2. The algebra of supersymmetry charges

In the previous lecture, I discussed the properties of the spin-$\frac{1}{2}$ representations of the Lorentz group in 4 dimensions. This gives us a basis to study algebras of supersymmetry charges.

A supersymmetry charge is a charge Q of spin-$\frac{1}{2}$ that commutes with $\mathcal{H}$. As I explained in the previous lecture, we can consider any such Q as an $(\frac{1}{2},0)$, together with its Hermitian conjugate, a $(0,\frac{1}{2})$. If there are N distinct charges, we have

$$Q^i_\alpha \qquad Q^{\dagger i}_\alpha \qquad\qquad i = 1 — N \qquad \alpha = 1,2$$

$N=1$ is "ordinary" supersymmetry in 4-dimensions, $N>1$ is "extended" supersymmetry.

There are actually profound restrictions on what the Q^i_α can be. To work these out, begin with the case $N=1$.

Consider the operator

$$\{Q_\alpha, Q_\beta^\dagger\}.$$

This operator belongs to the $(\frac{1}{2},\frac{1}{2})$ Lorentz representation.

Write

$$\{Q_\alpha, Q_\beta^\dagger\} = 2\,\sigma^a_{\alpha\beta} R_a$$

Now I claim ~~that~~ $R_a \neq 0$ and $[R_a, H] = 0$.

For the second property

$$[Q_\alpha, H] = 0 \Rightarrow [Q_\alpha^\dagger, H] = 0$$

so $$[H, \{Q_\alpha, Q_\beta^\dagger\}] = 0$$

For the first property, take the matrix element of the anticommutator for $\alpha = \beta$ (no sum) in the state $|\psi\rangle$

$$\langle\psi|\{Q_\alpha, Q_\alpha^\dagger\}|\psi\rangle$$

$$= \langle\psi| Q_\alpha Q_\alpha^\dagger |\psi\rangle + \langle\psi| Q_\alpha^\dagger Q_\alpha |\psi\rangle$$

$$= \| Q_\alpha^\dagger |\psi\rangle \|^2 + \| Q_\alpha |\psi\rangle \|^2$$

Unless $Q_\alpha|\psi\rangle = Q_\alpha^\dagger|\psi\rangle = 0$, this is

$$\langle\psi| \sigma^a_{\alpha\alpha} R_a |\psi\rangle > 0$$

There must be some state for which $Q_a|\psi\rangle \neq 0$; otherwise Q_a would be trivial. So R_a is nonzero.

However, there is a theorem of Coleman and Mandula that a quantum field theory with a nontrivial S-matrix cannot have a second conserved vector charge distinct from the energy-momentum 4-vector P_a. (Coleman and Mandula, O'Raifeartaigh, and others, derived such results to obliterate certain extended symmetries of the strong interactions proposed in the 1960's.) The proof of Coleman and Mandula is quite sophisticated, but here is a taste of the result.

Consider the scattering process $a+b \to c+d$, with conserved P_a and R_a.

matrix elements of P^a:

$$\langle a(p_a)|P^c|a(p_a')\rangle = p_b^c \cdot 2E_{p_a}(2\pi)^3\delta(\vec{p}-\vec{p}\,')$$

$$\langle b(p_b)|P^c|b(p_b')\rangle = p_b^c \cdot 2E_{p_b}(2\pi)^3\delta(\vec{p}-\vec{p}\,')$$

etc.

matrix elements of R^a:

$$\langle a(p_a)|R^c|a(p_a')\rangle = \underbrace{r(a)\,p_a^c}\ 2E_p(2\pi)^3\delta(\vec{p}-\vec{p})$$

the result must be of this form by the Wigner-Eckart theorem: p_a is the only 4-vector in the problem.

Then if $a+b \to c+d$

P conservation $\Rightarrow$ $p_a + p_b = p_c + p_d$

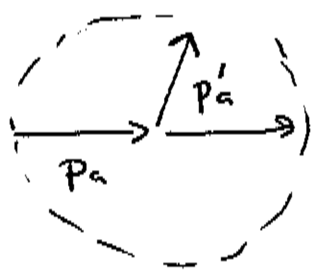
R conservation $\Rightarrow$ $r_a p_a + r_b p_b = r_c p_c + r_d p_d$

If $R \neq P$, there are two particles for which $r_a \neq r_b$.

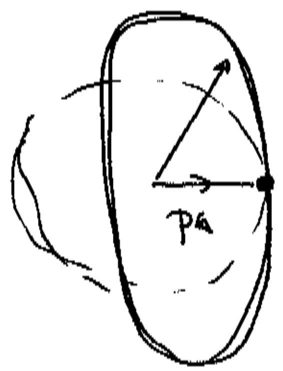
In elastic scattering $a+b \to a+b$, go to the CM frame

$$p_a \longrightarrow \quad \longleftarrow p_b$$

The solutions of $p_a' + p_b' = p_a + p_b$ lie on a circle



The solution of $r_a p_a + r_b p_b = r_a p_a' + r_b p_b'$ lie on an ellipse



So P and R are simultaneously conserved only in the forward direction, or at best at some discrete angles. But in quantum field theory, the scattering amplitude is analytic in $t = (p-p')^2$. This is possible only if the scattering amplitude is $\underline{0}$.

So at this point, we have almost totally destroyed the possibility of supersymmetry. There is only one consistent possibility

$$\{ Q_\alpha, Q_\beta^+ \} = 2\, \sigma^a_{\alpha\beta} P_a$$

In particular, on a state with

$P^0 |\psi\rangle = H|\psi\rangle = E|\psi\rangle \qquad \vec{P}|\psi\rangle = \vec{p}\,|\psi\rangle \qquad \vec{p} \parallel \hat{3}$

$$\{ Q_\alpha, Q_\beta^+ \} = 2\,(P^0 - \vec{P}\cdot\vec{\sigma})$$

$$\{ Q_\alpha, Q_\beta^+ \}|\psi\rangle = 2\,(P^0 - P^3\sigma^3)$$

so

$$\{ Q_1, Q_1^+ \} = 2\,(P^0 - P) = 2P^-$$

$$\{ Q_2, Q_2^+ \} = 2\,(P^0 + P) = 2P^+$$

Roughly, Q must be the square root of the Hamiltonian. If this sounds reminiscent of Dirac's derivation of the

Dirac eqn, we are going deeper into that territory.

The Coleman-Mandula theory actually excludes any ~~integer~~-spin charges with spin > 0 except P^a and M^{ab}. So it also restricts

$$\{Q_\alpha, Q_\beta\}$$

This is symmetric in $\alpha\beta$, so it is in $(1,0)$, so the only possible form is

$$\{Q_\alpha, Q_\beta\} = (\sigma^{ab}C)_{\alpha\beta} M_{ab}$$

But since this must commute with $H = P^0$, this form is impossible.

So

$$\{Q_\alpha, Q_\beta\} = 0$$

More generally, for N-extended SUSY:

$$\{Q^i_\alpha, Q^{\dagger j}_\beta\} = 2\,\delta^{ij}\,\sigma^a_{\alpha\beta} P_a$$

$$\{Q^i_\alpha, Q^j_\beta\} = Z^{ij} C_{\alpha\beta}$$

where the Z^{ij} are scalar $(0,0)$ charges s.t.

$$[Z^{ij}, H] = 0 \qquad Z^{ij} = -Z^{ji} \qquad \text{"central charges"}$$

Actually, in the above you might replace δ^{ij} by Δ^{ij}, a Hermitian positive matrix, but then we can change the basis of the Q^i_α and rescale to return $\Delta^{ij} \to \delta^{ij}$.

Let's now study the representations of the SUSY algebras on 1-particle states. Start with massless particles

$$|p\rangle \quad \text{s.t.} \quad p^a = (p, 0, 0, p)$$

Then

$$\{Q_1, Q_1^\dagger\} |p\rangle = 2(p-p)|p\rangle = 0$$

so $\quad Q_1 |p\rangle = Q_1^\dagger |p\rangle = 0$

$$\{Q_2, Q_2^\dagger\} |p\rangle = 4p \, |p\rangle$$

so let

$$a = \frac{Q_2}{\sqrt{4p}} \qquad a^\dagger = \frac{Q_2^\dagger}{\sqrt{4p}}$$

$$\{a, a^\dagger\} = 1 \qquad \{a, a\} = 0 = \{a^\dagger, a^\dagger\}$$

This is one set of fermion creation and annihilation operators. The representations of the algebra are 2-state systems!

$$|\psi\rangle \, , \quad a^\dagger |\psi\rangle$$

with

$$(a^\dagger)^2 |\psi\rangle = 0 \qquad a a^\dagger |\psi\rangle = |\psi\rangle$$

Now, $Q_2^\dagger$ lowers J^3 by $\frac{1}{2}$, Q_2 raises J^3 by $\frac{1}{2}$. So this representations have states of

Helicity $= \vec{S}\cdot\hat{P}$ equal to

h , $h+\frac{1}{2}$

The antiparticle of a state with helicity h has helicity $= -h$. In quantum field theory, the field that creates the particle state w. helicity h annihilates the antiparticle with helicity $-h$, so both states must be included in the theory. Then we can enumerate some possible particle representations:

$h=0$ $\qquad\qquad$ $h=-\frac{1}{2}$ $\quad$ $h=0$

$\qquad\qquad\qquad\qquad\qquad\qquad$ $h=0$ $\quad$ $h=+\frac{1}{2}$

This is a theory with the particles associated with

$\psi_\alpha \quad \psi_\alpha^\dagger$ $\qquad$ Weyl fermion ψ_L & antiparticle $\psi_R^\dagger$

and $\quad \phi \quad \phi^*$ $\qquad$ a complex scalar field or two real scalar fields.

$h=\frac{1}{2}$ $\qquad$ $h=-1$ $\quad$ $h=-\frac{1}{2}$ $\qquad$ $h=\frac{1}{2}$ $\quad$ $h=+1$

This theory contains the particles associated with

A_a or F_{ab} and $\psi_\alpha \ \psi_\alpha^\dagger$

This is the particle content of SUSY-QED.

All other representations have some state with $|h|>1$, i.e. spin > 1.

a representation that will be important to us later is

$$-2 \qquad -\tfrac{3}{2} \qquad\qquad\qquad +\tfrac{3}{2} \quad +2$$

a graviton plus a massless spin-$\tfrac{3}{2}$ field. This is the particle content of super<u>gravity</u>:

The same logic can be used to work out the massless representations of extended SUSY, at least when $Z^{ij}|p\rangle = 0$. In this case, we can write, for

$$P^a|p\rangle = (p,0,0,p)^a |p\rangle$$

$$Q^i_1 = Q^{\dagger i}_1 = 0 \quad \text{on } |p\rangle$$

$$a^i = \frac{Q^i_2}{\sqrt{4p}} \qquad a^{\dagger i} = \frac{Q^{\dagger i}_2}{\sqrt{4p}}$$

then $\{a^i, a^{j\dagger}\} = \delta^{ij} \qquad \{a^i, a^j\} = 0$

So there is a multiplet of states, for $N=2$

$h =$	h	$h+\frac{1}{2}$	$h+1$
	1	2	1
	$a^{1\dagger} a^{2\dagger}|p\rangle$	$a^{1\dagger}|p\rangle$, $a^{2\dagger}|p\rangle$	$|p\rangle$

for general N

$\underline{h}$	$\underline{h+1/2}$	$\cdots$	$h+(N-2)/2$	$\underline{h+(N-1)/2}$	$\underline{h+N/2}$
1	N	$\cdots$	$\frac{N(N-1)}{2}$	N	1
$\prod_i a^{i\dagger}\lvert p\rangle \uparrow$			$a^{i\dagger}a^{j\dagger}\lvert p\rangle$	$a^{i\dagger}\lvert p\rangle$	$\lvert p\rangle$

Let me enumerate the representations with $|h| \le 1$ for all states

$\underline{N=1}$

$h=$	$\underline{-\tfrac12}$	$\underline{0}$	$\underline{\tfrac12}$	
	1	2	1	chiral multiplet

$h=$	$\underline{-1}$	$\underline{-\tfrac12}$	$\underline{0}$	$\underline{\tfrac12}$	$\underline{1}$	
	1	1	0	1	1	vector or gauge multiplet

$\underline{N=2}$

$h=$	$\underline{-\tfrac12}$	$\underline{0}$	$\underline{\tfrac12}$	
	1	2	1	hyper multiplet

$h=$	$\underline{-1}$	$\underline{-\tfrac12}$	$\underline{0}$	$\underline{\tfrac12}$	$\underline{1}$	
	1	2	2	2	1	vector multiplet

a renormalizable field thy w. $N=3$ actually also has $N=4$,
so I'll skip to that case.

$N=4$ $\quad h = \underline{-1} \quad \underline{-\tfrac{1}{2}} \quad \underline{0} \quad \underline{\tfrac{1}{2}} \quad \underline{1}$

	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$N=4$	1	4	6	4	1

A theory with $N \geq 5$ SUSY necessarily contains particles with spin > 1.

It is also interesting to write down representations with the highest value of h being $h=2$. These are supergravity multiplets

h	-2	$-3/2$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$3/2$	2
$N=1$	1	1						1	1
$N=2$	1	2	1				1	2	1
$N=4$	1	4	6	4	2	4	6	4	1
$N=8$	1	8	28	56	70	56	28	8	1

A theory with $N > 8$ SUSY necessarily contains particles with spin > 2.

As the spin of fields increases, it becomes increasingly difficult to construct a consistent quantum theory. For spin 0 and spin $\frac{1}{2}$, this is straightforward. For spin 1, the particles created by A^0 apparently have negative norm. To remove these particles, we need a local gauge invariance. For higher spin, more intricate gauge invariances are needed. For spin 2, general coordinate invariance gives a positive quantum theory. For spin 5/2 or spin 3, there is no known solution. So it is possible that N=8 is the largest possible SUSY algebra.

Later in the course, I will discuss SUSY in higher dimensions. But obviously, any SUSY theory in dimension $d > 4$ must contain SUSY in 4-dimensions. An interesting question to ask is, if a theory contains the minimal SUSY algebra in d dimensions and this theory is reduced to 4-dimensions (by ignoring dependence on $x^4 \ldots x^{d-1}$), how many copies of the 4-d SUSY multiplet do we find. I will show later the course ~~that the~~ answer is:

$$d = 5, 6 \qquad N = 2$$

$$d = 7, 8, 9, 10 \qquad N = 4$$

$$d = 11 \qquad N = 8$$

$$d = 12 \ldots \qquad N = 16$$

So any SUSY theory in $d > 6$ necessarily contains spin > 1.

The maximal SUSY theory of scalars, vectors, and spinors is SUSY-Yang Mills theory in 6 dimensions any SUSY theory in $d > 11$ necessarily contains spin > 2.

The maximal supergravity without spin > 2 is supergravity in 11 dimensions

We will construct these theories later in the course.

The next step is to study the representations of the $N=1$ SUSY algebra on fields. For the rest of this lecture, I will develop some technology to assist us in doing this.

It will be useful to have a uniform notation for the action of SUSY on a field. So, define:

$$\delta_\xi \phi = [\xi^T C Q + \xi^\dagger C Q^*, \phi(x)]$$

where $Q, Q^\dagger$ are the SUSY charges and ξ_α is a classical anticommuting spinor. The object $\xi^T C Q$ is bosonic (spin 0)

Note also that

$$[\xi^T C Q + \xi^\dagger C Q^*]^* = -(\xi^T C Q + \xi^\dagger C Q^*)$$

$(\ )^*$ reverses the order of ξ and Q, it takes a (-1) to bring them back into the correct order. The minus sign here is useful, since it implies:

$$(\delta_\xi \phi)^* = ([\xi^T C Q + \xi^\dagger C Q^*, \phi])^*$$
$$= -[(\xi^T C Q + \xi^\dagger C Q^*)^*, \phi^*]$$

so

$$(\delta_\xi \phi)^* = + \delta_\xi \phi^*$$

Now we would like to construct a δ_ξ that satisfies the SUSY algebra. To set up this computation, compute

$$[\delta_\xi, \delta_\eta]\, \phi$$

The definition is

$$[\delta_\xi, \delta_\eta]\, \phi = [(\xi^T c Q + \xi^\dagger c Q), [(\eta^T c Q + \eta^\dagger c Q^*), \phi]]$$
$$- [(\eta^T c Q + \eta^\dagger c Q^*), [(\xi^T c Q + \xi^\dagger c Q^*), \phi]]$$

Before we compute this, I would like to talk about two aspects of such commutator relations:

① the <u>mysterious minus</u> signs

Everyone knows that P^a is represented on wavefunctions by $P^a = i\partial^a = (i\frac{\partial}{\partial t}, -i\vec{\nabla})^a$

However, there is a (-1) between this and the representation of P^a on fields. A field depends on x according to the Heisenberg picture $\phi(x) = e^{iP\cdot x}\, \phi(0)\, e^{-iP\cdot x}$

then $$-i\partial^a \phi(x) = P^a \phi(x) - \phi(x) P^a$$

or $$[P^a, \phi(x)] = -i\partial^a \phi(x)$$

a similar (-1) appears in the commutation relations of M^{ab}

Go back to the representation of M^{ab} on fields

$$M^{ab} = x^a \partial^b - x^b \partial^a$$

and commute w. $P^a = i\partial^a$

$$[M^{ab}, P^c] = -\eta^{ac} i\partial^b + \eta^{bc} i\partial^a$$

so $$[M^{ab}, P^c] = -(\eta^{ac} P^b - \eta^{bc} P^a)$$

compute $$\left[\tfrac{1}{2}\omega_{ab} M^{ab}, P^c\right] = -\omega^c{}_b P^b$$

with the action of M on a vector state

$$\left(\tfrac{1}{2}\omega_{ab} M^{ab}\right) V^c = +\omega^c{}_b V^b$$

This (-1) is reflected in all commutators of M with Heisenberg operators:

$$[M^{ab}, P^c] = -(\eta^{ac} P^b - \eta^{bc} P^a)$$
$$[M^{ab}, \psi_\alpha] = -(\sigma^{ab})_{\alpha\beta} \psi_\beta$$
$$[M^{ab}, \psi^+_\alpha] = +\psi^+_\beta \bar{\sigma}^{ab}_{\beta\alpha}$$

the last line follows from the second by conjugation.

Notice that

$$(\sigma^{ab})^{\dagger} = \frac{1}{4}\left(\sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a\right)^{\dagger} = \frac{1}{4}\left(\bar{\sigma}^b\sigma^a - \bar{\sigma}^a\sigma^b\right)$$

$$= -\bar{\sigma}^{ab}$$

Remember also that M^{ab} as I have defined it is <u>anti</u>-Hermitian

② <u>Jacobi identities</u>

Whenever there are a lot of commutators, it is often helpful to use the Jacobi identity. This is the relation, for any three operators A, B, C:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

The proof is tedious but straightforward. Write the left hand side at the top of a large piece of paper, expand all of the commutators, and cancel term by term. For example, the first double commutator has a term

$$-ACB \quad \leftarrow \quad [A,[B,C]]$$

and you can see that

$$+[B,[C,A]] \longrightarrow +ACB,$$ cancelling this.

All terms cancel in this way.

If some of A, B, C are fermions, we might like to use anticommutators in some places. So it is relevant that there is also a super-Jacobi identity.

Write the Jacobi identity of A, B, C

If there is a commutator of two fermionic operators, replace $[\,,\,]$ by $\{\,,\,\}$

If the order of two fermionic operators is reversed, give that term a (-1)

The resulting equation is $= 0$

e.g. J - bosonic $\quad Q_\alpha, Q_\beta^\dagger$ fermionic

$$[J, \{\underbrace{Q_\alpha, Q_\beta^\dagger}_{\text{bosonic}}\}] + \{Q_\alpha\, [\underbrace{Q_\beta^\dagger, J}_{\text{ferm. v. boson = fermion}}]\} - \{Q_\beta^\dagger, [\underbrace{J, Q_\alpha}_{\text{reversal of order}}]\} = 0$$

as a check

$$\left.\begin{array}{lll} [J, \{Q_\alpha, Q_\beta^\dagger\}] & \longrightarrow & +J\, Q_\beta^\dagger Q_\alpha \\ \{Q_\alpha, [Q_\beta^\dagger, J]\} & \longrightarrow & -J\, Q_\beta^\dagger\, Q_\alpha \end{array}\right\} \longrightarrow \text{cancel}$$

$$\left.\begin{array}{lll} [J, \{Q_\alpha, Q_\beta^\dagger\}] & \longrightarrow & -Q_\beta^\dagger\, Q_\alpha J \\ -\{Q_\beta^\dagger\, [J, Q_\alpha]\} & \longrightarrow & +Q_\beta^\dagger\, Q_\alpha J \end{array}\right\} \longrightarrow \text{cancel}$$

With some concentration, you can get the minus signs right, but it is often easier to work with combinations such as

$$(\xi^T C Q)$$

which are bosonic.

As an application of the Jacobi identity, note that

$$[A,[B,\phi]] + [B,[\phi,A]] + [\phi,[A,B]] = 0$$

implies

$$[A,[B,\phi]] - [B,[A,\phi]] = +[[A,B],\phi]$$

we can use this to compute

$$[\delta_\xi,\delta_\eta]\phi = (\text{expression on p. 14})$$

$$= [[\xi^T C Q + \xi^\dagger C Q^*, \eta^T C Q + \eta^\dagger C Q^*], \phi]$$

Now we can more easily compute the commutator

$$[\xi^T C Q, \eta^T C Q] = (\xi^T C Q)(\eta^T C Q) - (\eta^T C Q)(\xi^T C Q)$$

$$= (\xi^T C)_\alpha (\eta^T C)_\beta \underbrace{(-1)}_{\eta \text{ thru } Q} [Q_\alpha Q_\beta \underbrace{+}_{\text{reversal of order of } \eta + \xi} Q_\beta Q_\alpha]$$

$$= 0 \quad \text{since} \quad \{Q_\alpha, Q_\beta\} = 0$$

similarly, $[\xi^\dagger c Q^*, \eta^\dagger c Q^*] = 0$

next,

$$[\xi^\dagger c Q^*, \eta^T c Q]$$

$$= (\xi^\dagger c Q^*)(\eta^T c Q) - (\eta^T c Q)(\xi^\dagger c Q^*)$$

$$= (\xi^\dagger c)_\alpha (\eta^T c)_\beta (-1) \left(Q^*_\alpha Q_\beta + Q_\beta Q^*_\alpha \right)$$

$$= -(\xi^\dagger c)_\alpha (\eta^T c)_\beta \; \{Q_\beta, Q^*_\alpha\}$$

$$= -(\xi^\dagger c)_\alpha (\eta^T c)_\beta \, (2\sigma^a_{\beta\alpha} P_a)$$

$$= 2(\xi^\dagger c\, (\sigma^a)^T c^T \eta)\, P_a$$

$$= 2\,(-\xi^\dagger \bar\sigma^a \eta)\, P_a$$

using $c^T = -c = c^{-1}$ $\quad c(\sigma^a)^T c^{-1} = \bar\sigma^a$

then

$$[\delta_\xi, \delta_\eta]\phi = [(-2\xi^\dagger\bar\sigma^a\eta\, P_a + \cdots), \phi]$$

$$= -2(\xi^\dagger\bar\sigma^a\eta)(-i\,\partial_a) + \cdots$$

$$= 2i(\xi^\dagger\bar\sigma^a\eta)\,\partial_a\phi + \cdots$$

the term $[\xi^T c Q, \eta^\dagger c Q^*]$ leads to $-2i\,\eta^\dagger\bar\sigma^a\xi\,\partial_a\phi$

so, finally:

$$[\delta_\xi, \delta_\eta]\ \phi(x) = 2i\left(\xi^\dagger \bar{\sigma}^a \eta - \eta^\dagger \bar{\sigma}^a \xi\right) \partial_a \phi$$

In the next lecture, I'll construct a δ_ξ that satisfies this relation.