boson and 8 Fermion fields. Perhaps we missed something, but the explanation is actually different — $\Phi(x,\theta,\theta^+)$ is a <u>reducible</u> representation of $N=1$ SUSY. When we have a reducible representation of a group, Schur's Lemma states that there is an operator that commutes with the group actions that we can use to form projections onto the irreducible representations. In this case, it is not difficult to find such operators. Let

$$D_\alpha = \frac{\partial}{\partial\theta_\alpha} - i(\theta^+\bar\sigma^m)_\alpha\,\partial_m \qquad D^+_\alpha = -\frac{\partial}{\partial\theta^+_\alpha} + i(\bar\sigma^m\theta)_\alpha\,\partial_m$$

Then

$$[D_\alpha, Q_\xi] = -(+i\xi^+\bar\sigma^m)_\alpha\,\partial_m - (-i\xi^+\bar\sigma^m_\alpha)\,\partial_m = 0$$

similarly $[D^+_\alpha, Q_\xi] = 0$ and

$$\{D_\alpha, D^+_\beta\} = +2i\,\sigma^m_{\beta\alpha}\,\partial_m$$

From here on, so that the notation will not stray too far from ~~that~~ of Wess + Bagger, I will write $\theta^\dagger_\alpha \to \bar\theta_\alpha$ $D^\dagger_\alpha \to \bar D_\alpha$ so that

$$D_\alpha = \frac{\partial}{\partial\theta_\alpha} - i(\bar\theta\sigma^m)_\alpha\,\partial_m \qquad \bar D_\alpha = -\frac{\partial}{\partial\bar\theta_\alpha} + i(\bar\sigma^m\theta)_\alpha\,\partial_m$$

$$\{\bar D_\beta, D_\alpha\} = +2i\,\bar\sigma^m_{\beta\alpha}\,\partial_m$$

Now define a chiral superfield as a scalar field on superspace satisfying

$$\bar D_\alpha \Phi = 0$$

(An anti-chiral superfield satisfies $D_\alpha\bar\Phi = 0$. This is the complex-conjugate condition.) Since $\bar D_\alpha$ commutes with Q_ξ, the SUSY transform of Φ is also chiral, so that chiral superfield is itself a representation of SUSY.

Let's make this more explicit. First of all, let's work out the structure of Φ more clearly. The chiral condition is

$$0 = \bar D_\alpha\Phi = \left(-\frac{\partial}{\partial\bar\theta_\alpha} + i(\bar\sigma^m\theta)_\alpha\partial_m\right)\Phi$$

or

$$\frac{\partial}{\partial\bar\theta_\alpha}\Phi = +i(\bar\sigma^m\theta)_\alpha\,\partial_m\Phi$$

The most general solution is: $\Phi(x,\theta,\bar\theta) = \Phi(x+i\bar\theta\bar\sigma^m\theta, \theta)$ with general dependence on θ but only a particular dependence on $\bar\theta$.

Now Taylor expand in θ and $\bar\theta$. If we first ignore $(\bar\theta\bar\sigma^m\theta)$, we get the expansion at the top of p.4. Now expand out in powers of $\bar\theta\bar\sigma^m\theta$; we need to keep only terms with ~~no more that~~ 2 powers of θ and 2 powers of $\bar\theta$:

$$\Phi = \phi(x) + \sqrt{2}\,\theta^T c\,\psi(x) + \theta^T c\,\theta\, F'(x)$$
$$+ i\,\bar\theta\bar\sigma^m\theta\,\partial_m\phi(x) + i\,\bar\theta\bar\sigma^m\theta\,\sqrt{2}\,\theta^T c\,\partial_m\psi$$
$$+\frac{1}{2}(i\,\bar\theta\bar\sigma^m\theta)(i\,\bar\theta\bar\sigma^n\theta)\,\partial_m\partial_n\phi$$

Let $\theta^T c\,\theta = \theta^2$ $\qquad -\bar\theta^T c\,\bar\theta = \bar\theta^2$ $\qquad$ so $(\theta^2)^* = \bar\theta^2$

Notice that the θ_α obey a very simple calculus:

$$\theta_\alpha\theta_\beta = (\text{antisym in } (\alpha\leftrightarrow\beta)) = \frac{1}{2}c_{\alpha\beta}\,\theta^T c\,\theta$$

(proof: contract both sides w. $c_{\alpha\beta}$)

so $\qquad \theta_\alpha\theta_\beta = \frac{1}{2}c_{\alpha\beta}\,\theta^2$ $\qquad$ and similarly

$$\bar\theta_\alpha\bar\theta_\beta = -\frac{1}{2}c_{\alpha\beta}\,\bar\theta^2$$

then $\quad i\,\bar\theta\bar\sigma^m\theta\;\theta^T c\,\partial_m\psi = \frac{i}{2}\,\bar\theta\bar\sigma^m c\cdot c\,\partial_m\psi\;\theta^2 = -\frac{i}{2}\theta^2\,\bar\theta\bar\sigma^m\partial_m\psi$

$$\bar\theta\bar\sigma^m\theta\;\bar\theta\bar\sigma^n\theta = \underset{\text{interchange}}{(-1)}\left(-\tfrac{1}{2}c_{\alpha\beta}\right)\bar\theta^2\left(\tfrac{1}{2}c_{\gamma\delta}\theta^2\right)\bar\sigma^m_{\alpha\gamma}\bar\sigma^n_{\beta\delta}$$
$$= \frac{1}{4}\theta^2\bar\theta^2\,\mathrm{tr}\,c^T\bar\sigma^m c(\bar\sigma^n)^T = \frac{1}{2}\theta^2\bar\theta^2\,\eta^{mn}$$

with these simplifications:

$$\Phi = \phi(x) + \sqrt{2}\,\theta^T c \psi + \theta^2 F$$
$$+ i\,\bar\theta\bar\sigma^m\theta\,\partial_m\phi - \frac{i}{\sqrt{2}}\theta^2\,\bar\theta\bar\sigma^m\partial_m\psi$$
$$- \frac{1}{4}\,\theta^2\bar\theta^2\,\partial^2\phi$$

Now let's compute $\delta_\xi\Phi = \mathcal{Q}_\xi\Phi$. Since $\bar D_\alpha \mathcal{Q}_\xi\Phi = 0$, the result will be a chiral superfield and therefore completely determined by the terms proportional to $1, \theta, \theta^2$. So we need only compute these:

$$\mathcal{Q}_\xi\Phi = (1 \text{ term:}) \quad \left(-\frac{\partial}{\partial\theta}\xi\right)(\sqrt{2}\,\theta^T c\psi) = \sqrt{2}\,\xi^T c\psi$$

$$+ (\theta \text{ terms:}) \quad \left(-\frac{\partial}{\partial\theta}\xi\right)(\theta^2 F) + \left(\xi^+\frac{\partial}{\partial\bar\theta}\right)(i\,\bar\theta\bar\sigma^m\theta\,\partial_m\phi)$$
$$+ (i\,\xi^+\bar\sigma^m\theta\partial_m)\,\phi$$

$$= 2\,\xi^T c\,\theta F + 2i\,\xi^+\bar\sigma^m\theta\,\partial_m\phi$$
$$= 2\,\theta^T c\,\xi F - 2i\,\theta^T \underbrace{(-c^2)}_{=1}(\bar\sigma^m)^T\xi^*\,\partial_m\phi$$
$$= \sqrt{2}\,\theta^T c\left\{\sqrt{2}\,i\,\sigma^m c\,\xi^*\,\partial_m\phi + \sqrt{2}\,\xi F\right\}$$

$$+ (\theta^2 \text{ term:}) \quad \left(\xi^+\frac{\partial}{\partial\bar\theta}\right)\left(-\frac{i}{\sqrt{2}}\theta^2\,\bar\theta\bar\sigma^m\partial_m\psi\right) + i\,\xi^+\bar\sigma^m\theta\,\partial_m(\sqrt{2}\,\theta^T c\psi)$$

$$= -i\sqrt{2}\;\theta^2\,\xi^+\bar\sigma^m\partial_m\psi$$

so indeed

$$\mathcal{Q}_\xi \Phi = (\delta_\xi \phi) + \sqrt{2}\,\theta^T c\,(\delta_\xi \psi) + \theta^2 (\delta_\xi F) + \cdots$$

according to the transformation laws of the chiral supermultiplet.

Now it is time to explain the odd extra (-1) on p. 2.
δ_ξ is an operator that replaces some component fields by others.
$\mathcal{Q}_\xi$ is a differential operator on $(x, \theta, \bar\theta)$. So $\delta_\xi \mathcal{Q}_\eta \Phi$ is evaluated by passing δ_ξ through $\mathcal{Q}_\eta$ and modify the component fields.

Thus

$$\begin{aligned}
[\delta_\xi, \delta_\eta]\,\Phi &= (\delta_\xi \delta_\eta - \delta_\eta \delta_\xi)\,\Phi \\
&= (\delta_\xi \mathcal{Q}_\eta - \delta_\eta \mathcal{Q}_\xi)\,\Phi \\
&= (\mathcal{Q}_\eta \mathcal{Q}_\xi - \mathcal{Q}_\xi \mathcal{Q}_\eta)\,\Phi \\
&= -[\mathcal{Q}_\xi, \mathcal{Q}_\eta]\,\Phi
\end{aligned}$$

so if $\mathcal{Q}_\xi$ satisfies the CR on p. 2, δ_ξ does indeed satisfy the CR of N=1 SUSY.

Now that we have seen how to use superspace to automatically form representations of N=1 SUSY, let's try to form invariant actions.
A natural form of an action is

$$S = \int d^4x\, \mathcal{L} = \int d^4x\, d^2\theta\, d^2\bar\theta\, \mathbb{L}$$

We should recall the definition of the integral over an anticommuting variable. Consider a function of one θ: $g(\theta) = a + b\theta$

The integral over θ must be a linear function of $g(\theta)$, that is, a linear combination of a and b. It should also satisfy one of the following:

$$\int d\theta \; g(\theta) = \int d\theta \; g(\theta + \xi) \qquad \text{translation invariance}$$

$$\int d\theta \; \frac{d}{d\theta} g(\theta) = 0 \qquad \text{integration by parts.}$$

the definition $\qquad \int d\theta \; g(\theta) = (\text{const}) \cdot b$

satisfies all requirements. For superspace coordinates, I will adopt the normalization:

$$\int d^2\theta \;\; \theta^2 = 1 \qquad\qquad \int d^2\bar\theta \;\; \bar\theta^2 = 1$$

Now notice ~~that~~, for any function on superspace (chiral or not)

$$\delta_\xi \int d^4x \, d^2\theta \, d^2\bar\theta \quad A(x, \theta, \bar\theta)$$

$$= \int d^4x \; d^2\theta \; d^2\bar\theta \quad \partial_\xi A \quad = \quad 0$$

since every term has either $\frac{\partial}{\partial \theta_\alpha}$, $\frac{\partial}{\partial \bar\theta_\alpha}$ or $\frac{\partial}{\partial x^m}$.

As an example, consider

$$\int d^4x \; d^4\theta \quad \Phi^* \Phi \quad = \quad (\text{term } \Phi^* \Phi \text{ proportional to } \theta^2 \bar\theta^2)$$

From p. 7, the term in $\Phi^*\Phi$ with $\theta^2\bar\theta^2$ is:

$$\left[\Phi^*\Phi\right]_{\theta^2\bar\theta^2} = \phi^*\left(-\tfrac{1}{4}\theta^2\bar\theta^2\partial^2\phi\right) + \left(-i\bar\theta\bar\sigma^n\theta\partial_n\phi^*\right)\left(i\bar\theta\bar\sigma^m\theta\,\partial_m\phi\right)$$

$$\left(-\tfrac{1}{4}\theta^2\bar\theta^2\partial^2\phi^*\right)\phi$$

$$+\left(-\sqrt{2}\bar\theta^T c\,\psi^*\right)\left(-\tfrac{i}{\sqrt{2}}\theta^2\;\bar\theta\bar\sigma^m\partial_m\psi\right)$$

$$+\left(+\tfrac{i}{\sqrt{2}}\bar\theta^2\partial_m\psi^\dagger\bar\sigma^m\theta\right)\left(\sqrt{2}\,\theta^T c\,\psi\right)$$

$$+\quad\bar\theta^2F^*\;\theta^2F$$

using

$$\bar\theta\bar\sigma^m\theta\;\bar\theta\bar\sigma^n\theta = \tfrac{1}{2}\theta^2\bar\theta^2\eta^{mn}$$

$$\bar\theta^T c\,\psi^*\;\bar\theta\bar\sigma^m\partial_m\psi = -\tfrac{1}{2}\bar\theta^2\,\psi^\dagger c\,c\,\bar\sigma^m\partial_m\psi = +\tfrac{\bar\theta^2}{2}\psi^\dagger\bar\sigma^m\partial_m\psi$$

$$\partial_m\psi^\dagger\bar\sigma^m\theta\;\theta^T c\,\psi = \tfrac{1}{2}\theta^2\;\partial_m\psi^\dagger\bar\sigma^m c\,c\,\psi = -\tfrac{\theta^2}{2}\,\partial_m\psi^\dagger\bar\sigma^m\partial_m\psi$$

$$= \quad \phi^*(-\partial^2)\phi + \psi^\dagger i\bar\sigma^m\partial_m\psi + F^*F$$

after int. by parts

which is ~~just~~ the kinetic term of the chiral supermultiplet. To find the superpotential term, we need one more observation. Specifically if $B(x,\theta,\bar\theta)$ is a chiral superfield

$$\mathcal{Q}_\xi B = \left[\left(-\frac{\partial}{\partial\theta} - i\bar\theta\bar\sigma^m\partial_m\right)\xi + \xi^\dagger\left(\frac{\partial}{\partial\bar\theta} + i\bar\sigma^m\theta\,\partial_m\right)\right]B$$

$$= \left[\left(-\frac{\partial}{\partial\theta} - i\bar\theta\bar\sigma^m\partial_m\right)\xi + 2\xi^\dagger\left(i\bar\sigma^m\theta\,\partial_m\right)\right]B$$

so all terms contain either $\frac{\partial}{\partial\theta}$ or ∂_m and so

$$\delta_\xi \int d^4x\, d^2\theta\, B = \int d^4x\, d^2\theta\, \mathcal{Q}_\xi B = 0$$

so the θ^2 term in B is a supersymmetric Lagrangian. Note that there is no ambiguity about whether we should keep terms w. $\bar\theta$, since there are (see p. 7)

$$\theta^2 \left[-\tfrac{i}{\sqrt{2}} \bar\theta \bar\sigma^m \partial_m \psi - \tfrac{1}{4} \bar\theta^2 \partial^2 \phi \right]$$

and disappear when we integrate $\int d^4x$.

The most general Lagrangian term of this type is

$$\int d^2\theta\, W(\Phi)$$

where W is a general function of the chiral superfield Φ (but not of Φ^*). Expanding to order θ^2:

$$W(\Phi) = \cdots + \theta^2 F \frac{\partial W}{\partial \Phi} + \tfrac{1}{2}(\sqrt{2}\,\theta^T c\,\psi)(\sqrt{2}\,\theta^T c\,\psi) \frac{\partial^2 W}{\partial \Phi^2}$$

$$= \theta^2 \left\{ F \frac{\partial W}{\partial \Phi} \right\} + \tfrac{1}{2} \theta^2\, \psi^T c\, c\, \psi \frac{\partial^2 W}{\partial \Phi^2}$$

so

$$\int d^2\theta\, W(\Phi) = F \frac{\partial W}{\partial \Phi} - \tfrac{1}{2} \psi^T c\, \psi \frac{\partial^2 W}{\partial \Phi^2}$$

Thus we recover our previous expression for the action of a chiral superfield from the superspace Lagrangian

$$\mathcal{L} = \int d^4\theta\, \Phi^* \Phi + \int d^2\theta\, W(\Phi) + h.c.$$

Can we describe the vector multiplet in a superspace formalism. Let's now write the full content of a real scalar field in superspace. (I will henceforth call this object a "vector superfield")

$$
\begin{aligned}
V(x,\theta,\bar{\theta}) = \ & \phi(x) + \theta^T c\, \psi(x) + \theta^2 F(x) \\
& + \phi^*(x) - \bar{\theta}^T c\, \psi^*(x) + \bar{\theta}^2 F^*(x) \\
& + 2\bar{\theta}\bar{\sigma}^m\theta\, A_m(x) + 2\bar{\theta}^2\theta^T c\, \lambda(x) \\
& - 2\theta^2\bar{\theta}^T c\, \lambda^*(x) + \theta^2\bar{\theta}^2 D(x)
\end{aligned}
$$

The first two lines are the content of a chiral and an antichiral multiplet. What is left over is exactly the content of a vector multiplet. This suggests the following construction: Let $V(x,\theta,\bar{\theta})$ transform under local gauge symmetries in superspace as follows (Abelian case only)

$$\delta_{gauge} V = -\frac{i}{g}(\Lambda - \Lambda^+)$$

where Λ is a chiral superfield. Then we can use Λ to gauge away the first two lines of the expression above. Note that if $\Lambda = \left(\frac{\alpha + i\beta}{2}\right)(x) + \cdots$ α, β real scalar fields

$$-\frac{i}{g}(\Lambda-\Lambda^*) = \frac{-i}{g}\{(\alpha+i\beta) + i\,\bar\theta\bar\sigma^m\theta\,\partial_m(\alpha+i\beta)+\cdots$$
$$-(\alpha-i\beta) + i\,\theta\bar\sigma^m\theta\,\partial_m(\alpha-i\beta)+\cdots\}$$

$$= 2\beta/g + 2\bar\theta\bar\sigma^m\theta\,\frac{1}{g}\partial_m\alpha + \cdots$$

so β/g can be used to gauge away the real field $(\phi+\phi^*)$, but α is left over to transform A_m according to

$$\delta_{gauge} A_m = \frac{1}{g}\partial_m\alpha \qquad !$$

Let's now adjust ~~this~~ formalism so that it can generalize to the non-Abelian case and can act on matter fields. If Φ is a chiral field transforming under G in rep. r and t^a are the generators of G in this representation, let

$$\delta_\Lambda \Phi = e^{i\Lambda^a t^a}\Phi$$
$$\delta_\Lambda \Phi^* = \Phi^* e^{-i\Lambda^{*a}t^a}$$

$\delta_\Lambda(V^a t^a)$ be given by:

$$\delta_\Lambda e^{gV^a t^a} = e^{i\Lambda^{*a}t^a} e^{gV^a t^a} e^{-i\Lambda^a t^a}$$

Then

$$\int d^4x\ \Phi^* e^{gV^a t^a}\Phi$$

is a gauge-invariant action that couples Φ to a vector supermultiplet.

To work out the explicit form of this Lagrangian, it is easiest to gauge away the components $\phi\ \psi\ F\ \ \phi^*\ \psi^\dagger\ F^*$ of V. This gauge choice is called "Wess - Zumino gauge". The remaining terms in V have at least $\bar\theta\theta$ and so do not expand very far. The new terms beyond what appears on p. 10 are:

$$[\Phi^* e^{gV\cdot t}\Phi]_{\theta^2\bar\theta^2}$$

$$= \phi^*(2g\,\bar\theta\bar\sigma^m\theta\, A_m\cdot t)(i\,\bar\theta\bar\sigma^n\theta\,\partial_n\phi)$$

$$+ (-i\,\bar\theta\bar\sigma^n\theta\,\partial_n\phi^*)(2g\,\bar\theta\bar\sigma^m\theta\, A_m\cdot t)\,\phi$$

$$+ \tfrac{1}{2}\phi^*(2g\,\bar\theta\bar\sigma^m\theta\,A_m\cdot t)^2\phi + (-\sqrt{2}\psi^\dagger c\bar\theta)(2g\bar\theta\bar\sigma^m\theta\,A_m\cdot t)(\sqrt{2}\,\bar\theta c\psi)$$

$$+ \phi^*(2g\bar\theta^2\theta^T c\lambda\cdot t)(\sqrt{2}\,\theta^T c\psi)$$

$$+ (-\sqrt{2}\psi^\dagger c\bar\theta)(-2g\theta^2\,\bar\theta^T c\lambda^*\cdot t)\,\phi$$

$$+ \phi^*\cdot D\cdot t\,\phi\quad\theta^2\bar\theta^2$$

$$= \theta^2\bar\theta^2\Big\{\phi^* ig A_m\cdot t\,\partial^m\phi - i\partial_m\phi^*\,gA_m\cdot t\,\phi + g^2\phi^* A_m\cdot t A^m\cdot t\,\phi$$

$$+ \psi^\dagger g A_m\cdot t\,\psi - \sqrt{2}g(\phi^*\lambda^T t c\psi - \psi^\dagger c\lambda^*\cdot t\,\phi)$$

$$+ \phi^* D\cdot t\,\phi\Big\}$$

This is exactly what we need to build the matter coupling of the vector multiplet. Note that the first line combines with $\phi^*(-\partial^2)\phi$ to build

$$\phi^* -(\partial_m - igA_m t)(\partial^m - igA_m t)\,\phi \;=\; \phi^*(-\mathcal{D}_m \mathcal{D}^m)\phi$$

At this point, it has become clear why the treatment of gauge invariance was so awkward in our component field discussion of the vector multiplet. The superspace transformation

$$\delta_\xi V = \mathcal{Q}_\xi V$$

mixes up all of the component fields of V. If we want V to have a natural transformation law that closes off-shell, we need to keep the components ϕ, ψ, F etc. nonzero. We can define an alternative SUSY transformation of V that acts only on $(A_m, \lambda, \lambda^*, D)$ by

$$\delta_\xi V = \mathcal{Q}_\xi V + \delta_{\Lambda(\xi)} V$$

where $\Lambda(\xi)$ is a gauge transformation that restores $\mathcal{Q}_\xi V$ to Wess-Zumino gauge. This is not a symmetry unless we also write

$$\delta_\xi \Phi = \mathcal{Q}_\xi \Phi + \delta_{\Lambda(\xi)} \Phi$$

for all matter fields. The term $\delta_{\Lambda(\xi)}$ generates the new terms in the transformation laws and the SUSY CR.

Our final piece of business is to find the kinetic Lagrangian for the vector multiplet. To work toward this, I would like to write some identities for D_α and $\bar{D}_\alpha$

First

$$D^2 = D^T c D = \left[\left(\frac{\partial}{\partial\theta}\right)^T - i\bar{\theta}\bar{\sigma}\cdot\partial\right] c \left[\frac{\partial}{\partial\theta} - i(\bar{\sigma})^T\partial\bar{\theta}^*\right]$$

$$= \frac{\partial}{\partial\theta}c\frac{\partial}{\partial\theta} - 2i\,\bar{\theta}\bar{\sigma}\cdot\partial\, c\,\frac{\partial}{\partial\theta} - \left(-\tfrac{1}{2}\bar{\theta}^2\right)\mathrm{tr}\left[c^T\bar{\sigma}\cdot\partial\, c(\bar{\sigma}\cdot\partial)^T\right]$$

$$= \frac{\partial}{\partial\theta}c\frac{\partial}{\partial\theta} - 2i\,\bar{\theta}\bar{\sigma}c\frac{\partial}{\partial\theta}\partial + \bar{\theta}^2\partial^2$$

note that

$$\frac{\partial}{\partial\theta}c\frac{\partial}{\partial\theta}\,\theta^2 = \frac{\partial}{\partial\theta}c\;2c\theta = \mathrm{tr}\,2c^2 = -4$$

so $\left(-\frac{1}{4}\frac{\partial}{\partial\theta}c\frac{\partial}{\partial\theta}\right)$ or $-\frac{1}{4}D^2$ sends $\theta^2 \to 1$

The remaining terms in D^2 are total x derivatives, so

$$\int d^4x\, d^2\theta\; A(x,\theta,\bar{\theta}) = \int d^4x\left(-\tfrac{1}{4}D^2\right)A(x,\theta,\bar{\theta})$$

similarly let

$$\bar{D}^2 = -\bar{D}^T c\bar{D} = -\frac{\partial}{\partial\bar{\theta}}c\frac{\partial}{\partial\bar{\theta}} + 2i\frac{\partial}{\partial\bar{\theta}}c\,\bar{\sigma}^m\theta\,\partial_m$$
$$+\;\theta^T(\bar{\sigma}^m)^T c\,\bar{\sigma}^n\theta\,\partial_m\partial_n$$

or

$$\bar{D}^2 = -\frac{\partial}{\partial\bar{\theta}}c\frac{\partial}{\partial\bar{\theta}} + 2i\frac{\partial}{\partial\bar{\theta}}c\,\bar{\sigma}^m\theta\,\partial_m + \theta^2\partial^2$$

then

$$\int d^4x\, d^2\bar\theta\, A(x,\theta,\bar\theta) = \int d^4x \left(-\tfrac{1}{4}\bar D^2\right) A(x,\theta,\bar\theta)$$

Next consider

$$D\bar D^2 D = -C_{\alpha\beta}C_{\gamma\delta}\, D_\alpha \bar D_\gamma \bar D_\delta D_\beta$$

$$\bar D D^2 \bar D = -C_{\alpha\beta}C_{\gamma\delta}\, \bar D_\gamma D_\alpha D_\beta \bar D_\delta$$

$$D\bar D^2 D = -C_{\alpha\beta}C_{\gamma\delta}\left\{-\bar D_\gamma D_\alpha \bar D_\delta D_\beta + 2i\bar\sigma^m_{\gamma\alpha}\partial_m \bar D_\delta D_\beta\right\}$$

$$= \bar D D^2 \bar D - C_{\alpha\beta}C_{\gamma\delta}\left\{2i\,\bar\sigma^m_{\gamma\alpha}\partial_m \bar D_\delta D_\beta - 2i\,\bar\sigma^m_{\delta\beta}\partial_m \bar D_\gamma D_\alpha\right\}$$

$$= \bar D D^2 \bar D - 2i\,\bar D\, C^T \bar\sigma^m \partial_m C\, D + 2i\,\bar D\, C\bar\sigma^m\partial_m C^T D$$

so $$D\bar D^2 D = \bar D D^2 \bar D$$

notice that, since $D_\alpha D_\beta D_\gamma = \bar D_\alpha \bar D_\beta \bar D_\gamma = 0$, the three terms

$D^2\bar D^2$, $\bar D^2 D^2$, $D\bar D^2 D$ are all mutually orthogonal, in the sense that the product of any two gives 0.

Then,

$$D\bar D^2 D = -C_{\alpha\beta}C_{\gamma\delta}\left\{\bar D_\gamma \bar D_\delta D_\alpha D_\beta + 2i\,\bar\sigma^m_{\gamma\alpha}\partial_m \bar D_\delta D_\beta \times 2\right\}$$

$$= \bar D D^2 \bar D = -C_{\alpha\beta}C_{\gamma\delta}\left\{D_\alpha D_\beta \bar D_\gamma \bar D_\delta + 2i\,\bar\sigma^m_{\gamma\alpha}\partial_m D_\beta \bar D_\delta \times 2\right\}$$

so

$$-\bar D^2 D^2 - D^2\bar D^2 + 2D\bar D^2 D = -4i\,(C^T\bar\sigma\partial C)_{\delta\beta}\{D_\beta,\bar D_\delta\}$$

$$= 8\,\mathrm{tr}\left[C^T\bar\sigma\cdot\partial C(\bar\sigma\cdot\partial)^T\right] = 16\,\partial^2$$

So finally we have the identity:

$$\frac{1}{16}\bar{D}^2 D^2 + \frac{1}{16} D^2 \bar{D}^2 - \frac{1}{8} D \bar{D}^2 D = -\partial^2$$

This is a resolution of unity:

on a chiral superfield $\quad \frac{1}{16}\bar{D}^2 D^2 \Phi = -\partial^2 \Phi$

on an antichiral superfield $\quad \frac{1}{16} D^2 \bar{D}^2 \Phi^* = -\partial^2 \Phi^*$

on a vector superfield $\quad -\frac{1}{8} D\bar{D}^2 D V = -\partial^2 V$ + (gauge-variant)

We would like the gauge-fixed action for A_m to contain a term:

$$\int d^4x \left(+\frac{1}{2} A_m \partial^2 A^m \right)$$

Then, at least for the Abelian case, a reasonable guess for the action of V is:

$$\int d^4x\, d^4\theta \left\{ -\frac{1}{4} V \left(-\frac{1}{8} D\bar{D}^2 D V \right) \right\}$$

$$\underset{\text{contains}}{=} \int d^4x\, d^4\theta \left\{ -\frac{1}{4} (2\bar{\theta}\sigma^m\theta A_m)(-\partial^2)(2\bar{\theta}\sigma^n\theta A_n) \right\}$$

So this action contains the piece above and is invariant to gauge variations

$$\delta_\Lambda V = \frac{-i}{g}(\Lambda - \Lambda^*)$$

since $(D\bar{D}^2 D)\Lambda = (D\bar{D}^2 D)\Lambda^* = 0$.

Rewrite this in the following way: Let

$$W_\alpha = -\frac{1}{8}\bar{D}^2 (Dc)_\alpha V$$

Since $\bar{D}_\alpha \bar{D}^2 = 0$, W_α is a chiral superfield. Since $D_\alpha \Lambda^* = 0$, $\bar{D}^2 D_\alpha \Lambda = 0$, W_α is gauge-invariant. The leading component of W_α is:

$$-\frac{1}{8}\bar{D}^2 (Dc)_\alpha \quad (2\bar{\theta}^2 \theta^T c \lambda)$$

$$= -\frac{1}{8}\bar{D}^2 \cdot 2(c^T c \lambda)_\alpha \quad = \quad \lambda_\alpha$$

The action on p. 18 then can be written:

$$\int d^4x \, d^4\theta \, (+\frac{1}{32} V D\bar{D}^2 D V)$$

$$= \int d^4x \, d^4\theta \, [-\frac{1}{32}(D_\alpha V) c_{\alpha\beta} (\bar{D}^2 D_\beta V)]$$

$$= \int d^4x \, d^2\theta \, (-\frac{1}{4}\bar{D}^2) \{ -\frac{1}{32}(Dc)_\beta V \, (\bar{D}^2 D_\beta V) \}$$

$$= \int d^4x \, d^2\theta \, \frac{1}{128} (\bar{D}^2 (Dc)_\beta V) (\bar{D}^2 D \underbrace{c c^T}_{})_\beta V$$

$$= \int d^4x \, d^2\theta \, \frac{1}{2} W^T c W$$

$$= \int d^4x \, d^2\theta \, \frac{1}{2} W^2$$

In the non-Abelian case, we can write the same expression, with

$$W^a_\alpha t^a = -\frac{1}{8}\bar{D}^2 e^{-V^a t^a}(Dc)_\alpha e^{V^a t^a}$$

under a gauge transformation:

$$W^a_\alpha t^a \to -\frac{1}{8}\bar{D}^2 e^{+i\Lambda^a t^a} e^{-V^a t^a} e^{-i\Lambda^{*a} t^a} (Dc)_\alpha e^{+i\Lambda^* t^a} \times e^{V^a t^a} e^{-i\Lambda^a t^a}$$

now $\quad [D_\beta, \Lambda^*] = 0 \quad$ since $\quad D_\beta(\Lambda^*) = 0$

$\quad [\bar{D}_\beta, \Lambda] = 0$

so

$$= e^{i\Lambda^a t^a}\left\{-\frac{1}{8}\bar{D}^2 e^{-V\cdot t}(Dc)_\alpha e^{V\cdot t}\right\} e^{-i\Lambda^a t^a}$$

again

$$W_\alpha t \to e^{i\Lambda t}(W_\alpha t)e^{-i\Lambda\cdot t} = W_\alpha t + [i\Lambda\cdot t, W_\alpha t] + \cdots$$

That is, W^a_α transforms as a matter chiral superfield in the adjoint representation. The action for W^a_α is

$$\int d^4x\, d^2\theta \quad \frac{1}{2}[W^{aT}_\alpha c W^a_\alpha] = \int d^4x\, d^2\theta \ \mathrm{tr}[W^2]$$

(with $\mathrm{tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$ $\quad W_\alpha = (W^a_\alpha t^a)$).

To confirm what we have done, it would be good to work out the complete component-field expression for W_α, at least for the Abelian case. We can work in Wess-Zumino gauge.

I leave this as an exercise for you. The result is

$$W_\alpha = \lambda_\alpha + [(i\sigma^{mn}F_{mn} + D)\,\theta]_\alpha + \theta^2\,(\partial_m\lambda^* i\bar{\sigma}^m c)_\alpha$$

Then

$$\begin{aligned}
\tfrac{1}{2}W^TcW = \;& \tfrac{1}{2}\lambda^T c\lambda \\
& + \theta^T c\,(-i\sigma^{mn}F_{mn} + D)\,\lambda \\
& + \theta^2\Big\{\tfrac{1}{2}\big(D^2 - \tfrac{1}{2}(F_{mn})^2 + \tfrac{i}{4}\epsilon^{mnpq}F_{mn}F_{pq}\big) \\
& \quad + \partial_m\lambda^*(-i\bar{\sigma}^m)\,\lambda\Big\}
\end{aligned}$$

We recognize the chiral supermultiplet (Φ, ψ, F) that we discovered by accident in our previous discussion of the vector multiplet.

W_α is essentially the superfield generalization of the gauge-covariant field strength.