4. Physics of chiral supermultiplets

In the previous lecture, we studied the representation of the SUSY algebra on the chiral supermultiplet (ϕ, ψ, F). We showed that actions of the following general form are supersymmetric:

$$\mathcal{L} = \partial_m \phi_k^* \partial^m \phi_k + \psi_k^\dagger i \bar{\sigma}^m \partial_m \psi_k + F_k^* F_k$$

$$+ \left(F_k \frac{\partial W}{\partial \phi_k} - \frac{1}{2} \psi_k^T c \psi_\ell \frac{\partial^2 W}{\partial \phi_k \partial \phi_\ell} \right) + \text{h.c.}$$

where $W(\{\phi_k\})$ is an analytic function of the ϕ_k, called the superpotential. This is not actually the most general supersymmetric kinetic term (as we will discuss in a few lectures), but it is the most general nonderivative term. In this lecture, I would like to discuss some of the physics of this Lagrangian.

The F_k are auxiliary fields, and so there is not a loss of particle states if we integrate them out. The F terms in $\mathcal{L}$ are:

$$F_k^* F_k + F_k \frac{\partial W}{\partial \phi_k} + F_k^* \left(\frac{\partial W}{\partial \phi_k}\right)^*$$

$$= \left| F_k + \left(\frac{\partial W}{\partial \phi_k}\right)^* \right|^2 - \left|\frac{\partial W}{\partial \phi_k}\right|^2$$

setting $F_k + \left(\frac{\partial W}{\partial \phi_k}\right)^* = 0$ according to the F_k field equation, or just functionally integrating over F_k, this becomes

$$= \quad - \sum_k \left|\frac{\partial W}{\partial \phi_k}\right|^2$$

That is, our field theory has a potential energy

$$V = + \sum_k \left|\frac{\partial W}{\partial \phi_k}\right|^2$$

Note ~~that~~ V is ≥ 0 and $= 0$ only if $\frac{\partial W}{\partial \phi_k} = 0$ for all k. There is a reason for this. In lecture 2, we proved that

$$\frac{1}{4}\left(\{Q_1, Q_1^\dagger\} + \{Q_2, Q_2^\dagger\}\right) = H$$

we also proved ~~that~~ the matrix element of the LHS in any state is ≥ 0. In fact,

$$\langle\psi|\{Q_1,Q_1^\dagger\} + \{Q_2,Q_2^\dagger\}|\psi\rangle = 0$$

only if $\quad Q_1|\psi\rangle = Q_1^\dagger|\psi\rangle = Q_2|\psi\rangle = Q_2^\dagger|\psi\rangle = 0$

so

$$\langle\psi|H|\psi\rangle \geq 0 \quad \text{and} \quad \langle\psi|H|\psi\rangle = 0 \text{ only if}$$

$|\psi\rangle$ is a supersymmetric state.

It is also possible to argue that $\langle\psi|\frac{\partial W}{\partial\phi_k}|\psi\rangle \neq 0$ signals SUSY breaking. Look at the equation

$$\delta_\xi\psi_k = \sqrt{2}i\,\sigma^m c\,\xi^*\,\partial_m\phi_k + \sqrt{2}\,F_k\xi$$

this implies

$$[\xi^T c Q, \psi_k] = \sqrt{2}\,F_k\,\xi$$

Take the matrix element in $|\psi\rangle$: $\quad Q|\psi\rangle = \langle\psi|Q = 0$ if $|\psi\rangle$ is supersymmetric. Then, if $|\psi\rangle$ is supersymmetric,

$$\langle\psi|F_k|\psi\rangle = 0$$

If our theory can have a nontrivial potential, it can have spontaneous symmetry breaking. For example, consider the superpotential

$$W = \lambda\,X(\Phi_1\Phi_2 - f^2) \qquad f = \text{const.}$$

the potential for this theory is

$$V = |\partial X \Phi_1|^2 + |\partial X \Phi_2|^2 + |\partial(\Phi_1\Phi_2 - f^2)|^2$$

The minimum of the potential is at

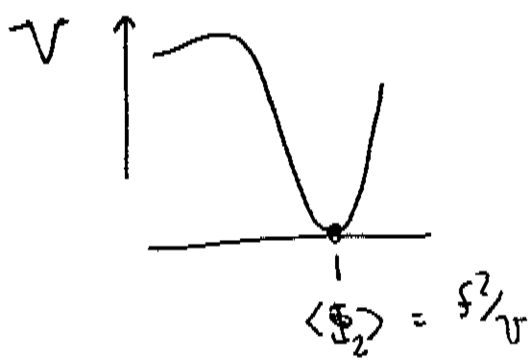
$$\langle X \rangle = 0 \qquad \langle \Phi_1 \rangle = v \qquad \langle \Phi_2 \rangle = f^2/v$$

where v is a complex number. The original theory had a global symmetry

$$\Phi_1 \to e^{i\alpha}\Phi_1 \qquad \Phi_2 \to e^{-i\alpha}\Phi_2$$

which is spontaneously broken if $v \neq 0$.

This broken symmetry state is remarkable for two reasons. First, its energy is exactly zero. For fixed $\langle \Phi_1 \rangle = v$, the potential for $\langle \Phi_2 \rangle$ has the form



The exact zero is required by supersymmetry, as I have already discussed.

Second, the vacuum state is part of a complex manifold of vacuum states — a "moduli space". We have a vacuum for every v in the complex plane. For a fixed choice

v_0,
$$\langle \Phi_1 \rangle = v_0 \qquad \langle \Phi_2 \rangle = f^2/v_0$$

The global symmetry tells us that we can find additional vacuum states by moving along the symmetry direction

$$v_0 \to e^{i\alpha} v_0 \qquad f^2/v_0 \to e^{-i\alpha}\,(f^2/v_0)$$

But the condition for the vacuum state is

$$0 = F_X = \lambda\,(\Phi_1 \Phi_2 - f^2)$$

which is analytic in Φ_1, Φ_2. This means that the condition is still satisfied if we let α be a complex number

$$v_0 \to e^{i(\alpha+i\beta)}\, v_0 = e^{i\alpha} e^{\beta} v_0$$
$$f^2/v_0 \to e^{-i(\alpha+i\beta)}(f^2/v_0) = e^{-i\alpha} e^{-\beta} (f^2/v_0)$$

The field fluctuation along α is the Goldstone boson. The field fluctuation along β is another massless boson. Both fields must be massless if SUSY is unbroken, and there must also be a massless Weyl fermion, so that

$$((\alpha + i\beta)\ ,\ g\alpha)$$

can fit together into a Goldstone chiral multiplet.

Let's check the statement about fermion masses. The fermion mass term is

$$+\tfrac{1}{2}\Psi_k^T c\,\Psi_\ell \;\frac{\partial^2 W}{\partial\phi_k\partial\phi_\ell}$$

$$= \lambda\,\Psi_X^T c\,\Psi_1\,\Phi_2 + \lambda\,\Psi_X^T c\,\Psi_2\,\Phi_2 + \lambda\,\Psi_1^T c\,\Psi_2\,X$$

where Ψ_1, Ψ_2 are the fermionic partners of $\Phi_1\,\Phi_2$. In the vacuum state

$$= \left(\lambda\frac{f}{v^2}\Psi_1^T + \lambda v\,\Psi_2^T\right) c\,\Psi_X$$

so Ψ_X forms a Dirac fermion with $\Psi = \dfrac{(f/v^2\,\Psi_1 + v\,\Psi_2)}{\sqrt{|f^2/v^4|^2 + |v|^2}}$

This state has mass

$$m = \lambda\sqrt{|f^2/v|^2 + |v|^2}$$

The combination of Ψ_1 and Ψ_2 orthogonal to Ψ is massless and can be identified with $g\alpha$.

By this point, you see that it is not so easy to find a potential that breaks SUSY spontaneously. On one hand, if there exists a state with $\mathcal{H} = 0$, this is necessarily the minimum of the energy, and it is also a supersymmetric state. On the other hand, the conditions for a vacuum state are

$$F_k^* = \frac{\partial W}{\partial \phi_k} = 0$$

This is a set of N complex equations in N unknowns. Generically, there is always a solution. But SUSY is spontaneously broken only in those cases where there is not a solution.

Just to show that there does exist a superpotential that gives spontaneous SUSY breaking, consider a theory with one supermultiplet and

$$W(\phi) = A\phi + B \qquad A, B = \text{consts.}$$

Then $$F^* = A \neq 0$$

However, the Lagrangian is $\mathcal{L} = \partial_m \phi^* \partial^m \phi + \psi^\dagger i \bar{\sigma}^m \partial_m \psi - |A|$ so that the physics of SUSY breaking in this case is rather trivial.

We will see later in the course that this superpotential is not so trivial in the context of supergravity.

Here is a less trivial example:

(the O'Raifeartaigh model)

$$W = \lambda\phi_0 + m\phi_1\phi_2 + g\phi_0\phi_1^2$$

The $F=0$ conditions are

$$F_0^* = \lambda + g\phi_1^2 = 0$$
$$F_1^* = m\phi_2 + 2g\phi_0\phi_1 = 0$$
$$F_2^* = m\phi_1 = 0$$

The F_0 and F_2 equations depend only on ϕ_1 and contradict each other. If we minimize $|F_1|^2$ by setting $F_1 = 0$

$\left(\phi_2 = -\frac{1}{m}\cdot 2g\phi_0\phi_1\right)$,

$$V = |m\phi_1|^2 + |\lambda + g\phi_1^2|^2$$

take λ, g, m real

$$= |\lambda|^2 + \mathrm{Re}\,(2\lambda g\phi_1^2) + m^2|\phi_1|^2 + \cdots$$

so if $m^2 > 2\lambda g$, V is minimized at $\phi_1 = 0$

If we write $\quad \phi_1 = \frac{1}{\sqrt{2}}(\sigma + i\pi)$

$$V = \lambda^2 + \frac{1}{2}(m^2 + 2\lambda g)\sigma^2 + \frac{1}{2}(m^2 - 2\lambda g)\pi^2 + \cdots$$

The remaining term of V is

$$|m\phi_2 + 2g\,\phi_0\phi_1|^2$$

For $\phi_1 = 0$, this is minimized at $\phi_2 = 0$, $\phi_0 = \underline{\text{anything}}$.
Let's choose the simplest case $\phi_0 = 0$ and compute the mass matrix. (The boson and the fermion masses depend on $\langle\phi_0\rangle$.)

$$V = \lambda^2 + \frac{1}{2}(m^2 + 2\lambda g)\sigma^2 + \frac{1}{2}(m^2 - 2\lambda g)\pi^2 + m^2|\phi_2|^2 + \cdots$$

so the boson masses are

$$0,\ 0,\ \sqrt{m^2 - 2\lambda g},\ \sqrt{m^2 + 2\lambda g},\ m,\ m$$

The fermion mass term is

$$\frac{1}{2}\psi_k^T C\psi_\ell \frac{\partial W}{\partial\phi_k\partial\phi_\ell} = m\psi_1^T C\psi_2 + g\,\psi_1^T C\psi_1\,\phi_0 + 2g\,\psi_1^T C\psi_0\,\phi_1$$

At $\phi_0 = \phi_1 = 0$, there is a Dirac fermion mass for ψ_1 with ψ_2, and ψ_0 is massless.

There are three indications that SUSY is spontaneously broken.

(1) The vacuum energy is $V = \lambda^2 > 0$

(2) The boson and fermion mass spectra are different

(3) The boson masses do not form pairs (complex fields)

If SUSY is spontaneously broken, since it is a continuous generate symmetry, Goldstone's theorem implies that there is a massless particle created by the symmetry current. This particle is a massless fermion, called the Goldstino. The proof of Goldstone's theorem tells us that this fermion must also be created by the ψ_k field such that

$$< [\xi^T C Q, \psi_k] > = \sqrt{2} < F_k > \xi \neq 0$$

In this model, $F_0 \neq 0$ so ψ_0 should be the Goldstino, and, indeed ψ_0 is massless. Even for other vacuum states for which $\phi_0 \neq 0$, as long as $\phi_1 = 0$ (so that $F_2 = 0$), ψ_0 is massless.

Now let me say a few words about the quantum theory associated with chiral supermultiplets. For simplicity, let's look first at the simplest model with 1 multiplet (ϕ, ψ, F) and

$$W = \frac{\lambda}{3!}\phi^3$$

so

$$\mathcal{L} = \partial_m\phi^*\partial^m\phi + \psi^\dagger i\bar{\sigma}^m\partial_m\psi + F^*F$$
$$+ \frac{\lambda}{2}F\phi^2 + \frac{\lambda}{2}F^*\phi^{*2} - \frac{\lambda}{2}\psi^T c\psi\phi + \frac{\lambda}{2}\psi^\dagger c\psi^*\phi^*$$

If I integrated out F, we would find a ϕ^4 interaction

$$-\left|\tfrac{1}{2}\lambda\phi^2\right|^2$$

But the following analysis will be slightly easier if we do not integrate out F. Instead, treat F as a quantum field. The propagators are:

$$\phi\,\phi^* = \frac{i}{k^2} \qquad \psi\,\psi^\dagger = \frac{i\sigma\cdot k}{k^2} \qquad F\,F^* = +i$$

In $\mathcal{L}$, ϕ and ψ are massless. But, maybe we could generate a mass in perturbation theory. Actually, it is easy to see that it is not possible to generate a mass for ψ. $\mathcal{L}$ has a symmetry

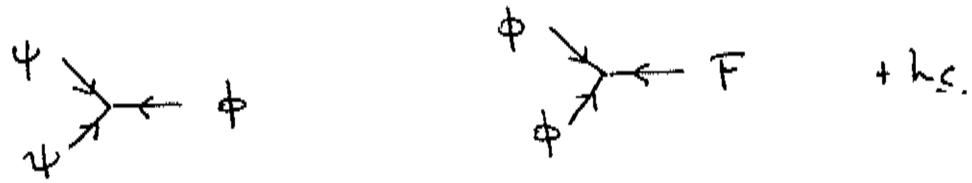
$$\psi \to e^{i\alpha/2}\psi \qquad \phi \to e^{-i\alpha}\phi \qquad F \to e^{2i\alpha}F$$

which forbids a mass term

$$\delta\mathcal{L}_m = m\,\psi^T C \psi$$

In perturbation theory, there is not Feynman diagram of the form

that we can build out of the vertices



The boson mass term is a different story, however. There is no obvious symmetry that forbids the mass term

$$\delta\mathcal{L}_\mu = \mu^2 \phi^* \phi$$

and there are Feynman graphs of the form 

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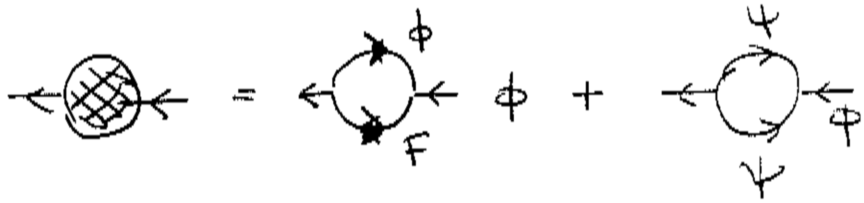
Let's compute them:



(If we integrated out F, the first graph would look like

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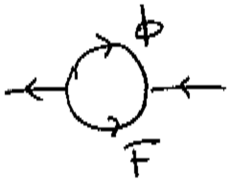
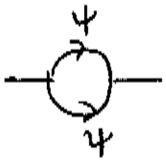


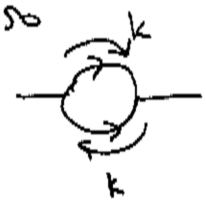
$$= (i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{i}{k^2}\right)\cdot(+i) = +\lambda^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}$$

for 

we must be careful about the overall sign

$$\left(-i\tfrac{\lambda}{2}\psi^T c\, \psi\, \phi\right)\left(+i\tfrac{\lambda}{2}\psi^\dagger c\, \psi^* \phi^*\right)\times 2$$

so



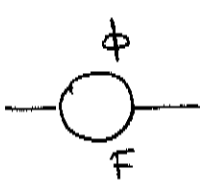
$$= \frac{(-i\lambda)(i\lambda)}{2}\int\frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[c\,\frac{i\sigma\cdot k}{k^2}\, c\left(\frac{i\sigma\cdot(-k)}{k^2}\right)^T\right]$$

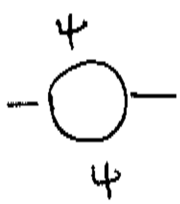
$$= \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[\frac{\sigma\cdot k}{k^2}\, c\left(\frac{\sigma\cdot k}{k^2}\right)^T c\right]$$

$$= \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[\frac{\sigma\cdot k}{k^2}\,\frac{\bar\sigma\cdot k}{k^2}\,(-1)\right]$$

$$= -\lambda^2 \int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2} \qquad \sigma\cdot k\;\bar\sigma\cdot k = k^2$$

so

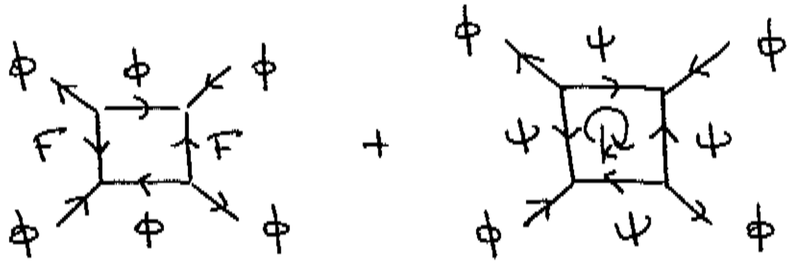


$+$


$= 0$!

This cancellation had better occur; if ψ cannot get a mass, neither can ϕ, by supersymmetry.

The above cancellation, however, turns out to be only the tip of the iceberg. Consider the 1-loop correction to the ϕ^4 interaction



the first graph is

$$(i\lambda)^4 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2}\, i\, \frac{i}{k^2}\, i = \lambda^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2}$$

the second is

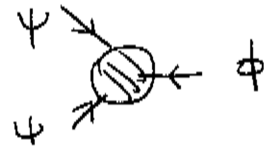
$$(-i\lambda\, \psi^T c\, \psi\, \phi)(i\lambda\, \psi^\dagger c\, \psi^* \phi^*)(-i\lambda\, \psi^T c\, \psi\, \phi)(i\lambda\, \psi^\dagger c\, \psi^* \phi^*)$$

$$= \lambda^4 \cdot \underbrace{(-1)}_{\substack{\text{fermion interchange} \\ \text{for } \psi^* \psi^T}} \cdot \underbrace{\tfrac{1}{2}}_{\substack{\text{symmetry} \\ \text{factor}}} \cdot \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left[\left(\frac{-i\sigma\cdot k}{k^2}\right) c \left(\frac{i\sigma\cdot k}{k^2}\right)^T c\; \frac{-i\sigma\cdot k}{k^2}\, c \left(\frac{i\sigma k}{k^2}\right)^T c\right]$$

$$= -\frac{1}{2}\lambda^4 \int \frac{d^4k}{(2\pi)^4} \operatorname{tr} \frac{\sigma\cdot k}{k^2}\, \frac{\bar\sigma\cdot k}{k^2}(-1)\, \frac{\sigma\cdot k}{k^2}\, \frac{\bar\sigma k}{k^2}(-1)$$

$$= -\lambda^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2}$$ <u>cancels the above!</u>

As for the 1-loop corrections to the Yukawa vertex



or to the F vertex

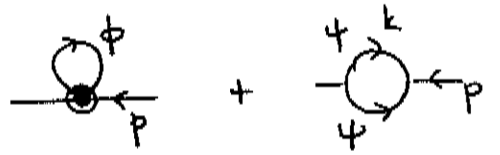
you can check ~~that~~ there are no such diagrams!

The thy is not completely finite. There are self-energy (field-strength renormalization) graphs for ϕ, ψ, and F. Introduce Feynman parameters for 1-loop integrals:

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2\,(k+p)^2} = \int_0^1 dx \int \frac{d^4\mathbb{k}}{(2\pi)^4}\,\frac{1}{[\mathbb{k}^2 - x(1-x)p^2]^2}$$

with $\quad \mathbb{k} = k + xp \quad$ or $\quad k = \mathbb{k} - xp \quad k+p = \mathbb{k} + (1-x)p$

Then, for ϕ:



$$= \lambda^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2} - \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\,\frac{\sigma\cdot k\ \ \bar{\sigma}\cdot k+p}{k^2\,(k+p)^2}$$

$$= \lambda^2 \int_0^1 dx \int \frac{d^4\mathbb{k}}{(2\pi)^4}\,\frac{(\mathbb{k}+(1-x)p)^2 - (\mathbb{k}-xp)\cdot(\mathbb{k}+(1-x)p)}{[\mathbb{k}^2 - x(1-x)p^2]^2}$$

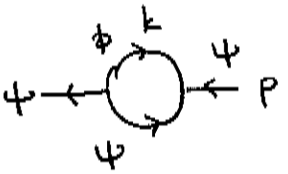
$$= \lambda^2 \int_0^1 dx \int \frac{d^4\mathbb{k}}{(2\pi)^4}\,\frac{[(1-x)^2 + x(1-x)]\,p^2}{[\mathbb{k}^2 - x(1-x)p^2]^2}$$

so

$$= \lambda^2 \int_0^1 dx \; (1-x)\, p^2 \, \frac{i}{(4\pi)^2} \log \Lambda^2$$

$$= i p^2 \cdot \left(\frac{\lambda^2}{2\cdot(4\pi)^2} \log \Lambda^2 \right)$$

for ψ:



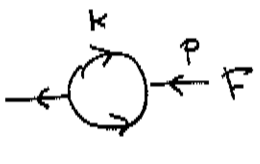
$$(+i\lambda\, \psi^\dagger c\, \psi^* \phi^*)(-i\lambda\, \psi^T c\, \psi\, \phi)$$

$$= \underbrace{\lambda^2 \cdot (-1)}_{\text{fermion order}} \int \frac{d^4k}{(2\pi)^4}\; c \left[\frac{-i\sigma\cdot(k+p)}{(k+p)^2} \right]^T c \;\; \frac{i}{k^2}$$

$$= -\lambda^2 \cdot \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{-\bar{\sigma}\cdot(k + (1-x)p)}{[k^2 - x(1-x)p^2]^2}$$

$$= i\,\bar{\sigma}\cdot p \left(\frac{\lambda^2}{2(4\pi)^2} \log \Lambda^2 \right)$$

for F:



$$= (i\lambda)^2 \cdot \underbrace{\frac{1}{2}}_{\text{symmetry factor}} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{(k+p)^2}$$

$$= i \left(\frac{\lambda^2}{2} \frac{1}{(4\pi)^2} \log \Lambda^2 \right)$$

All three fields receive the same field strength renormalization. We find for all three fields

$$\gamma = \frac{1}{2}\Lambda \frac{\partial}{\partial\Lambda} Z(\Lambda) = \frac{\lambda^2}{32\pi^2}$$

As for the β-functions, the contribution from the 3- or 4-point vertices vanishes, leaving only the contribution from field strength renormalization

$$\beta_\lambda = -\lambda \sum_{\text{legs}} \frac{1}{2}\Lambda \frac{\partial}{\partial\Lambda} Z_i(\Lambda)$$

$$\beta_\lambda = -\frac{3\lambda^3}{32\pi^2}$$

both for the coefficient of $\psi_c^T \psi \phi$ and for the coefficient of $\phi^2 F$.

We see, then, that renormalization preserves the structure of $\mathcal{L}$ given by SUSY. We also see ~~that~~ this is enforced in a very strong way: there are no direct perturbative corrections to the superpotential, only the corrections that come from field rescaling.

This turns out to be a general result : the superpotential is not directly renormalized in any supersymmetric theory .