5. The vector multiplet

In the previous two lectures, we have discussed the chiral supermultiplet of N=1 SUSY, the multiplet containing only fermions and scalars. In our general discussion of the N=1 SUSY multiplets, the next example, and the last one with all spins ≤ 1, had a Weyl fermion, its antiparticle, and a massless vector. In this lecture, I will construct the corresponding representation on fields and discuss some of its properties.

The fields we will need are a Weyl fermion field λ_α and a vector field A_m. To obtain a sensible quantum theory, the vector field must be a gauge field. So, before discussing SUSY, I should remind you of some properties of gauge fields and gauge invariance.

We start from a continuously generated group G (a Lie group). The elements of G can be written as

$$g = e^{i\alpha^a T^a}$$

where α^a are parameters and T^a are the "generators" of G.

The multiplication law of the group is determined by the commutation relations of the T^a (the "Lie algebra")

$$[T^a, T^b] = i f^{abc} T^c$$

If $\qquad f^{abc} f^{abd} = M^{cd} > 0 \quad$ a positive matrix

the Lie algebra is "compact". In this case, the generated group is a compact manifold, and the Lie algebra has finite-dimensional unitary representations. In this case, we can rescale the T^c so that

$$f^{abc} f^{abd} = C \cdot \delta^{cd}$$

and, since $\quad f^{abc} \propto \mathrm{tr}\left[[t^a, t^b] t^c\right]$ for t^c a finite dimensional Hermitian representation of the T^a, f^{abc} is totally antisymmetric. The Jacobi identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0$$

implies $\quad f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$

We can put scalar (or fermion) fields into a finite-dimensional representation of G. Then, if t^a represents T^a in this representation, the action of G on the scalar is

$$\phi \to e^{i\alpha^a t^a}\phi$$

It is easy enough to construct Lagrangians invariant to this transformation ("global gauge invariance"). However, it is a little trickier to construct Lagrangians invariant to

$$\phi \to e^{i\alpha^a(x)t^a}\phi$$

where $\alpha^a(x)$ is space-time dependent ("local gauge invariance"). To do this, we must construct a covariant derivative $\mathcal{D}_m\phi$ which transforms as

$$\mathcal{D}_m\phi \to e^{i\alpha^a(x)t^a}\mathcal{D}_m\phi$$

This is not so easy, because

$$\partial_m\phi \to \partial_m(e^{i\alpha\cdot t}\phi) = e^{i\alpha\cdot t}(\partial_m\phi + i\partial_m\alpha^a t^a\phi)$$

The trick is to introduce vector fields with the transformation under local gauge invariance

$$\delta_\alpha A_m^a = \frac{1}{g}(\partial_m\alpha^a + g f^{abc}A_m^b\alpha^c)$$

to go with $\quad \delta_\alpha\phi = i\alpha^a t^a\phi$

Then if we write $\quad \mathcal{D}_m\phi = (\partial_m - igA_m^a t^a)\phi$

$$\delta_\alpha(\mathcal{D}_m\phi) = (\partial_m - ig A_m^a t^a)(i\alpha^a t^a \phi) - ig \frac{1}{g}(\partial_m \alpha^a + g f^{abc} A_m^b \alpha^c) t^a \phi$$

$$= (i\alpha^a t^a)(\mathcal{D}_m\phi) + \cancel{i(\partial_m \alpha^a) t^a \phi} + ig [A_m^a t^a, \alpha^b t^b]\phi - \cancel{i\,\partial_m \alpha^a t^a \phi} - ig f^{abc} A_m^b \alpha^c t^a \phi$$

so

$$\delta_\alpha(\mathcal{D}_m\phi) = i\alpha^a t^a(\mathcal{D}_m\phi)$$ as required, since

$$g[A_m^a t^a, \alpha^b t^b]\phi = ig f^{abc} A_m^a \alpha^b t^c \phi$$

and f^{abc} is totally antisymmetric.

Among the various representations of the Lie algebra of G, there is a special one: The generators of G must belong to a finite-dimensional representation of this algebra. The representation matrices are not hard to find:

$$(t^a)_{bc} = i f^{bac}$$ "adjoint representation"

To prove this, we need to show that

$$([t^a, t^b])_{de} = i f^{abc}(t^c)_{de}$$

$$f^{dac} f^{cbe} - f^{dbc} f^{cae} = f^{abc} f^{dce}$$

rearrange: $-f^{ace} f^{bdc} - f^{bce} f^{dac} - f^{dce} f^{abc} = 0$

which is just the Jacobi identity, swapping the indices d & c.

Anyway, in the adjoint representation

$$D_m \alpha^a = \partial_m \alpha^a - ig (if)^{abc} A_m^b \alpha^c$$

$$= \partial_m \alpha^a + g f^{abc} A_m^b \alpha^c$$

so the gauge transformation of A_m^a is in fact

$$\delta_\alpha A_m^a = \frac{1}{g} D_m \alpha^a \quad \text{in the adjoint representation.}$$

The commutator of covariant derivatives

$$[D_m, D_n] = [\partial_m - ig A_m^a t^a, \partial_n - ig A_n^b t^b]$$

$$= -ig \left(\partial_m A_n^a t^a - \partial_n A_m^a t^a - ig [A_m^a t^a, A_m^b t^b] \right)$$

$$= -ig \left(\partial_m A_n^a - \partial_n A_m^a + f^{abc} A_m^a A_n^b \right) t^a$$

$$= -ig \; F_{mn}^c t^c$$

acts on a multiplet of fields

$$\delta_\alpha \quad [\mathcal{D}_m, \mathcal{D}_n]\,\phi \quad = \quad (i\alpha^a t^a)\;[\mathcal{D}_m, \mathcal{D}_n]\,\phi$$

by the property of the covariant derivative. Since $\delta_\alpha \phi = i\alpha^a t^a \phi$,

we must have

$$\delta_\alpha \left(\; [\mathcal{D}_m, \mathcal{D}_n] \right) = \left[i\alpha^a t^a, \quad [\mathcal{D}_m, \mathcal{D}_n] \right]$$

$$\delta_\alpha \;\left(F_{mn}^c t^c \right) \;=\; i[\alpha^a t^a \,,\; F_{mn}^c t^c]$$

$$= \quad i \quad i f^{acb} \alpha^a F_{mn}^c \, t^b$$

or

$$\delta_\alpha \quad F_{mn}^c \quad = \quad i \quad i f^{cab} \alpha^a F_{mn}^b$$

$$= \quad i \; (t^a)_{cb} \, \alpha^a \quad F_{mn}^b$$

with (t^a) in the adjoint rep. Since f^{abc} is totally

antisymmetric

$$\delta_\alpha \left(F_{mn}^a \right)^2 = F_{mn}^a \;\; i\,(t^b)_{ac} F_{mn}^c \; + (i t^b)_{ca} F_{mn}^a F_{mn}^c$$

$$= 0$$

so

$$\mathcal{L} = \int d^4x \left(-\tfrac{1}{4} \left(F_{mn}^a \right)^2 \right)$$

is a gauge-invariant kinetic term for A_m^a

Now let's generalize this formalism to SUSY. The SUSY transformation relates A_m^a and a multiplet of Weyl fermions λ_α^a ("gauginos"). The simplest relation is

$$\delta_\xi A^{ma} = (\xi^\dagger \bar\sigma^m \lambda^a + \lambda^{a\dagger} \bar\sigma^m \xi)$$ since A_m^a is Hermitian

$$\delta_\xi \lambda^a = i \sigma^{mn} F_{mn}^a \xi$$

$$\delta_\xi \lambda^{a\dagger} = \xi^\dagger (i \bar\sigma^{mn} F_{mn}^a)$$

Notice that this is consistent with the gauge transformation properties: λ^a, $\lambda^{a\dagger}$, and F_{mn}^a transform as fields in the adjoint representation. A^{ma} transforms differently, but we can write this as

$$\delta_\alpha A^{ma} = \frac{1}{g} \partial^m \alpha^a + i (i f^{abc}) \alpha^b A_m^c$$

take δ_ξ

$$\delta_\xi (\delta_\alpha A^{ma}) = i (t^b)_{ac} \, \delta_\xi A_m^c$$

so it is fine for the RHS of $\delta_\xi A$ to transform as a multiplet in the adjoint rep.

To check the above relations, compute

$$[\delta_\xi, \delta_\eta] A^{ma} = \delta_\xi (\eta^\dagger \bar\sigma^m \lambda^a + \lambda^{a\dagger} \bar\sigma^m \eta) - (\eta \leftrightarrow \xi)$$

$$= \eta^\dagger \bar\sigma^m \, i \sigma^{pq} F^a_{pq} \xi - (\eta \leftrightarrow \xi)$$

$$+ \xi^\dagger i \bar\sigma^{pq} F^a_{pq} \bar\sigma^m \eta - (\eta \leftrightarrow \xi)$$

$$= -i \, \xi^\dagger [\bar\sigma^m \sigma^{pq} - \bar\sigma^{pq} \bar\sigma^m] \eta F^a_{pq} + h.c.$$

I claimed in the first lecture (have you proved it yet?) that

$$\bar\sigma^a \sigma^{bc} - \bar\sigma^{bc} \bar\sigma^a = \eta^{ab} \bar\sigma^c - \eta^{ac} \bar\sigma^b$$

so

$$= -2i \, \xi^\dagger \bar\sigma^q \eta \, F^{m}{}_{q}{}^{a} + h.c.$$

This doesn't look quite right. But let's write it explicitly:

$$[\delta_\xi, \delta_\eta] A^{ma} = 2i (\xi^\dagger \bar\sigma^q \eta - \eta^\dagger \bar\sigma^q \xi)$$

$$\cdot \; \partial_q A^a_m - \partial_m A^a_q - g f^{abc} A^b_m A^c_q$$

let $\Xi^q = i(\xi^\dagger \bar\sigma^q \eta - \eta^\dagger \bar\sigma^q \xi)$

$$= 2 \Xi^q \partial_q A^a_m - \partial_m (2 \Xi^q A^a_q)$$

$$- g f^{abc} A^b_m (2 \Xi^q A^c_q)$$

so the relation we have found is

$$[\delta_\xi, \delta_\eta] A^{ma} = 2i(\xi^\dagger \bar\sigma^q \eta - \eta^\dagger \bar\sigma^q \xi)\, \partial_q A_m^a$$
$$+ \delta_{\alpha(\xi,\eta)} A_m^a$$

where the second term is a gauge transformation with parameter

$$\alpha(\xi\eta) = (-2g\, \Xi^q A_q^a)$$

This transformation is odd, but actually there is nothing wrong with it, since all that we need to have a supersymmetric, unitary quantum theory is ~~that~~ $[\delta_\xi, \delta_\eta] = 2i(\xi^\dagger \bar\sigma^m \eta - \eta^\dagger \bar\sigma^m \xi)\, \partial_m$ on gauge-invariant states. To ensure this, we just need

$$[\delta_\xi, \delta_\eta]\, \phi = 2i(\xi^\dagger \bar\sigma^q \eta - \eta^\dagger \bar\sigma^q \xi)\, \partial_q \phi + \delta_{\alpha(\xi,\eta)} \phi$$

on all other fields. If ϕ is in a nontrivial representation of G, this reads

$$= 2\Xi^q \partial_q \phi + i(-2g\, \Xi^q A_q^a t^a)\phi$$

$$= 2\, \Xi^q (\partial_q - ig A_q^a t^a)\, \phi$$

or

$$[\delta_\xi, \delta_\eta]\phi = 2i(\xi^\dagger \bar\sigma^q \eta - \eta^\dagger \bar\sigma^q \xi)\, D_q \phi$$

which is more in tune with the structure of the theory there the corresponding relation with $\partial_a \Phi$.

The desired form of the SUSY commutator on λ^a is then

$$[\delta_\xi, \delta_\eta]\, \lambda^a = 2i(\xi^\dagger \bar{\sigma}^m \eta - \eta^\dagger \bar{\sigma}^m \xi)\, \mathcal{D}_m \lambda$$

We could check whether this is true from the transformations on p.7. The answer turns out to be that it is not — unless

$$i\bar{\sigma}^m \mathcal{D}_m \lambda = 0$$

That is, the relations on p.7 satisfy the SUSY CR's only <u>on shell</u>.

In the case of a chiral supermultiplet, we closed the CR's off-shell by adding an auxiliary field to balance the number of field degrees of freedom between Bose and Fermi fields. Let's now do the counting for the case of the vector multiplet:

	Particle states	(real) fields
A_m	2	4 − 1 (gauge freedom)
λ_α	− 2	− 4
we need → D	0	1
one <u>real</u> field	0	0

The SUSY variation of D should be proportional to $\bar{\sigma}^m \mathcal{D}_m \lambda$. So here is a set of SUSY variations:

$$\delta_\xi A^{ma} = \xi^\dagger \bar{\sigma}^m \lambda^a + \lambda^{a\dagger} \bar{\sigma}^m \xi$$

$$\delta_\xi \lambda^a = (i \sigma^{mn} F^a_{mn} + D^a)\xi$$

$$\delta_\xi \lambda^{a\dagger} = \xi^\dagger (i \bar{\sigma}^{mn} F^a_{mn} + D^a)$$

$$\delta_\xi D^a = -i(\xi^\dagger \bar{\sigma}^m \mathcal{D}_m \lambda^a - \mathcal{D}_m \lambda^{a\dagger} \bar{\sigma}^m \xi)$$

Do these satisfy the SUSY CR's? You can check that there is no change in the relation for

$$[\delta_\xi, \delta_\eta] A^{ma}$$

since D^a adds an extra term

$$\eta^\dagger \bar{\sigma}^m D^a \xi + \xi^\dagger \bar{\sigma}^m D^a \eta - (\eta \leftrightarrow \xi)$$

$$= \xi^\dagger (\bar{\sigma}^m D^a - D^a \bar{\sigma}^m)\eta + \text{h.c.} = 0$$

Next, check

$$[\delta_\xi, \delta_\eta] D^a = \delta_\xi(-i(\eta^\dagger \bar{\sigma}^m \mathcal{D}_m \lambda - \mathcal{D}_m \lambda^\dagger \bar{\sigma}^m \eta)) - (\xi \leftrightarrow \eta)$$

there are two contributions : one in which δ_ξ acts on λ and one in which δ_ξ acts on A_m in $\mathcal{D}_m$. The first is

$$+ i \mathcal{D}_m \left(\xi^\dagger (i \bar\sigma^{pq} F_{pq}{}^a + D^a) \bar\sigma^m \eta \right)$$

$$+ i \, \xi^\dagger \bar\sigma^m \, (i \sigma^{pq} F_{pq}{}^a + D^a) \eta \qquad + \text{h.c.}$$

$$= \; 2i \; \xi^\dagger \bar\sigma^m \eta \;\; \mathcal{D}_m D^a$$

$$+ (i)^2 \, \xi^\dagger (\bar\sigma^{pq} \bar\sigma^m + \bar\sigma^m \sigma^{pq}) \eta \;\; \mathcal{D}_m F_{pq}{}^a$$

Now, it follows from the σ identities that

$$\bar\sigma^m \sigma^{pq} + \bar\sigma^{pq} \bar\sigma^m \; = \; -i \, \epsilon^{mpqr} \bar\sigma_r$$

Now $\epsilon^{mpqr} \mathcal{D}_m F_{pq}{}^a$ looks like part of a field equation, but actually it is identically zero : (the Bianchi identity)

$$\epsilon^{mpqr} \mathcal{D}_m (-ig F_{pq}^a) \; t^a$$

$$= \; \epsilon^{mpqr} \left[\mathcal{D}_m , (-ig F_{pq}^a t^a) \right]$$

$$= \; \epsilon^{mpqr} \left[\mathcal{D}_m , \left[\mathcal{D}_p , \mathcal{D}_q \right] \right]$$

$$= \quad 0 \qquad \text{by the Jacobi identity.}$$

the second contribution to the CR at the bottom of p.11 is

$$[\delta_\xi, \delta_\eta] D^a = (\text{p.12})$$

$$+ \; i \xi^\dagger \bar\sigma^m g f^{abc} (\delta_\eta A^b_m) \lambda^c$$

$$+ \; i \lambda^{\dagger c} g f^{abc} (\delta_\xi A^b_m) \bar\sigma^m \eta \quad + \text{h.c.}$$

$$= (\text{p.12}) + i g f^{abc} \xi^\dagger \bar\sigma^m \lambda^c (\eta^\dagger \bar\sigma_m \lambda^b + \lambda^{\dagger b} \bar\sigma_m \eta)$$

$$+ \; i g f^{abc} \lambda^{\dagger c} \bar\sigma^m \eta (\xi^\dagger \bar\sigma_m \lambda^b + \lambda^{\dagger b} \bar\sigma_m \xi)$$

$$+ \text{h.c.}$$

the structure $\quad \xi^\dagger \bar\sigma^m \lambda^c \eta^\dagger \bar\sigma_m \lambda^b \underset{\text{Fierz}}{=} \eta^\dagger \bar\sigma^m \lambda^c \xi^\dagger \bar\sigma_m \lambda^b$

is symmetric under $(\eta \leftrightarrow \xi)$ and so must vanish. Similarly for the term with η and ξ. The remaining terms Fierz into

$$+ i g f^{abc} \xi^\dagger \bar\sigma^m \eta (\lambda^{\dagger b} \bar\sigma_m \lambda^c + \lambda^{\dagger c} \bar\sigma_m \lambda^b)$$

$$= 0 \quad \text{by antisymmetry of } f^{abc}$$

so finally

$$[\delta_\xi, \delta_\eta] D^a = 2i (\xi^\dagger \bar\sigma^m \eta - \eta^\dagger \bar\sigma^m \xi) \mathcal{D}_m D^a$$

as desired.

It is also possible to show explicitly that

$$[\delta_\xi, \delta_\eta]\, \lambda^a = 2i\,(\xi^\dagger \bar\sigma^m \eta - \eta^\dagger \bar\sigma^m \xi)\, \mathcal{D}_m \lambda^a$$

off-shell. I leave that to you as an exercise.

Now that we have found SUSY variations for (A, λ, D), our next task is to find a Lagrangian invariant to these variations. One method is to guess a Lagrangian and check its invariance. However, I would like to employ a special ~~trick that~~ will have some interesting additional consequences.

Let $\quad \Phi = \frac{1}{2}(\lambda^{aT} c\, \lambda^a) \quad$ gauge-invariant, scalar

(the Majorana mass term for the gaugino λ^a). Compute

$$\delta_\xi \Phi = \lambda^{aT} c\,(i\sigma^{pq} F^a_{pq} + D^a)\, \xi$$

$$= \xi^T c\,(-i\sigma^{pq} F^a_{pq} + D^a)\, \lambda^a$$

by $\quad (\sigma^{pq})^T c = c\, \sigma^{qp} = -\, c\, \sigma^{pq}$

or $\quad \delta_\xi \Phi = \sqrt{2}\, \xi^T c\, \Psi \quad$ with

$$\Psi = \left(\frac{-i\sigma^{pq} F^a_{pq} + D^a}{\sqrt{2}}\right) \lambda^a$$

next:

$$\delta_\xi \Psi = \frac{1}{\sqrt{2}} (-i\sigma^{pq} F_{pq}{}^a + D^a)(i\sigma^{rs} F_{rs}{}^a + D^a)\xi$$

$$+ \frac{1}{\sqrt{2}} (-i\sigma^{pq}) \cdot 2\, \mathcal{D}_p (\xi^\dagger \bar\sigma^q \lambda^a + \lambda^{a\dagger} \bar\sigma^q \xi)\ \lambda^a$$

$$+ \frac{1}{\sqrt{2}} (-i)(\xi^\dagger \bar\sigma^m \mathcal{D}_m \lambda^a - \mathcal{D}_m \lambda^{a\dagger} \bar\sigma^m \xi)\ \lambda^a$$

the first term here is

$$\sqrt{2}\left(\frac{1}{2}(D^a)^2 \xi + \frac{1}{2}(\sigma^{pq} F_{pq}{}^a\, \sigma^{rs} F_{rs}{}^a)\ \xi\right)$$

now

$$\sigma^{pq}\sigma^{rs} = \frac{1}{4}[-\eta^{pr}\eta^{qs} + \eta^{qr}\eta^{ps} + i\epsilon^{pqrs}]$$

+ terms antisymmetric under $pq \leftrightarrow rs$

$$= \sqrt{2}\left(\frac{1}{2}(D^a)^2 - \frac{1}{4} F^{pq\,a} F_{pq}{}^a + \frac{i}{8}\epsilon^{pqrs} F_{pq}{}^a F_{rs}{}^a\right)\xi$$

the next 2 lines organize into

$$\frac{-i}{\sqrt{2}} (\xi^\dagger \bar\sigma^q \mathcal{D}_p \lambda^a)(\eta^{pq} + 2\sigma^{pq})\lambda^a$$

$$+ \frac{i}{\sqrt{2}} (\mathcal{D}_p \lambda^{a\dagger} \bar\sigma^q \xi)(\eta^{qp} + 2\sigma^{qp})\lambda^a$$

$$= \frac{-i}{\sqrt{2}}\left\{(\xi^\dagger \bar\sigma^q \mathcal{D}^p \lambda^a)\ \sigma^p \bar\sigma^q \lambda^a - (\mathcal{D}_p \lambda^{a\dagger} \bar\sigma^q \xi)\ \sigma^q \bar\sigma^p \lambda^a\right\}$$

Now use the Fierz identities in the form

$$(\bar\sigma^q)_{\alpha\beta} \quad (\bar\sigma_q)_{\gamma\delta} \quad = \quad 2\, C_{\alpha\gamma}\, C_{\beta\delta}$$

$$(\bar\sigma^q)_{\alpha\beta} \quad (\sigma_q)_{\gamma\delta} \quad = \quad 2\, \delta_{\alpha\delta}\, \delta_{\gamma\beta}$$

to rewrite this as

$$= \quad \frac{-i\cdot 2}{\sqrt{2}} (\sigma^p C^T \xi^*)\, (D_p \lambda^{aT}) C\, \lambda^a \; + \frac{i\cdot 2}{\sqrt{2}} D_p \lambda^{a\dagger} \bar\sigma^p \lambda^a\, \xi\, (-1)$$

$$= \quad i\sqrt{2}\;\; \sigma^p C \xi^* \;\; D_p \left(\tfrac{1}{2}\lambda^{aT} C \lambda^a\right) \; + \sqrt{2}\, \left(-i D_p \lambda^\dagger \bar\sigma^p \lambda^a\right) \xi$$

i.e.

$$\delta_\xi \Psi = \sqrt{2} i \;\; \sigma^p C \xi^* \, D_p \Phi \quad + \sqrt{2}\; \mathcal{F} \;\; \xi$$

where $\quad \mathcal{F} = \left\{ \frac{1}{2}(D^a)^2 - \frac{1}{4}(F^a_{pq})^2 + \frac{i}{8}\epsilon^{pqrs} F^a_{pq} F^a_{rs} \right.$
$\left. \qquad\qquad\qquad - i\, D_p \lambda^{\dagger a} \bar\sigma^p \lambda^a \right\}$

We recognize

$$(\Phi, \Psi, \mathcal{F})$$

as fields of a chiral supermultiplet. The F-term of a chiral supermultiplet satisfies

$$\delta_\xi F = -\sqrt{2} i \;\; \xi^\dagger \bar\sigma^m \; \partial_m F$$

(if F is a gauge singlet $\mathcal{D}_m = \partial_m$) and so

$$\delta_\xi \int d^4x\, F = 0$$

so the F term of a chiral supermultiplet is an invariant Lagrangian term. We have then shown that

$$\int d^4x\, \mathcal{L} = \int d^4x \left(-\tfrac{1}{4}(F^a_{pq})^2 + i\,\lambda^{\dagger a}\bar\sigma^m \mathcal{D}_m \lambda^a + \tfrac{1}{2}(D^a)^2 \right)$$

[recall that $\int d^4x\, \epsilon^{pqrs} F^a_{pq} F^a_{rs}$ is a total derivative]

is an invariant action. This is just what we would have guessed — kinetic terms for the gauge field and Weyl fermion, and a term w. no derivatives for D^a!

Now we have the pure gauge action; can we couple these gauge fields to matter? Matter should be scalars and spinors — chiral supermultiplets — in representations r of G. As a starting point, we can take the action and transformation laws of the chiral multiplet, as discussed last week, and replace $\partial_m \to \mathcal{D}_m$ everywhere. This gets us a big way, but not quite to the right place.

For the transformation laws, we actually need

$$\delta_\xi \phi = \sqrt{2}\,\xi^T c\, \psi$$

$$\delta_\xi \psi = \sqrt{2}\, i\, \sigma^q c\, \xi^* \mathcal{D}_q \phi + \sqrt{2} F \xi$$

$$\delta_\xi F = -\sqrt{2}\, i\, \xi^\dagger \bar\sigma^p \mathcal{D}_p \psi - \underbrace{2g\, \xi^\dagger c \lambda^{a*} t^a \phi}_{\text{extra term}}$$

The relation

$$[\delta_\xi, \delta_\eta]\phi = 2i(\xi^\dagger \bar\sigma^m \eta - \eta^\dagger \bar\sigma^m \xi)\, \mathcal{D}_m \phi$$

goes through just as in lecture 3, with $\partial_m \to \mathcal{D}_m$. For λ and F, the analysis is a little trickier. If we were to compute with just the first term of $\delta_\xi F$, we would find

$$[\delta_\xi, \delta_\eta] F = \delta_\xi(-\sqrt{2}\, i\, \eta^\dagger \bar\sigma^p \mathcal{D}_p \psi) - (\xi \leftrightarrow \eta) + \ldots$$

$$= -\sqrt{2}\, i\, \eta^\dagger \bar\sigma^p \mathcal{D}_p (\sqrt{2}\, i\, \sigma^q c\, \xi^* \mathcal{D}_q \phi + \sqrt{2} F \xi) + \cdots$$

the second term here gives

$$-2i\, \eta^\dagger \bar\sigma^p \xi\, \mathcal{D}_p F$$

which is the answer we are looking for. The first term vanished in our previous analysis, but now it gives

$$2\ \eta^\dagger \bar\sigma^p \sigma^q c\ \xi^*\ \mathcal{D}_p \mathcal{D}_q \phi \quad - \ (\xi \leftrightarrow \eta)$$

If we keep the part antisymmetric under $(\xi \leftrightarrow \eta)$, we must replace

$$\bar\sigma^p \sigma^q = \eta^{pq} + 2\bar\sigma^{pq}$$

& discard $\eta^\dagger c\, \eta^{pq} \xi^*$, which is symmetric. This leaves

$$2 \cdot 2 \cdot \eta^\dagger \bar\sigma^{pq} c\ \xi^*\ \mathcal{D}_p \mathcal{D}_q \phi$$

$$= 2\eta^\dagger\ \bar\sigma^{pq} c\ \xi^*\ [\mathcal{D}_p, \mathcal{D}_q]\, \phi$$

$$= 2\eta^\dagger \bar\sigma^{pq} c\ \xi^*\ (-ig F^a_{pq} t^a)\, \phi$$

This term is nonzero; & it needs the extra term in $\delta_\xi F$ to be cancelled:

$$[\delta_\xi, \delta_\eta] F \ = \ \cdots \ + \ \delta_\xi \left(-2g\, \eta^\dagger c\, \lambda^{a*} t^a \phi\right) \quad - \ (\xi \leftrightarrow \eta)$$

$$= \quad -2g\, \eta^\dagger c\ (i\, \bar\sigma^{pq} F^a_{pq})^T \xi^*\ t^a \phi \quad + \cdots$$

$$= \quad -2ig\ \eta^\dagger c (\bar\sigma^{pq})^T \xi^*\ F^a_{pq} t^a \phi \quad + \cdots$$

$$= \quad -2ig\ \eta^\dagger (-\bar\sigma^{pq}) c\ \xi^*\ F^a_{pq} t^a \phi + \cdots$$

cancels the term above.

With more detailed analysis, one can show that

$$[\delta_\xi, \delta_\eta]\ (\phi, \psi, F) \ = \ 2i(\xi^\dagger \bar\sigma^m \eta - \eta^\dagger \bar\sigma^m \xi)\ \mathcal{D}_m (\phi, \psi, F)$$

Next, we need to extend the kinetic energy term for the chiral multiplet to a Lagrangian that couples the chiral multiplet to the vector multiplet. The first step is to convert derivatives to covariant derivatives:

$$\mathcal{L} = (D_m\phi)^* D^m\phi + \psi^\dagger i\bar\sigma^m D_m \psi + F^* F$$

but, if ϕ, ψ couple to A, they must also couple to λ. An appropriate term would be a Lorentz scalar, with $\lambda^a t^a$ acting between ϕ^* and ψ:

$$(\text{const})\cdot g\ \phi^* \lambda^{aT} t^a c\, \psi \quad + \text{h.c.}$$

Note that this is dimensionally correct: in 4-d

$\phi \sim (\text{mass})$ $\quad \psi \sim (\text{mass})^{3/2}$ $\quad F \sim (\text{mass})^2$

$A \sim (\text{mass})$ $\quad \lambda \sim (\text{mass})^{3/2}$ $\quad D^a \sim (\text{mass})^2$ $\qquad g \sim$ dim-less.

There is one more dimensionally consistent scalar term:

$$g\ \phi^* D^a t^a\ \phi$$

and this is required because λ^a transforms into D^a. The relative coefficients in the final Lagrangian are fixed by demanding that $\delta_\xi \mathcal{L} = 0$. This gives:

$$\mathcal{L} = \mathcal{D}_m\phi^* \mathcal{D}^m\phi + \psi^\dagger i\bar\sigma^m \mathcal{D}_m\psi + F^*F$$
$$- \sqrt{2}g\,(\phi^*\, \lambda^{aT} t^a c\,\psi - \psi^\dagger c\, \lambda^{a*} t^a \phi)$$
$$+ g D^a\, \phi^* t^a \phi$$

Let me just check a few pieces of $\delta_\xi \mathcal{L} = 0$.

terms w. $\psi^\dagger \lambda \psi$:

$$\delta_\xi \;(\psi^\dagger i\bar\sigma^m(-igA_m^a t^a)\psi - \sqrt{2}g\, \phi^*\lambda^{aT} t^a c\psi)$$
$$= \psi^\dagger i\bar\sigma^m t^a \psi\,(-ig)(\xi^\dagger\bar\sigma_m\lambda^a) - \sqrt{2}g\,(-\sqrt{2}\,\psi^\dagger c\,\xi^*)\,\lambda^{aT} t^a c\psi$$
$$= 2g(-1)\,\psi^\dagger c\,\xi^*\, t^a\,(\psi^T c\lambda^a) + 2g(\psi^\dagger c\xi^*)(\lambda^{aT} t^a c\psi)$$
$$= 0 \quad \checkmark$$

terms w. $\phi^* D \psi$

$$\delta_\xi \;(- \sqrt{2}g\, \phi^*\, \lambda^{aT} t^a c\psi + g D^a\, \phi^* t^a\phi)$$
$$= - \sqrt{2}g\, \phi^*\, \xi^T D^a t^a c\psi + g D^a\, \phi^* t^a (\sqrt{2}\xi^T c\psi)$$
$$= 0 \quad \checkmark$$

These two exercises already confirm the relative normalization of the three terms. I leave the rest of the proof of $\delta_\xi\mathcal{L} = 0$ to you as an exercise.

When we combine the gauge and matter terms, we find the following terms containing the auxiliary field D^a:

$$\mathcal{L}_D = \frac{1}{2}(D^a)^2 + g\, D^a \phi^* t^a \phi$$

We can eliminate D^a from the Lagrangian in the same way that we eliminated F, by ~~integrating~~ out or solving for D^a from its field equation. Complete the square

$$= \frac{1}{2}\left(D^a + g\, \phi^* t^a \phi\right)^2 - \frac{g^2}{2}\left(\phi^* t^a \phi\right)^2$$

then

$$-\langle D^a \rangle = g\, \phi^* t^a \phi$$

If there are several matter fields, in reps. r_k

$$-\langle D^a \rangle = g \sum_k \phi_k^* (t^a)_{r_k} \phi_k$$

and

$$V = + \frac{g^2}{2}\left(\sum_k \phi_k^* t^a \phi_k\right)^2$$

This "D term" of the potential has the same interesting properties as the F term of the potential:

$$V \geq 0$$

$$V = 0 \quad \text{if and only if} \quad \langle D^a \rangle = 0$$

Then it seems that $\langle D^a \rangle \neq 0$ signals spontaneous SUSY breaking. To see that this is correct, consider the relation

$$\delta_\xi \lambda^a = [\xi^T c Q, \lambda^a] = (i\sigma^{pq} F_{pq} + D)\xi$$

now $\langle F_{pq} \rangle = 0$ in any Lorentz invariant state, so in such a state

$$\langle [\xi^T c Q, \lambda^a] \rangle = \langle D^a \rangle \xi$$

and if $\langle D^a \rangle \neq 0$, SUSY is spontaneously broken.

As with the F-term potential, we can find models in which either a global symmetry or supersymmetry itself is spontaneously broken. As a first example, consider a theory in which the matter contains a Dirac fermion with mass term

$$m\, \psi_2^T c\, \psi_1 + \text{h.c.}$$

If ψ_1 belongs to the representation r of G, ψ_2 must belong to $\bar{r}$. If ψ_1 transforms under G according to

$$\psi_1 \to e^{i\alpha^a t^a}\psi_1 \quad \text{or} \quad \delta_\alpha \psi_1 = i\alpha^a t^a \psi_1$$

then $$\psi_2 \to \psi_2 e^{-i\alpha^a t^a} \quad \text{or} \quad \delta_\alpha \psi_2 = \psi_2(-i\alpha^a t^a)$$

that is, if t^a are the representation matrices in r, the representation matrices in $\bar{r}$ are $(t^a)_{\bar{r}} = -(t^{aT}_r)$

This means that the D terms for the scalar partner fields ϕ_1, ϕ_2 have the form

$$-D^a = g\left(\phi_1^* t^a \phi_1 - \phi_2 t^a \phi_2^*\right)$$

so that

$$V_D = \frac{g^2}{2}\left(\phi_1^* t^a \phi_1 - \phi_2 t^a \phi_2^*\right)^2$$

$D^a = 0$ can be arranged by setting $\langle\phi_1\rangle = \langle\phi_2^*\rangle$, while these quantities take arbitrary nonzero values. This is a line of supersymmetric minima of V_D running out to infinity, with G spontaneously broken everywhere except at $\langle\phi_1\rangle = \langle\phi_2^*\rangle = 0$.

For definiteness, consider the SU(2) case, and

$$\langle\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} = \langle\phi_2^*\rangle$$

This is like the Standard Model with two Higgs doublets and $\sin^2\theta_W = 0$: All three SU(2) gauge bosons get masses equal to

$$m_W^2 = \frac{g^2 v^2}{4} \times 2 \qquad \text{or} \qquad m_W = \frac{g v}{\sqrt{2}}$$

the expectation values of $\phi_1 \phi_2$ also give masses to the gauginos λ^a. Consider

$$-\sqrt{2}\, g \left(\langle \phi_1^* \rangle \lambda^{aT} t^a c \psi_1 - \psi_2^T c\, t^a \lambda^a \langle \phi_2^* \rangle \right)$$

with $t^a = \sigma^a/2$ for $SU(2)$.

$$= -\sqrt{2} g \frac{v}{\sqrt{2}} \left[(1\ 0)\, \lambda^{aT} \frac{\sigma^a}{2} c \,(\psi_1) - \psi_2^T c \frac{\sigma^a}{2} \lambda^a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

for $a = 3$ $\quad \psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix} \quad \psi_2 = (\psi_{21}\ \psi_{22})$

$$= -\frac{g}{2} v \quad \lambda^{3T} c \,(\psi_{11} - \psi_{21})$$

$$= -\left(\frac{g}{\sqrt{2}} v\right) \lambda^{3T} c \left(\frac{\psi_{11} - \psi_{21}}{\sqrt{2}}\right)$$

so λ^3 and $\dfrac{\psi_{11} - \psi_{21}}{\sqrt{2}}$ obtain a Dirac mass equal to $\dfrac{gv}{\sqrt{2}}$

similarly λ^- and ψ_{12}, λ^+ and ψ_{22} pair up and obtain mass $\dfrac{gv}{\sqrt{2}}$.

Now we have, for each a, a massive vector (3 degrees of freedom) and a massive Dirac fermion (4 degrees of freedom). To form a supersymmetric spectrum, there should be one more real massive scalar field. It is not hard to identify this field for $a = 3$, it is the fluctuation

$$\phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h/\sqrt{2} \\ 0 \end{pmatrix} \qquad \phi_2^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v - h/\sqrt{2} \\ 0 \end{pmatrix}$$

In the ~~potential~~ on p. 24, this field contributes

$$\frac{1}{2} g^2 \left(\frac{1}{2} \frac{1}{2} (v + h/\sqrt{2})^2 - (v - h/\sqrt{2})^2 \right)^2$$

$$= \frac{1}{4} g^2 v^2 h^2 + \cdots = \frac{1}{2} m^2 h^2$$

so $m^2 = \dfrac{g^2 v^2}{2}$ ✓

the partners of $\lambda^1 \lambda^2$ come from $\psi_{12} - \psi_{22}^*$.

In this example, we find the massive vector supermultiplet

$$\begin{array}{ccccc} W & + & \Psi & + & h \\ \text{spin 1} & & \text{Dirac fermion} & & \text{real scalar} \end{array}$$

This multiplet has a number of particle states

$$3 \quad - \quad 4 \quad + \quad 1 \quad = \quad 0 \checkmark$$

The multiplet is formed when a vector supermultiplet eats a chiral supermultiplet in the Higgs mechanism.

Now here is an example with spontaneously broken SUSY. Note ~~fist~~ that, in a U(1) gauge theory, D is gauge invariant. Its transformation law is

$$\delta_\xi D = -i\,[\xi^\dagger \bar\sigma^m \partial_m \lambda - \partial_m\lambda^\dagger \bar\sigma^m \xi]$$

so $$\delta_\xi \int d^4x\, D = 0$$

so we can add a term $+\kappa D$ to a supersymmetric Lagrangian. This is called the "Fayet-Iliopoulos term".

The D potential is now generated by

$$\tfrac{1}{2}D^2 + \kappa D + gD\sum_k \phi_k^* t_k \phi_k$$

in $U(1)$, $t_k = Q_k$, the electric charges.

$$= \tfrac{1}{2}\Big(D + \kappa + g\sum_k Q_k|\phi_k|^2\Big)^2$$

$$- \tfrac{1}{2}\Big(\kappa + g\sum_k Q_k|\phi_k|^2\Big)^2$$

so $$V_D = \tfrac{1}{2}\Big(\kappa + g\sum_k Q_k|\phi_k|^2\Big)^2$$

In the following, assume κ — a constant — is > 0.

then there are two options :

① some $Q_k < 0$. Then the corresponding ϕ_k can get a v.e.v., allowing $D = 0$ and a supersymmetric vacuum state. The $U(1)$ symmetry is spontaneously broken

② all $Q_k > 0$. Then the minimum of the potential is at $\langle \phi_k \rangle = 0$ for all k,

$$V_D = \frac{\kappa^2}{2} > 0$$

and SUSY is spontaneously broken.

In this case, there must be a massless fermion to be the Goldstino. In fact, since

$$\langle [Q, \lambda] \rangle \cong \langle D \rangle \neq 0$$

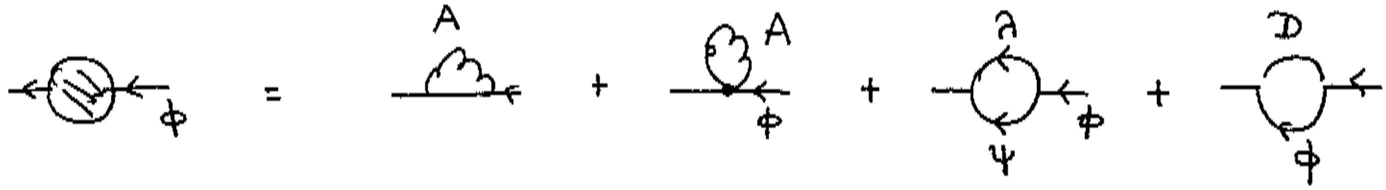
The Goldstino must be λ. In fact, the mass term for λ is

$$- \sqrt{2} g \left(\langle \phi_k^* \rangle \lambda Q_k \psi_k + h.c. \right)$$

$$= 0 \quad \text{since} \quad \langle \phi_k \rangle = 0 \text{ for all } k.$$

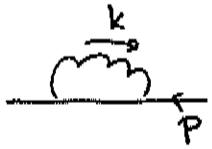
So λ is massless as required.

Let me now say a bit about the quantum theory of the vector multiplet. First of all, can the radiative corrections due to the vector multiplet give mass to a matter scalar ϕ? We need to compute



Recall $$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k+p)^2} = \int_0^1 dx \int \frac{d^4\mathbb{k}}{(2\pi)^4} \frac{1}{[\mathbb{k}^2 - x(1-x)p^2]^2}$$

with $k = \mathbb{k} - xp \qquad k+p = \mathbb{k} + (1-x)p$ (Feynman says)



$$= \int \frac{d^4k}{(2\pi)^4} \left(ig(k+2p)^m\right) t^a \frac{i}{(k+p)^2} \left(ig(k+2p)_m\right) t^a \frac{-i}{k^2}$$

$t^a t^a = C_2(r)\cdot \mathbb{1}$

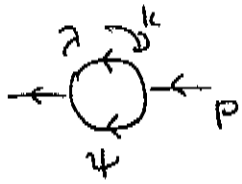
$$= -g^2 C_2(r) \int_0^1 dx \int \frac{d^4\mathbb{k}}{(2\pi)^4} \frac{(\mathbb{k} + (2-x)p)^2}{[\mathbb{k}^2 - x(1-x)p]^2}$$

$$= \int \frac{d^4k}{(2\pi)^4} \left(+\frac{2ig^2}{2}\right) \frac{-i}{k^2} t^a t^a (\eta^m{}_m) \cdot \frac{(k+p)^2}{(k+p)^2}$$

$\left(\text{set } \eta^m{}_m = 4\right) \rightarrow$
$$= g^2 C_2(r) \int_0^1 dx \int \frac{d^4\mathbb{k}}{(2\pi)^4} \cdot 4 \cdot \frac{(\mathbb{k} + (1-x)p)^2}{[\mathbb{k}^2 - x(1-x)p^2]^2}$$

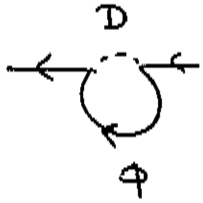
by
$$(-\sqrt{2}g\, \phi^* \lambda^T c\, \psi)(+\sqrt{2}g\, \psi^\dagger c\, \lambda^* \phi)$$

= no fermion ints.



$$= (-\sqrt{2}gi)(\sqrt{2}gi)\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\frac{i\sigma\cdot(k+p)}{(k+p)^2}\,c\left(\frac{-i\sigma\cdot k}{k^2}\right)^T c\right]t^a t^a$$

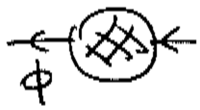
$$= -2\cdot 2g^2 C_2(r)\int dx\int\frac{d^4\mathbb{k}}{(2\pi)^4}\,\frac{(\mathbb{k}+(1-x)p)\cdot(\mathbb{k}-xp)}{[\mathbb{k}^2 - x(1-x)p^2]^2}$$



$$= (+ig)^2\,t^a t^a\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{(k+p)^2}\cdot i\cdot\frac{k^2}{k^2}$$

$$= +g^2 C_2(r)\int_0^1 dx\int\frac{d^4\mathbb{k}}{(2\pi)^4}\,\frac{(\mathbb{k}-xp)^2}{[\mathbb{k}^2-x(1-x)p^2]^2}$$

in all



$$= g^2 C_2(r)\int_0^1 dx\int\frac{d^4\mathbb{k}}{(2\pi)^4}\,\frac{1}{[\mathbb{k}^2-x(1-x)p^2]^2}$$

$$\cdot\Big\{-(\mathbb{k}^2+(2-x)^2p^2) + 4(\mathbb{k}^2+(1-x)^2p^2) - 4(\mathbb{k}^2 - x(1-x)p^2)$$

$$+ (\mathbb{k}^2 + x^2p^2)\Big\}$$

$\mathbb{k}^2$'s cancel, also

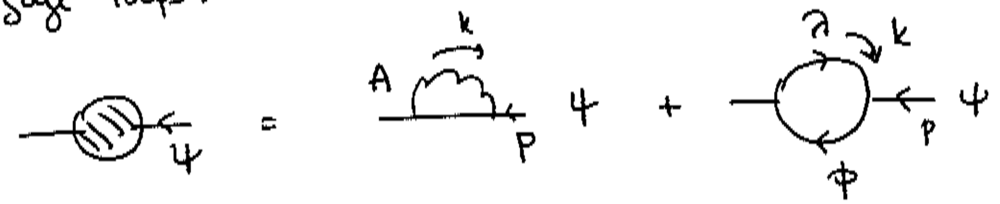
$$\int_0^1 dx\,\left\{-(2-x)^2 + 4(1-x)^2 + 4x(1-x) + x^2\right\}$$

$$= \int_0^1 dx\,\left\{-4+4x-x^2+4-4x+x^2\right\} = 0$$

so  has no log divergence either in the mass or in the field strength renormalization. (in Feynman gauge)

Note that the mass correction completely cancels, but only if we set $\eta^m{}_m = 4$, In dimensional regularization, one sets $\eta^m{}_m = 4-\epsilon = d$. So dimensional regularization does not preserve SUSY. Actually this is obvious; we need to keep the same number of Bose and Fermi degrees of freedom at all stages. Siegel introduced an alternative regulator called "dimensional reduction" — analytically continue in the dimensionality of moments, but sum field indices over their complete set of values: A_m w. $m = 0,1,2,3$. (This prescription has inconsistencies at 3 loops but up to there it is quite effective.)

Next compute the field strength renormalization of ψ due to gauge loops.



$$= (ig)^2 \int \frac{d^4k}{(2\pi)^4}\, t^a \bar\sigma^m \frac{i\sigma\cdot(k+p)}{(k+p)^2}\, t^a \bar\sigma_m \frac{-i}{k^2}$$

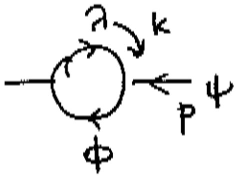
$\bar\sigma^m \sigma^p \bar\sigma_m = -2\sigma^p$

$$= -g^2 C_2(r) \int dx \int \frac{d^4k}{(2\pi)^4} \frac{\bar\sigma^m \sigma\cdot(k+(1-x)p)\,\bar\sigma_m}{[k^2 - x(1-x)p^2]^2}$$

$$= +2g^2 C_2(r) \int dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - x(1-x)p^2]^2} \left((1-x)\bar\sigma\cdot p\right)$$

Using $(+\sqrt{2}g\ \psi_c^\dagger \lambda^* \phi)(-\sqrt{2}g\ \phi^* \lambda_c^T \psi)$

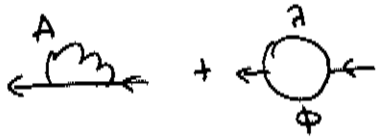
$\rightarrow$ (-1) for 1 fermion interchange.



$$=(-1)(\sqrt{2}g i)(-\sqrt{2}g i)\int\frac{d^4k}{(2\pi)^4}\ ct^a\left(\frac{i\sigma\cdot k}{k^2}\right)^T ct^a\ \frac{i}{(k+p)^2}$$

$$= +2g^2 C_2(r)\int dx \int\frac{d^4\ell}{(2\pi)^4}(-1)\frac{\bar\sigma\cdot(\ell - xp)}{[\ell^2 - x(1-x)p^2]^2}$$

in all



$$= 2g^2 C_2(r)\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-x(1-x)p^2]^2}$$

$$\cdot\,[(1-x)+x]\ \bar\sigma\cdot p$$

$$= (i\bar\sigma\cdot p)\cdot\left(\frac{2g^2C_2(r)}{(4\pi)^2}\log\Lambda^2\right)$$

this is different from the result for ϕ. Is it right?

Try the following exercise: consider a superpotential term

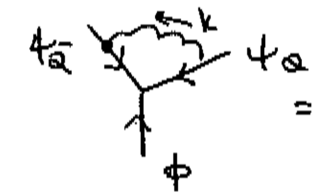
$$W = \lambda\,\phi\,\bar Q\,Q$$

where Q, $\bar Q$ are Weyl fermions and their partners in representation $r, \bar r$ of G, and ϕ is a singlet of G. If $\langle\phi\rangle = v/\sqrt{2}$, the fermion $(\psi_Q, \psi_{\bar Q})$ will obtain a Dirac mass $\frac{\lambda v}{\sqrt{2}}$. This is a simplified form of the Higgs couplings in a supersymmetric version of the Standard Model.

This superpotential generates three Yukawa couplings:

$$\mathcal{L} = -\lambda\left(\phi\, \psi_{\bar{Q}}^T c\, \psi_Q + \psi_\phi^T c\, \psi_{\bar{Q}}\, Q + \psi_\phi^T c\, \bar{Q}\, \psi_Q\right) + \text{h.c.}$$

The renormalization of the vertex $(\phi\, \psi_{\bar{Q}}^T c\, \psi_Q)$ is:



$$= (ig)^2(-i\lambda)\int \frac{d^4k}{(2\pi)^4}\, (-t^a)(\bar{\sigma}^m)^T \left(\frac{-i\sigma\cdot k}{k^2}\right)^T c\, \frac{i\sigma\cdot k}{k^2}\, t^a \bar{\sigma}_m$$

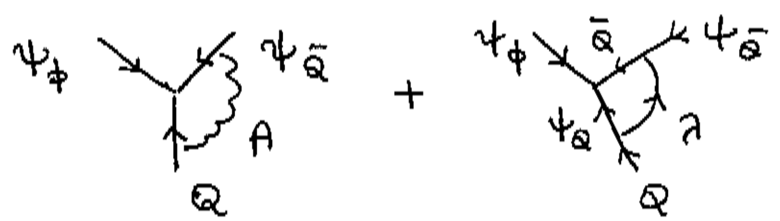
$$= (-i/k^2)$$

$$= (-i\lambda)\, c\, (-ig^2)\, C_2(r) \int \frac{d^4k}{(2\pi)^4}\, \frac{\sigma^m k^2 \bar{\sigma}_m}{(k^2)^2}$$

$$= (-i\lambda c)\cdot -i\, 4g^2 C_2(r) \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2}$$

$$= (-i\lambda c)\, \frac{4g^2 C_2(r)}{(4\pi)^2}\, \log \Lambda^2$$

The renormalization of the vertex $(\psi_\phi^T c\, \psi_{\bar{Q}}\, Q)$ is:



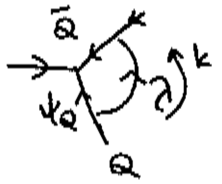
$$= \int \frac{d^4k}{(2\pi)^4} (-i\lambda c) \frac{i\sigma\cdot k}{k^2}\; ig\,\bar\sigma_m (-t^a) \frac{i}{k^2} (-igk^m)\, t^a \;\frac{-i}{k^2}$$

$$= (-i\lambda c)\; (+ig^2 C_2(r)) \int \frac{d^4k}{(2\pi)^4} \frac{-1}{(k^2)^2}$$

$$= (-i\lambda c) \;\frac{g^2 C_2(r)}{(4\pi)^2} \log \Lambda^2$$

$$(-i\lambda\; \psi_\phi^T c\, \bar Q\, \psi_Q)\; [-\sqrt{2} g\; \bar Q^*\, \lambda^T c\, \psi_{\bar Q} (-t)]\; [\sqrt{2} g\; \psi_Q^\dagger c \lambda^* \cdot t\, Q]$$

$\to$ (−1) for fermi interchange



$$= (-i\lambda c) \int \frac{d^4k}{(2\pi)^4} (-1)(-\sqrt{2} g i)(\sqrt{2} g i)\; \frac{i}{k^2}$$

$$\frac{i\sigma\cdot(-k)}{k^2}\; c \left(\frac{i\sigma\cdot k}{k^2}\right)^T c \quad (-t)^a\, t^a$$

$$= (-i\lambda c) \quad +i2g^2 C_2(r) \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left(-\frac{1}{k^2}\right)$$

$$= (-i\lambda c) \quad -2ig^2 C_2(r) \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2}\right)^2$$

$$= (-i\lambda c) \;\frac{2g^2 C_2(r)}{(4\pi)^2} \log \Lambda^2$$

The relation of these vertices is not very obvious. But, compute the β-functions

$\beta = -\Lambda \frac{\partial}{\partial \Lambda}$  $+ \lambda \sum_{\text{legs}} \Lambda \frac{\partial}{\partial\Lambda}$ 

for $\phi \psi_{\bar{Q}}^T c\, \psi_Q$:

$$\beta_\lambda = -\frac{8\lambda g^2 C_2(r)}{(4\pi)^2} + 2\cdot\frac{1}{2}\,\frac{4g^2 C_2(r)}{(4\pi)^2}$$

$$= -4\,\frac{\lambda g^2}{(4\pi)^2}\,C_2(r)$$

for $\psi_\phi^T c\, \psi_{\bar{Q}}\, Q$:

$$\beta_\lambda = -6\,\frac{\lambda g^2 C_2(r)}{(4\pi)^2} + \frac{1}{2}\cdot\frac{4g^2 C_2(r)}{(4\pi)^2}$$

$$= -4\,\frac{\lambda g^2}{(4\pi)^2}\,C_2(r)$$

so the two couplings λ (actually, all three) do run together. However, the formalism we have so far does not make this especially obvious.