3. The chiral supermultiplet

At the end of the previous lecture, I defined the action of an $N=1$ SUSY on fields

$$\delta_\xi \phi = [\xi^T c Q + \xi^\dagger c Q^*, \phi(x)]$$

where Q, Q^* are the SUSY charges and ξ_α is a constant anticommuting spinor parameter. I showed ~~that~~

$$(\delta_\xi \phi)^* = \delta_\xi \phi^*$$

and that the δ_ξ obey the algebraic relation

$$[\delta_\xi, \delta_\eta] \phi = 2i (\xi^\dagger \bar\sigma^a \eta - \eta^\dagger \bar\sigma^a \xi) \partial_a \phi$$

Beginning in this lecture, I would like to find representations of this commutation relation on multiplets of fields.

In the previous lecture, we classified representations of $N=1$ SUSY on massless particles. The simplest representation was the chiral supermultiplet: a left-handed Weyl fermion + a scalar, and their antiparticles. These 4 particles are associated with fields

$$\psi_\alpha \quad \phi \qquad \phi^* \quad \psi^*_\alpha$$

Let's try to find a representation of δ_ξ on these fields. That is, we need a set of formulae

$$\delta_\xi \phi = \ \ldots \qquad \delta_\xi \psi = \ \ldots$$

with the RHS linear in $\xi, \xi^\dagger$ with the correct Lorentz symmetry. The first line is obviously

$$\delta_\xi \phi = \sqrt{2}\ \underbrace{\xi^T c\, \psi}_{\text{scalar combination}}$$

In 4-dimensions, the units of ϕ are $[\text{mass}]^1$

ψ $[\text{mass}]^{3/2}$

$$\text{so} \qquad \xi \sim [\text{mass}]^{-\frac{1}{2}} \sim [\text{cm}]^{\frac{1}{2}}$$

so $\xi \sim [\text{distance}]^{\frac{1}{2}}$, as we might have guessed. Then

$$\delta_\xi \psi = \text{combination of } \xi, \phi, \partial_a$$

we need to add $\partial_a \sim [\text{mass}]^1$ to make the dimensions come out right

$$\delta_\xi = \sqrt{2}\, i\ \sigma^n c\, \xi^* \, \partial_n \phi$$

Check, then

$$[\delta_\xi, \delta_\eta]\,\phi = \delta_\xi\,(\sqrt{2}\,\eta^T c\,\psi) - (\eta \leftrightarrow \xi)$$

$$= \sqrt{2}\,\eta^T c\,(\sqrt{2}\,i\,\sigma^n c\,\xi^* \partial_n \phi) - (\eta \leftrightarrow \xi)$$

$$= 2i\;\xi^\dagger (-1)\; c^T (\sigma^n)^T c^T\, \eta\;\partial_n \phi - (\eta \leftrightarrow \xi)$$

$$= -2i\;\xi^\dagger c\,(\sigma^n)^T c\;\eta\;\partial_n\phi - (\eta \leftrightarrow \xi)$$

$$= +2i\;\xi^\dagger \bar\sigma^n \eta\;\partial_n\phi - (\eta \leftrightarrow \xi)$$

excellent! Checking this relation on ψ is a little more complicated:

$$[\delta_\xi, \delta_\eta]\,\psi_\alpha = \delta_\xi\,(\sqrt{2}\,i\,(\sigma^n c\,\eta^*)_\alpha\,\partial_n\phi) - (\eta \leftrightarrow \xi)$$

$$= \sqrt{2}\,i\,(\sigma^n c\,\eta^*)_\alpha\;\partial_n(\sqrt{2}\,\xi^T c\,\psi) - (\eta \leftrightarrow \xi)$$

On the RHS of this equation, the spinor indices are not flowing in the right way. We need to redirect the flow of indices using a Fierz identity. Recall

$$\xi_\alpha\,\eta^\dagger_\beta = -\tfrac{1}{2}(\sigma^a)_{\alpha\beta}\,(\eta^\dagger\bar\sigma_a\,\xi)$$

So the above can be rewritten:

$$= 2i\;(\sigma^n c)_{\alpha\gamma}\,(-\xi_\beta\,\eta^\dagger_\gamma)\,\partial_n(c\,\psi)_\beta - (\eta \leftrightarrow \xi)$$

$$= 2i\;[\sigma^n c\;\tfrac{1}{2}(\sigma^a)^T c\;\partial_n\psi]_\alpha\,(\eta^\dagger\bar\sigma_a\,\xi) - (\eta \leftrightarrow \xi)$$

$$= -i\,(\sigma^n \bar\sigma^a \partial_n \psi)_\alpha \; \eta^\dagger \bar\sigma_a \xi \; - (\eta \leftrightarrow \xi)$$

now $\quad \sigma^n \bar\sigma^a + \sigma^a \bar\sigma^n = 2\eta^{an}$

$$= -2i\,(\eta^\dagger \bar\sigma_a \xi)\; \partial^a \psi \; + \; i\,\sigma^a \bar\sigma^n \partial_n \psi\, (\eta^\dagger \bar\sigma_a \xi)$$

$$-(\eta \leftrightarrow \xi)$$

This is not exactly what we were looking for. But it is very tempting here to use the Dirac equation

$$i\,\bar\sigma^n \partial_n \psi = 0$$

If ψ obeys the Dirac equation, then indeed

$$[\delta_\xi, \delta_\eta]\,\psi = 2i\,(\xi^\dagger \bar\sigma^a \eta - \eta^\dagger \bar\sigma^a \xi)\,\partial_a \psi$$

So, on the whole multiplet, we have shown that

$$[\delta_\xi, \delta_\eta]\,(\phi, \psi) = 2i\,(\xi^\dagger \bar\sigma^a \eta - \eta^\dagger \bar\sigma^a \xi)\,\partial_a\,(\phi, \psi)$$

$$+ \left(\begin{array}{c}\text{terms that are} = 0 \text{ when} \\ \text{the eq. of motion are satisfied}\end{array}\right)$$

In this circumstance, we say that δ_ξ satisfies the SUSY commutation relations <u>on-shell</u>.

Can this δ_ξ be a symmetry of a Lagrangian? The simplest Lagrangian for this set of fields is

$$\mathcal{L} = \partial_m\phi^* \partial^m\phi + \psi^\dagger i\bar{\sigma}^m \partial_m \psi$$

This Lagrangian implies the Klein-Gordon eq. for ϕ

$$\partial_m \partial^m \phi = 0$$

and the Weyl eq. for ψ:

$$i\bar{\sigma}^m \partial_m \psi = 0$$

Compute:

$$\begin{aligned}
\delta_\xi \mathcal{L} = \; & \partial_m(\sqrt{2}\,\xi^T c\,\psi)^* \partial^m\phi + \partial_m\phi^* \partial^m(\sqrt{2}\,\xi^T c\,\psi) \\
& + \psi^\dagger i\bar{\sigma}^m \partial_m (\sqrt{2}\, i\sigma^n c\,\xi^* \partial_n\phi) \\
& + (-\sqrt{2}\, i\, \partial_n\phi^* \,\xi^T c^\dagger (\sigma^n)^\dagger)\, i\bar{\sigma}^m \partial_m \psi \\
= \; & -\sqrt{2}(\xi^\dagger c\, \partial_m\psi^*)\partial^m\phi - \sqrt{2}\,\psi^\dagger \bar{\sigma}^m \sigma^n c\, \xi^* \partial_m\partial_n\phi \\
& + \sqrt{2}\,\partial_m\phi^* (\xi^T c\,\partial^m\psi) + \sqrt{2}\,\partial_n\phi^*\, \xi^T(-c)\,\sigma^n\bar{\sigma}^m \partial_m\psi
\end{aligned}$$

Integrate by parts and use

$$\partial_m\partial_n\phi = \underline{\text{symmetric}} \text{ in } (m \leftrightarrow n)$$

and

$$\{\bar{\sigma}^m, \sigma^n\} = \eta^{mn}\cdot 1$$

$$= +\sqrt{2}\,\xi^{\dagger} c\, \psi^{*}\, \partial^m \partial_m \phi - \sqrt{2}\,\psi^{\dagger} c\, \xi^{*}\, \partial_m \partial^m \phi$$

$$-\sqrt{2}\,\partial_m \partial^m \phi^{*}\, \xi^{T} c\, \psi + \sqrt{2}\,\partial_m \partial^m \phi^{*}\, \xi^{T} c\, \psi$$

$$= 0 \quad !$$

Notice that we did not use the equation of motion in this analysis. However, the two calculations ~~fit~~ together, because $\mathcal{L}$ implies the equation of motion for ψ that we need to justify the commutation relations for δ_ξ.

If you ~~find this~~ structure of a symmetry, in which the symmetry relies on the form of the Lagrangian, a little odd, you are in good company. We would really like to find a symmetry that can stand on its own. Then we can ask what Lagrangians are invariant under this symmetry. For this, we need a symmetry that is realized off-shell i.e. without the use of the equations of motion.

In some cases, we can add fields to the multiplet in such a way as to allow ~~the~~ commutation relations to be satisfied off-shell. If these fields lead to no new particles, we say that they are "auxiliary fields". A common problem of formal SUSY is to find the set of auxiliary fields that allows the SUSY commutation relation to "close" off-shell.

However, such problems typically have straightforward solutions only for N=1 SUSY. Sets of auxiliary fields even for N=2 SUSY are quite complicated. The N=4 Yang-Mills theory that we discussed last time requires an infinite number of auxiliary fields if we want to write a δ_ξ that satisfies the correct commutation relation off-shell. In general, the formal problems of supersymmetry become worse as N increases. There are even theories — "Type IIB" N=4 supergravity is an example — that do not seem to be described by Lagrangians. Most interesting theories, though, are described by representations of SUSY on fields that close on-shell, so I will emphasize this component field method in this course. For N=1 SUSY, there is a more automatic method for finding the off-shell representation that I will discuss next week.

Let's continue now with the formalism of the chiral supermultiplet. We wrote a free massless supersymmetric Lagrangian. Can we write a massive or interacting theory?

A massive thy would have the form $\mathcal{L}_{boson} + \mathcal{L}_{fermion}$

$$\mathcal{L}_{boson} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

$$\mathcal{L}_{fermion} = \psi^\dagger i \bar{\sigma}^m \partial_m \psi - \frac{m}{2} (\psi^T c \psi - \psi^\dagger c \psi^*)$$

The mass term of the Klein Gordon thy is hopefully familiar to you. The mass term in the Weyl thy needs a little explanation. The term I wrote is scalar and Hermitian, so it must be right. Note that

$$\psi^T c \chi$$

is symmetric in the fields ψ, χ — if they are anticommuting — so it is reasonable to write $\psi^T c \psi$. The equation of motion following from $\mathcal{L}_{fermion}$ is

$$i \bar{\sigma}^m \partial_m \psi + m c \psi^* = 0$$

$$-i \partial_m \psi^\dagger \bar{\sigma}^m - m \psi^T c = 0$$

Then

$$(-i \sigma^n \partial_n) \, i \bar{\sigma}^m \partial_m \psi = \partial^m \partial_m \psi$$

$$= -i \sigma^n \partial_n (-mc \psi^*)$$

$$= i m c \, (\bar{\sigma}^n)^T \partial_n \psi^*$$

$$= mc \, (i \partial_n \psi^\dagger \bar{\sigma}^n)^T$$

$$= mc\,[-m(\psi^T c)^T]$$

$$= m^2 c\, c\, \psi = -m^2 \psi$$

so it indeed follows that $[\partial_m \partial^m + m^2]\psi = 0$ i.e. that ψ has mass m. The physical content of the theory is one massive fermion with two states corresponding to $J^3 = \pm\frac{1}{2}$. These states are their own antiparticles. The term $\psi^T c \psi$ has fermion number $+2$, so fermion number is not conserved. This structure is called a Majorana fermion.

I'd like to digress and show you how this equation fits together with more familiar massive fermion equations. Consider, for example, the massive electron of quantum electrodynamics:

$$\mathcal{L} = \bar{\Psi} i \gamma^m \partial_m \Psi - m \bar{\Psi}\Psi$$

where $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\quad \gamma^m = \begin{pmatrix} & \sigma^m \\ \bar{\sigma}^m & \end{pmatrix}$ $\quad \bar{\Psi} = \Psi^\dagger \gamma^0 = \Psi^\dagger \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$$= \psi_L^\dagger i \bar{\sigma}^m \partial_m \psi_L + \psi_R^\dagger i \sigma^m \partial_m \psi_R$$

$$- m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)$$

As I explained in the first lecture, we can replace

$$\psi_R = -c\chi_L^* \qquad \psi_R^\dagger = \chi_L^T c$$

then $\mathcal{L}$ becomes

$$\mathcal{L} = \psi_L^\dagger i\bar{\sigma}^m \partial_m \psi_L + \chi_L^\dagger i\bar{\sigma}^m \partial_m \chi_L$$
$$-m\left(\chi_L^T c\,\psi_L - \psi_L^\dagger c\,\chi_L^*\right)$$

From this, we can guess the most general form of a free fermion Lagrangian in 4 dimensions:

$$k, l = 1 - N_f$$

$$\mathcal{L} = \psi_k^\dagger i\bar{\sigma}^m \partial_m \psi_k$$
$$-\frac{1}{2}\left(m_{kl}\,\psi_k^T c\,\psi_l - m_{kl}^*\,\psi_k^\dagger c\,\psi_l^*\right)$$

where m_{kl} is a complex symmetric $N_f \times N_f$ matrix.

If m_{kl} is 1×1 $\quad m_{kl} = m$

we have a Majorana fermion

If m_{kl} is 2×2 and $m_{kl} = \begin{pmatrix} & m \\ m & \end{pmatrix}$

we have the QED structure, a Dirac fermion.

If there is a charge Q s.t.

$$Q^T m + m Q = 0$$

Q is a conserved fermion number. In the QED or Dirac case

$$Q = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

The most general Klein-Gordon + Dirac Lagrangian is then

$$\mathcal{L} = \partial_m \phi_k^* \partial^m \phi_k - \mu^2_{k\ell} \phi_k^* \phi_\ell$$
$$+ \psi_k^\dagger i \bar{\sigma}^m \partial_m \psi_k - \tfrac{1}{2}(m_{k\ell} \psi_k^T c \psi_\ell + \text{h.c.})$$

with μ^2 Hermitian, m complex symmetric. We should be able to find a supersymmetric of this Lagrangian, but only under certain circumstances: the number of ϕ_k must be equal to the number of ψ_k, and the eigenvalues of μ^2 must be the squares of the eigenvalues of m. It is not very obvious in the presentation how such a symmetry would work.

To make it more obvious, I will introduce an auxiliary field that allows the SUSY algebra to close off-shell. Note that we have one complex Bose field

ϕ, describing 2 particles, and two complex Fermi fields, each describing 1 particle. Let's add one more complex Bose field, F, having it describe 0 particles. Assign F the units

$$\phi \sim [\text{mass}]^1 \qquad \psi \sim [\text{mass}]^{3/2} \qquad F \sim [\text{mass}]^2$$

Then we can extend the previous formulae for δ_ξ to:

$$\delta_\xi \phi = \sqrt{2}\,\xi^T c \psi$$

$$\delta_\xi \psi = \sqrt{2}i\,\sigma^n c\,\xi^* \partial_n \phi + \sqrt{2}\,F\xi$$

$$\delta_\xi F = -\sqrt{2}i\,\xi^\dagger \bar{\sigma}^m \partial_m \psi$$

On shell in the massless case we would have $F=0$, and note ~~that~~, for consistency $\delta_\xi F = 0$. Check

$$[\delta_\xi, \delta_\eta] F = \delta_\xi\left(-\sqrt{2}i\,\eta^\dagger \bar{\sigma}^m \partial_m \psi\right) - (\eta \leftrightarrow \xi)$$

$$= (\phi \text{ term}) + \left(-\sqrt{2}i\,\eta^\dagger \bar{\sigma}^m \partial_m\right) \sqrt{2} F \xi$$

$$- (\eta \leftrightarrow \xi)$$

$$= (\phi \text{ term}) - 2i\left(\eta^\dagger \bar{\sigma}^m \xi\right) \partial_m F - (\eta \leftrightarrow \xi)$$

so if we can get rid of the ϕ term we do have

$$[\delta_\xi, \delta_\eta] F = 2i\,(\xi^\dagger \bar\sigma^a \eta - \eta^\dagger \bar\sigma^a \xi)\, \partial_a F$$

as required. The ϕ term is

$$-\sqrt{2}i \;\; \eta^\dagger \bar\sigma^m \partial_m (\sqrt{2} i\, \sigma^n C\, \xi^* \,\partial_n \phi) \;\; (\xi \leftrightarrow \eta)$$

$$= \;\; 2\, \eta^\dagger \bar\sigma^m \sigma^n C\, \xi^* \;\partial_m \partial_n \phi \; - \; (\xi \leftrightarrow \eta)$$

$$= \;\; 2\, \underbrace{\eta^\dagger C \xi^*}\;\; \partial_m \partial^m \phi \; - \; (\xi \leftrightarrow \eta)$$

but this structure is <u>symmetric</u> under $\eta \leftrightarrow \xi$!

$$= 0$$

Similarly, the F term in $\delta_\xi \psi$ gives an extra term in

$$[\delta_\xi, \delta_\eta]\, \phi \;=\; \sqrt{2}\, \eta^T C\, (\sqrt{2} F \xi) \; - \; (\xi \leftrightarrow \eta)$$

$$= 0$$

The real test is the action of $[\delta_\xi, \delta_\eta]$ on ψ:

$$[\delta_\xi, \delta_\eta]\, \psi \;=\; (\text{terms from pp. 3-4})$$

$$+ \;\; \delta_\xi\, (\sqrt{2} F \eta) \; - \; (\eta \leftrightarrow \xi)$$

$$= (\text{pp. 3-4}) \; + \; \sqrt{2}\;\eta\, (-\sqrt{2} i\, \xi^\dagger \bar\sigma^m \partial_m \psi) \; - (\eta \leftrightarrow \xi)$$

$$= (\text{pp } 3\text{-}4) \;-\; 2i\, \eta\, \xi^\dagger \bar\sigma^m \partial_m \psi \;-\; (\eta \leftrightarrow \xi)$$

apply the Fierz transformation

$$= (\text{pp } 3\text{-}4) \;+\; i\, (\xi^\dagger \bar\sigma^a \eta)\, \sigma_a \bar\sigma^m \partial_m \psi \;-\; (\eta \leftrightarrow \xi)$$

in all

$$\begin{aligned}
[\delta_\xi, \delta_\eta]\, \psi \;=\;& 2i\, (\xi^\dagger \bar\sigma^a \eta - \eta^\dagger \bar\sigma^a \xi)\, \partial_a \psi \\
& - i\, (\xi^\dagger \bar\sigma^a \eta - \eta^\dagger \bar\sigma^a \xi)\, \sigma_a \bar\sigma^n \partial_n \psi \quad \text{pp.3-4} \\
& + i\, (\xi^\dagger \bar\sigma^a \eta - \eta^\dagger \bar\sigma^a \xi)\, \sigma_a \bar\sigma^m \partial_m \psi \quad \text{) above}
\end{aligned}$$

so the SUSY commutation relation is satisfied on shell for all fields.

It is not difficult to show that the Lagrangian

$$\mathcal{L} = \partial_m \phi^* \partial^m \phi + \psi^\dagger i \bar\sigma^m \partial_m \psi + F^* F$$

is invariant under this extended SUSY transformation. The new terms, not on pp 5-6, are

$$\begin{aligned}
\delta_\xi \mathcal{L} =\;& (\text{previous}) + \psi^\dagger i \bar\sigma^m \partial_m (\sqrt{2} F \xi) + \sqrt{2} F^* \xi^\dagger i \bar\sigma^m \partial_m \psi \\
& + F^* (-\sqrt{2} i \xi^\dagger \bar\sigma^m \partial_m \psi) + \sqrt{2} i\, \partial_m \psi^\dagger \bar\sigma^m \xi F
\end{aligned}$$

$$\delta_\xi \mathcal{L} = (\text{previously} = 0) + (= 0 \text{ after integrating by parts})$$

The new term, though, allows us to write a nonderivative Lagrangian that can contain a mass term. Consider, for example, the structure

$$\mathcal{L}_m = m\left(F\phi - \tfrac{1}{2}\psi^T c\psi\right) + h.c.$$

The SUSY transformation of this term is

$$\delta_\xi \mathcal{L}_m = m\big(F\sqrt{2}\,\xi^T c\psi - \sqrt{2}i(\xi^\dagger \bar\sigma^m \partial_m \psi)\phi$$
$$- \psi^T c\,(\sqrt{2}i\,\sigma^n c\,\xi^* \partial_n \phi + \sqrt{2}F\xi)\big)$$

$$= m\big(\sqrt{2}F\,\xi^T c\psi - \sqrt{2}F\,\psi^T c\xi$$
$$+ \sqrt{2}i\,\xi^\dagger \bar\sigma^m \psi\,\partial_m \phi - \sqrt{2}i\,\psi^T c\,\sigma^m c\,\xi^* \partial_m \phi\big)$$

$$= 0 + m\big(\sqrt{2}i(\xi^\dagger \bar\sigma^m \psi)\partial_m \phi$$
$$+ \sqrt{2}i\,\psi^T (\bar\sigma^m)^T \xi^* \partial_m \phi\big)$$

$= 0$ after interchanging the fermion order in the 2nd line.

this gives as the complete Lagrangian

$$\mathcal{L} = \partial_m \phi^* \partial^m \phi + F^* F + m(F\phi + F^* \phi^*)$$
$$+ \psi^\dagger i \bar{\sigma}^m \partial_m \psi - \frac{m}{2}(\psi^T C \psi - \psi^\dagger C \psi^*)$$

To analyze this structure, complete the square in the first line

$$\mathcal{L} = \partial_m \phi^* \partial^m \phi + (F^* + m\phi)(F + m\phi^*) - m^2 \phi^* \phi$$
$$+ \psi^\dagger i \bar{\sigma}^m \partial_m \psi - \frac{m}{2}(\psi^T C \psi - \psi^\dagger C \psi^*)$$

We now eliminate F:

classically, by using the equation of motion

$$F = -m\phi^*$$

quantum mechanically, by ~~functionally~~ integrating over F.

either way, F is a pure constrained field. It contains no new particles not already present in ϕ. This is what we mean by an "auxiliary field". After eliminating F, we have

$$\mathcal{L} = \partial_m \phi^* \partial^m \phi - m^2 \phi^* \phi + \psi^\dagger i \bar{\sigma}^m \partial_m \psi - \frac{m}{2}(\psi^T C \psi - \psi^\dagger C \psi^*)$$

a Lagrangian with a massive boson particle & antiparticle and a massive Majorana fermion (2 states each), with all particles having mass m.

A more general supersymmetric Lagrangian is

$$\mathcal{L} = \partial_m \phi_k^* \partial^m \phi_k + \psi_k^\dagger i \bar{\sigma}^m \partial_m \psi_k + F_k^* F_k$$

$$+ \left[m_{k\ell} F_k \phi_\ell - \tfrac{1}{2} m_{k\ell} \psi_k^T c \psi_\ell \right]$$

$$+ \text{h.c.}$$

$m_{k\ell}$ complex symmetric

The equation of motion for F_k is

$$F_k^* + m_{k\ell} \phi_\ell = 0$$

Eliminating $F_k F_k^*$, we have

$$\mathcal{L} = \partial_m \phi^* \partial^m \phi - \phi_k^* (m^\dagger m)_{k\ell} \phi_\ell$$

$$+ \psi_k^\dagger i \bar{\sigma}^m \partial_m \psi_k - \tfrac{1}{2} m_{k\ell} \psi_k^T c \psi_\ell + \text{h.c.}$$

If m is complex symmetric, we can write it as

$$m = U^T d\, U$$

where d is complex diagonal. The diagonal elements of d are of the form

$$d_i = M_i e^{i\alpha}$$

α can be removed by the redefinition $e^{i\alpha/2}\psi_i \to \psi_i$

U diagonalizes $m^\dagger m$; the eigenvalues are $|M_i|^2$.

So this Lagrangian has complex bosons and massive fermions with equal masses M_i.

We can generalize further. Notice that the mass term of $\mathcal{L}$ contains ϕ, ψ, F separately from $\phi^* \psi^\dagger F^*$. The most general such structure is built from an <u>analytic</u> <u>function</u> of the ϕ_k. Let $W(\phi)$ be such an analytic function. Then the structure

$$\mathcal{L}_F = F_k \frac{\partial W}{\partial \phi_k} - \frac{1}{2} \psi_k^T c \psi_\ell \frac{\partial^2 W}{\partial \phi_k \partial \phi_\ell}$$

is invariant to SUSY

$$\delta_\xi \mathcal{L}_F = 0 \qquad \text{after integrating by parts}$$

$W(\phi)$ is called the superpotential. Setting

$$W(\phi) = \tfrac{1}{2} m_{kl}\, \phi_k \phi_l$$

gives the previous Lagrangian. For a more interesting example, consider

$$W(\phi) = \frac{\lambda}{3!}\phi^3 \qquad (1 \text{ field})$$

This gives

$$\mathcal{L} = \partial_m \phi^* \partial^m \phi + \psi^\dagger i \bar{\sigma}^m \partial_m \psi + F^* F$$

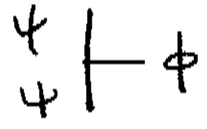
$$+ \tfrac{1}{2} F \phi^2 - \tfrac{1}{2} \psi^T c \psi \cdot \lambda \phi$$

eliminating F : $\quad F^* = \tfrac{1}{2}\phi^2$

$$\mathcal{L} = \partial_m \phi^* \partial^m \phi - \frac{\lambda^2}{4} |\phi^2|^2$$

$$+ \psi^\dagger i \bar{\sigma}^m \partial_m \psi - \frac{\lambda}{2} \psi^T c \psi \phi + \frac{\lambda}{2} \psi^\dagger c \psi^* \phi^*$$

The second line contains the Yukawa interaction



which is a renormalizable coupling in 4 dimensions.
The first line contains the ϕ^4 interaction, which is also renormalizable in 4-d. And, the coefficients are related by the constraint of SUSY. We will study this relation further in the next lecture.