# 12. Supergravity in higher dimensions

In our study of supersymmetric Yang-Mills theory, we found that the 4-dimensional theory could be extended to higher dimensions, giving the foundation for the construction of theories with $N=2$ and $N=4$ SUSY. It is natural to hope that supergravity might have a similar extension. In this case, a theory that we might particularly like to find is the 11-dimensional supergravity which would dimensionally reduce to a 4-d theory with $N=8$ supersymmetry, the largest degree of supersymmetry possible for a theory with all spins $\leq 2$.

It turns out that the theory of supergravity in higher dimensions is considerably more complicated than for Yang-Mills theory. In this lecture, I would like to derive a few of its aspects. In particular, I would like to discuss the particle content of supergravity in 5-d and in the 11-dimensional theory.

In the previous lecture, I gave a rather complete accounting of pure supergravity in 4 dimensions. To begin this lecture, I would like to rederive a few of the main results using Majorana fermions and trying as much as possible to give arguments that do not use special properties of 4 dimensions. These arguments will not give the full structure of the theory but will suffice for an analysis of the linearized theory. Then I will try to generalize these arguments to higher dimensions.

Consider, then, the supersymmetric theory of a vierbein and a Majorana (4-component) gravitino in 4 dimensions. The transformation laws are:

$$\delta_\xi e_m^a = -i\kappa\, \bar\xi \gamma^a \psi_m$$

$$\delta_\xi \psi_m = \frac{2}{\kappa} D_m \xi \qquad D_m\xi = \partial_m \xi + \tfrac{1}{4}\omega_m^{bc}\gamma^{bc}\xi$$

The Lagrangian is

$$\mathcal{L} = -\frac{1}{2\kappa^2} eR - \frac{i}{4} e\left[\bar\psi_k \gamma^{klm} D_l \psi_m - D_l \bar\psi_k \gamma^{klm}\psi_m\right]$$

Notice that I have rewritten the Rarita-Schwinger term in a way that does not use $\epsilon^{mnpq}$. So now we will write the RS equation as

$$-i\,\gamma^{klm} D_l \psi_m = 0$$

I would like to derive some other forms of this equation that we will need. In the process, I would also like to prove a general result that will be useful for dealing with products of $\gamma^k$ matrices.

To begin, I would like to write

$$\gamma^a \gamma^{b_1 \cdots b_n}$$

as a sum of terms fully antisymmetrized in the indices of $\gamma$-matrices. To do this, first drop the antisymmetrization on $b_1 \cdots b_n$ and write

$$\gamma^a \gamma^{b_1} \cdots \gamma^{b_n}$$

Now (anti)-symmetrize the position of $\gamma^a$ in the product:

$$\gamma^a\gamma^{b_1}\cdots\gamma^{b_n} = \frac{1}{(n+1)}\left((n+1)\gamma^a\gamma^{b_1}\cdots\gamma^{b_n}\right)$$

$$= \frac{1}{(n+1)}\Big(\gamma^a\gamma^{b_1}\cdots\gamma^{b_n}$$

$$+ 2\eta^{ab_1}\gamma^{b_2\cdots b_n} - \gamma^{b_1}\gamma^a\gamma^{b_2}\cdots\gamma^{b_n}$$

$$+ 2\eta^{ab_1}\gamma^{b_2\cdots b_n} - 2\eta^{ab_2}\gamma^{b_1 b_3\cdots b_n} + \gamma^{b_1}\gamma^{b_2}\gamma^a\gamma^{b_3}\cdots\gamma^{b_n}$$

$$+ \cdots\Big)$$

$$= \frac{1}{(n+1)}\left(\gamma^a\gamma^{b_1}\cdots\gamma^{b_n} - \gamma^{b_1}\gamma^a\gamma^{b_2}\cdots\gamma^{b_n} + \gamma^{b_1}\gamma^{b_2}\gamma^a\gamma^{b_3}\cdots - \cdots\right)$$

$$+ \left(\frac{2}{n+1}\right)\cdot\left[\eta^{ab_1}\gamma^{b_2\cdots b_n} + \cdots\right]$$

in the [ ] there are $\frac{(n+1)(n)}{2}$ terms, all equivalent to

$$\eta^{ab_1}\gamma^{b_1}\cdots\gamma^{b_n}$$

after antisymmetrizing on $b_1\cdots b_n$. So

$$\gamma^a\gamma^{b_1\cdots b_n} = \gamma^{ab_1\cdots b_n}$$

$$+ \underbrace{\left[\eta^{ab_1}\gamma^{b_2\cdots b_n} - \eta^{ab_2}\gamma^{b_1 b_3\cdots b_n} + \cdots\right]}_{n \text{ terms.}}$$

A convenient way to express ~~this~~ is to write

$$\overbrace{\gamma^a\gamma^{b_1}} = \eta_{ab_1} \quad \text{"the contraction"}$$

$$\gamma^a\gamma^{b_1\cdots b_n} = \gamma^{ab_1\cdots b_n} + (\text{sum of all possible contractions})$$

where, in calculating the value of a contraction, the anticommutation of the $b_i$ is taken into account:

$$\overbrace{\gamma^a\gamma^{b_1 b_2}} = -\eta^{ab_2}\gamma^{b_1}$$

This result easily generalizes to products of $\gamma$'s and becomes the "$\gamma$-squeezing identity"

$$\gamma^{a_1 \cdots a_m}\,\gamma^{b_1 \cdots b_n} = \gamma^{a_1 \cdots a_m b_1 \cdots b_n} + (\text{all possible contractions})$$

Here are some consequences of this identity:

$$\gamma^c \gamma^{a_1 a_2 \cdots a_n} \gamma_c = \underbrace{\gamma^{c a_1 a_2 \cdots a_n}{}_c}_{0} + \gamma^{\overline{c a_1 \cdots a_n}}{}_c + \gamma^{\overline{c a_1} \cdots a_n}{}_c + \gamma^{c a_1 \overline{\cdots a_n}}{}_c$$

$$= 0 + (-1)^n d\,\gamma^{a_1 \cdots a_n} + n\,\eta^{a_1 c}\gamma^{a_2 \cdots a_n}{}_c + n\,\gamma^{c a_1 \cdots a_{n-1}}\eta^{a_n c}$$

$$= (-1)^n (d - 2n)\,\gamma^{a_1 \cdots a_n}$$

which we found in another way in lecture 8. Also,

$$\gamma_a \gamma^{a a_1 \cdots a_n} = (d-n)\,\gamma^{a_1 \cdots a_n}$$

Now analyze the R-S equation, in the form

$$\gamma^{klm}\psi_{lm} = 0 \qquad \psi_{lm} = D_l\psi_m - D_m\psi_l$$

$$0 = \gamma_k \gamma^{klm}\psi_{lm} = (d-2)\,\gamma^{lm}\psi_{lm} \Rightarrow \gamma^{lm}\psi_{lm} = 0$$

$$0 = \gamma^k \gamma^{lm}\psi_{lm} = \underbrace{\gamma^{klm}\psi_{lm}}_{=0} + \gamma^m\psi_{km} - \gamma^l\psi_{lk} = 0$$

$$\Rightarrow \gamma^l\psi_{lm} = 0$$

$$0 = \gamma^k\gamma^l\psi_{lm} = \gamma^{kl}\psi_{lm} + \eta^{kl}\psi_{lm} \Rightarrow \gamma^{kl}\psi_{lm} = -\psi^k{}_m$$

$$0 = \gamma^j\gamma^{kl}\psi_{lm} + \gamma^j\psi_{km} = \gamma^{jkl}\psi_{lm} + \eta^{jk}\underbrace{\gamma^l\psi_{lm}}_{=0} + (-\gamma^k\psi_{jm}) + \gamma^j\psi_{km}$$

$$0 = \gamma^i \gamma^{jkl} \psi_{lm} + \gamma^i \gamma^j \psi_{km} - \gamma^i \gamma^k \psi_{jm}$$

$$= \gamma^{ijkl} \psi_{lm} + \eta^{ij} \gamma^{kl} \psi_{lm} - \eta^{ik} \gamma^{jl} \psi_{lm} + \eta^{il} \gamma^{jk} \psi_{lm}$$

$$+ \eta^{ij} \psi_{km} + \gamma^{ij} \psi_{km} - \gamma^{ik} \psi_{jm} - \eta^{ik} \psi_{jm}$$

$$= \gamma^{ijkl} \psi_{lm} - \cancel{\eta^{ij} \psi_{km}} + \cancel{\eta^{ik} \psi_{jm}} + \gamma^{jk} \psi_{im}$$

$$+ \cancel{\eta^{ij} \psi_{km}} - \cancel{\eta^{ik} \psi_{jm}} + \gamma^{ij} \psi_{km} - \gamma^{ik} \psi_{jm}$$

so!

$$\gamma^{klm} \psi_{lm} = 0 \qquad \gamma^{lm} \psi_{lm} = 0$$

$$\gamma^{l} \psi_{lm} = 0$$

$$\gamma^{kl} \psi_{lm} = - \psi_{km}$$

$$\gamma^{jkl} \psi_{lm} = - \gamma^{j} \psi_{km} + \gamma^{k} \psi_{jm}$$

$$\gamma^{ijkl} \psi_{lm} = - \gamma^{ij} \psi_{km} + \gamma^{ik} \psi_{jm} - \gamma^{jk} \psi_{im}$$

etc.

Now let's go back to supergravity. Consider first the commutator of SUSY variations:

$$[\delta_\xi, \delta_\eta]\, e_m^a = \delta_\xi ( -i\kappa\, \bar\eta \gamma^a \psi_m ) - (\xi \leftrightarrow \eta)$$

$$= -i\kappa\, \bar\eta \gamma^a \frac{2}{\kappa} D_m \xi + i\kappa\, \bar\xi \gamma^a D_m \eta \cdot \frac{2}{\kappa}$$

$$= 2i \; \bar\xi \gamma^a D_m \eta + (D_m \bar\xi) \gamma^a \eta$$

using the Majorana property $\bar\xi \gamma^a \eta = -\bar\eta \gamma^a \xi \qquad \bar\xi \gamma^a \gamma^b \eta = +\bar\eta \gamma^b \gamma^a \xi$

$$= 2i\, D_m ( \bar\xi \gamma^a \eta )$$

$$[\delta_\xi, \delta_\eta]\, \psi_m = \delta_\xi\left( \tfrac{2}{K} D_m \eta \right) - (\xi \leftrightarrow \eta)$$

$$= \tfrac{1}{2K} (\delta_\xi \omega_m{}^{bc}) \gamma^{bc} \eta - (\xi \leftrightarrow \eta)$$

As I reported in the previous lecture, a rather involved computation gives the "supercovariant" result: (written now for Majorana fermions)

$$\delta_\xi \omega_m{}^{bc} = -\tfrac{i}{2} K e_m^a \left[ \bar\xi \gamma_c \psi_{ab} + \bar\xi \gamma_b \psi_{ca} - \bar\xi \gamma_a \psi_{bc} \right]$$

In 4-dimensions, we could write this as

$$\delta_\xi \omega_m{}^{bc} = e_m^a \; \bar\xi \gamma_c \psi_{ab} - (b \leftrightarrow c) \qquad \text{using the RS equation}$$

so $\quad [\delta_\xi, \delta_\eta]\, \psi_m = -i\, \gamma^{bc} \eta\, \bar\xi \gamma_c \psi_{mb} - (\xi \leftrightarrow \eta)$

using the RS eq. again $\quad = -i\, \gamma^b \gamma^c \eta\, \bar\xi \gamma_c \psi_{mb} - (\xi \leftrightarrow \eta)$

At this point we need a Fierz identity. Actually, what we need is the "magic Fierz identity" from p. 22 of lecture 8.

$$\bar\psi \gamma^c \eta\, \bar\xi \gamma_c \chi - (\xi \leftrightarrow \eta) = \bar\xi \gamma^c \eta\, \bar\psi \gamma_c \chi$$

then $\quad [\delta_\xi, \delta_\eta]\, \psi_m = -i (\bar\xi \gamma^c \eta)\, \gamma^b \gamma_c \psi_{mb}$

$$= -2i\, \bar\xi \gamma^c \eta\, \psi_{mc}$$

or

$$(*) \quad \left[ \begin{aligned} [\delta_\xi, \delta_\eta]\, e_m^a &= 2i\, D_m (\bar\xi \gamma^a \eta) \\ [\delta_\xi, \delta_\eta]\, \psi_m &= 2i\, \bar\xi \gamma^k \eta\, (D_k \psi_m - D_m \psi_k) \end{aligned} \right.$$

In the previous lecture, I showed that these relations could be written in a form in which the right-hand side was a sum of gauge variations in supergravity. I will not repeat that here.

We can also look back at the proof that $\delta_\xi \mathcal{L} = 0$

In our new notation

$$\delta_\xi \mathcal{L}_H = -\frac{1}{2\kappa^2} e\,(-i\kappa\bar{\xi}\gamma^a\psi_m)\,[\,e^m_a R - 2e^m_b e^n_a R_n{}^b\,]$$

For the RS Lagrangian, I will write the variation as

$$\delta_\xi \mathcal{L}_{RS} = -\frac{i}{4} e\,\frac{2}{\kappa}\{\,\bar{\psi}_k\gamma^{klm}D_l D_m\xi - \bar{\xi}\gamma^{klm}D_k D_l\psi_m$$
$$- D_l D_k\bar{\xi}\gamma^{klm}\psi_m + D_m D_l\bar{\psi}_k\gamma^{klm}\xi\,\}$$

$$+ \cdots$$

where $\cdots$ includes terms generated by integration by parts and variation of $e^a_m$ factors. In the previous lecture, I showed how these terms work out in 4-dimensions. The story gets more complicated in higher dimensions. Suffice it to say that all of these terms have at least 3 powers of the gravitino field, so I will prove $\delta_\xi \mathcal{L}_{RS} = 0$ only up to this level.

Anyway, with this restriction

$$\delta_\xi \mathcal{L}_{RS} = -i\,\frac{e}{2\kappa}\{\bar{\psi}_k\gamma^{klm}\,\frac{1}{2}\cdot\frac{1}{4}R_{lm}{}^{ab}\gamma^{ab}\xi - \bar{\xi}\gamma^{klm}\frac{1}{8}R_{kl}{}^{ab}\gamma^{ab}\psi_m$$
$$+ \frac{1}{8}R_{lk}{}^{ab}\bar{\xi}\gamma^{ab}\gamma^{klm}\psi_m + \frac{1}{8}R_{ml}{}^{ab}\bar{\psi}_l\gamma^{ab}\gamma^{klm}\xi\}$$

$$= +i\,\frac{e}{8\kappa}\,R_{kl}{}^{ab}\,\bar{\xi}\,[\gamma^{klm}\gamma^{ab} + \gamma^{ab}\gamma^{klm}]\,\psi_m$$

now $\gamma^{ab}\gamma^{klm} + \gamma^{klm}\gamma^{ab} = 2\gamma^{klmab} + 2(-e^{ka}e^{lb}\gamma^m - e^{la}e^{mb}\gamma^k - e^{ma}e^{kb}\gamma^l)$
$+ (a \leftrightarrow b)$

Now the total antisymmetrized

$$D_{[k} D_l e_{m]}^{a} = \frac{1}{2} R_{[kl}{}^{a}{}_{b} e_{m]}^{b} = \frac{1}{2} D_{[k} T_{lm]}^{a}$$

so $R_{kl}{}^{ab} \gamma^{klmab} \propto$ torsion $\propto (\psi)^2$. At the level of this analysis, we can drop ~~that~~ term. The rest is

$$\delta_\xi \mathcal{L}_{RS} = -i \frac{e}{2\kappa} R_{kl}{}^{ab} ( e_a^k e^l_b \, e^m_c \, \bar{\xi} \gamma^c \psi_m - 2 e^l_b \, e^m_a e^k_c \, \bar{\xi} \gamma^c \psi_m )$$

which just cancels $\delta_\xi \mathcal{L}_H$.

I have now shown that

(1) $[\delta_\xi, \delta_\eta] e_m^a = 2i D_m (\bar{\xi} \gamma^a \eta)$

$[\delta_\xi, \delta_\eta] \psi_m = 2i \bar{\xi} \gamma^n \eta (D_n \psi_m - D_m \psi_n)$

(2) $\delta_\xi \mathcal{L} = 0$ up to terms of order $(\xi \psi^3)$

Except in the SUSY commutator on $\psi_m$, for which we needed a Fierz-~~identity~~, we used only arguments that are true in <u>any</u> <u>dimensionality</u>. So this gives the hope that we could construct higher-dimensional supergravity theories by a simple generalization of this technology.

In super-Yang Mills theory, we were able to identify several dimensions in which the field content

$$\lambda + A_m$$

contains equal numbers of fermion and boson particles. Could this also be true for the supergravity content $e_m^a + \psi_m$? To see, we need to count the particle content of each multiplet.

Begin with the bosons. As a warm-up recall that the gauge field $A_m$ contains $(d-2)$ particles, corresponding to polarization vectors transverse to the light-cone directions $p, \tilde{p}$. For $e_m^a$, we saw that the physical particles correspond to traceless symmetric tensors like polarized transverse to $p, \tilde{p}$. Thus:

$$e_m^a \rightarrow \frac{(d-2)(d-1)}{2} - 1 \quad \text{particles.}$$

In lecture 8, we analyzed the size of a minimal fermion - terms of particles. Call this $\chi$:

$$\chi = 2 \text{ in } d=4, \ 4 \text{ in } d=5,6 \ , \ldots$$

In 4 dimensions, we saw that the solutions to the Rarita Schwinger equation corresponded to (solution to the Dirac equation) $\times$ (vectors $u_m$ transverse to $p, \tilde{p}, \varepsilon_+$). This continues to be true in higher dimensions

$$\psi_m \rightarrow (d-3) \cdot \chi \quad \text{particles.}$$

Another way to see this is to consider $\psi_m$ as $\frac{\chi}{2}$ 4-d RS fields + $(d-4) \cdot \frac{\chi}{2}$ spin-1/2 fields in 4-dimensions.

$$\psi_m \rightarrow 2 \cdot \frac{\chi}{2} + (d-4) \cdot 2 \cdot \frac{\chi}{2} = (d-3) \cdot \chi$$

In any event, we can now tabulate the number of particles for $e_m^a$ and $\psi_m$ in a series of dimensions. I give also the results for $A_m$ and $\chi$ for comparison:

| $d$: | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|
| $A_m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $e_m^a$ | 2 | 5 | 9 | 14 | 20 | 27 | 35 | 44 |
| $\chi$ | 2 | 4 | 4 | 8 | 8 | 8 | 8 | 16 |
| $\psi_m$ | 2 | 8 | 12 | 32 | 40 | 48 | 56 | 128 |

So above 4 dimensions, there is no dimensionality in which $e_m^a$ and $\psi_m$ have the same number of particles. We must add bosons to obtain a supergravity theory with equal numbers of bosons and fermions.

In $d=5$, there is a relatively simple solution to the counting problem: add a vector boson!

$$e_m^a : 5 \quad + \quad A_m : 3 \quad = \quad \psi_m : 8$$

Cremmer and Chamseddine & Nicolai showed that this gives a complete supergravity theory in 5 dimensions.

Here are the transformation laws and the Lagrangian:

$$\mathcal{L} = -\frac{1}{2\kappa^2} R - \frac{i}{4} e \left[ \bar{\psi}_k^i \gamma^{klm} D_l \psi_m^i - D_l \bar{\psi}_k^i \gamma^{klm} \psi_m^i \right]$$

$$- \frac{1}{4} e \, F_{mn} F_{pq} \, g^{mp} g^{nq} \quad + \cdots$$

$$\delta e_m^a = -i\kappa\, \bar{\xi}^i \gamma^a \psi_m^i$$

$$\delta \psi_m = \frac{2}{\kappa} D_m \xi^i - \frac{1}{2\sqrt{6}}\left(\gamma^{pq}{}_m + 4\gamma^p \delta^q_m\right) F_{pq}\, \xi^i + \cdots$$

$$\delta A_m = +i\,\frac{\sqrt{6}}{2}\, \bar{\xi}^i \psi_m^i$$

Where ... refers to terms nonlinear in the gravitino and A fields that I will not keep track of here. $\xi^i$ is a symplectic-Majorana spinor in 5d, and $\psi_m^i$ is also symplectic-Majorana. s-M spinors in 5d satisfy:

$$\bar{\xi}^i \eta^i = -\bar{\eta}^i \xi^i \qquad \bar{\xi}^i \gamma^{abc} \eta^i = +\bar{\eta}^i \gamma^{abc} \xi^i$$
$$\bar{\xi}^i \gamma^a \eta^i = -\bar{\eta}^i \gamma^a \xi^i \qquad \bar{\xi}^i \gamma^{abcd} \eta^i = -\bar{\eta}^i \gamma^{abcd} \xi^i$$
$$\bar{\xi}^i \gamma^{ab} \eta^i = +\bar{\eta}^i \gamma^{ab} \xi^i \qquad \bar{\xi}^i \gamma^{abcde} \eta^i = -\bar{\eta}^i \gamma^{abcde} \xi^i$$

The (-1) signs are a test of consistency, since $\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 = -1$, so e.g. $\gamma^{abcd} = -\epsilon^{abcde}\gamma_e$

If we check $\delta_\xi \mathcal{L} = 0$, the cancellation between $e_m^a$ and $\psi_m$ occurs just as above. There are new terms involving $A_m$:

$$\delta_\xi \mathcal{L} = +ie\, D_l \bar{\psi}_k^i\, \gamma^{klm}\left(-\frac{1}{2\sqrt{6}}\right)\left(\gamma^{pq}{}_m + 4\gamma^p \delta^q_m\right) F_{pq}\, \xi^i$$
$$- e F^{mn}\left(+i\frac{\sqrt{6}}{2}\right) \bar{\xi}^i D_m \psi_n^i + \cdots$$

consider in particular the term in which, in the first line, all $\gamma$'s are contracted:

$$\gamma^{klm}\gamma^{pq}{}_m = -g^{kp}g^{lq}g^n{}_m - g^{lp}g^{mq}g^k{}_m - g^{mp}g^{kq}g^l{}_m + (p \leftrightarrow q)$$
$$= -g^{kp}g^{lq}(5-1-1)\cdot 2$$

so this gives

$$+ie\, D_\ell \bar{\Psi}^i_k \xi^i \left(-\frac{1}{2\sqrt{6}}\right)(-6)\, F^{k\ell}$$

$$= +ie\, F^{k\ell} \frac{\sqrt{6}}{2} (\bar{\xi}^i D_k \psi_\ell^i)$$

which just cancels the 2nd line above.

Now look at the SUSY commutators. On $e_m^a$, there is an additional term:

$$[\delta_\xi, \delta_\eta]\, e_m^a = D_m(2i\bar{\xi}\gamma^a\eta) + i\frac{K}{2\sqrt{6}} \bar{\eta}^i \gamma^a(\gamma^{pq}{}_m + 4\gamma^p \delta^q_m) F_{pq} \xi^i$$
$$- (\eta \leftrightarrow \xi)$$

In simplifying this term, we note that $\bar{\eta}\gamma^{cd}\xi - (\eta\leftrightarrow\xi) = 0$, so we only need to keep the terms w. 0 or 4 antisymmetrized $\gamma$'s. These are

$$+i\frac{K}{2\sqrt{6}} \bar{\eta}(\gamma^{apqm} + 4g^{ap}\delta^q_m)\, F_{pq}\, \xi - (\eta\leftrightarrow\xi)$$

$$= -\frac{iK}{\sqrt{6}}(\bar{\xi}\eta F^a{}_b e^b_m + \bar{\xi}\gamma^{pqab}\eta F_{pq} e^b_m)$$

$$= \delta_L \Omega^a{}_b e^b_m$$

so this term adds to the local Lorentz variation on the RHS of the SUSY CR. Similarly

$$[\delta_\xi, \delta_\eta]\, A_m = \delta_\xi\left(+i\frac{\sqrt{6}}{2} \bar{\eta}^i \psi_m^i\right) - (\xi\leftrightarrow\eta)$$

$$= +i\frac{\sqrt{6}}{2}\left(\bar{\eta}^i \frac{2}{K} D_m \xi^i - \frac{1}{2\sqrt{6}}\bar{\eta}^i(\gamma^{pq}{}_m + 4\gamma^p\delta^q_m) F_{pq}\xi^i\right.$$
$$- (\xi\leftrightarrow\eta)$$

by antisymmetry, we can drop the term w. $\gamma^{pq}{}_m$. Then

$$[\delta_\xi, \delta_\eta] A_m = \frac{\sqrt{6}}{K} \partial_m (i \bar{\xi} \eta) - i [\bar{\eta} \gamma^p \xi - \bar{\xi} \gamma^p \eta] F_{pm}$$

or

$$[\delta_\xi, \delta_\eta] A_m = 2i \bar{\xi} \gamma^p \eta F_{pm} + \partial_m \left( \frac{\sqrt{6}}{K} i \bar{\xi} \eta \right)$$

the first term has the desired form; the second is an additional U(1) gge transformation on $A_m$. Finally, look at the SUSY commutation on $\psi_m$:

$$[\delta_\xi, \delta_\eta] \psi_m = \frac{1}{2K} (\delta_\xi \omega_m^{bc}) \gamma^{bc} \eta^i - \frac{1}{2\sqrt{6}} (\gamma^{pq}{}_m + 4 \gamma^p \delta^q_m) \eta^i (\delta_\xi F_{pq}) - (\eta \leftrightarrow \xi)$$

assuming both variations can be made supercovariant, we obtain

$$= -\frac{i}{4} \gamma^{bc} \eta^i \left( 2 \bar{\xi}^j \gamma_c \psi_{mb} - \bar{\xi}^j \gamma_m \psi_{bc} \right)$$

$$- \frac{i}{4} (\gamma^{bc}{}_m + 4 \gamma^b e^c_m) \eta^i \; \bar{\xi}^j \psi^j_{bc} \quad - (\xi \leftrightarrow \eta)$$

Let me assume for the moment that this can be Fierz'd into the form

$$= +2i \; \bar{\xi}^j \gamma^c \eta^i \; \psi^j_{cm} - (\xi \leftrightarrow \eta)$$

then we could use:

$$\bar{\xi}^j \gamma^c \eta^i - \bar{\eta}^j \gamma^c \xi^i = \bar{\xi}^j \gamma^c \eta^i - (\eta^k)^T C c_{kj} \gamma^c c_{il} (\bar{C}^{-1}) [\bar{\xi}]^l$$

$$= \bar{\xi}^j \gamma^c \eta^i - \bar{\xi}^l \gamma^c \eta^k (\delta_{ki} \delta_{jl} - \delta_{kl} \delta_{ij})$$

$$= \bar{\xi}^j \gamma^c \eta^i - \bar{\xi}^j \gamma^c \eta^i + \bar{\xi} \gamma^c \eta \, \delta_{ij}$$

to obtain $[\delta_\xi, \delta_\eta] \psi_m = 2i \, \bar{\xi} \gamma^p \eta \; \psi_{pm}$ as desired.

to achieve this, we need to show that

$$-\tfrac{1}{4}\gamma^{bc}\eta\cdot(2\bar{\xi}\gamma_c\psi_{mb} - \bar{\xi}\gamma_m\psi_{bc})$$

$$-\tfrac{1}{4}\gamma^{bc}{}_m\eta\,\bar{\xi}\psi_{bc} - \gamma^b\eta\,\bar{\xi}\psi_{bm} = 2\bar{\xi}\gamma^c\eta\,\psi_{cm}$$

We may use the RS equation, since the relation is expected to be true on shell only. To check the relation, we need to check it for $\eta\bar{\xi} = I$ where $I$ is each element of a basis of $4\times 4$ matrices. A convenient set of choices is $\eta\bar{\xi} = (1, \gamma^d, \gamma^{ef})$; this is $1+5+10 = 16$ choices. So

for $\eta\bar{\xi} = 1$ $\quad$ RHS $= -2\,\mathrm{tr}[\gamma^c]\psi_{cm} = 0$

$$\text{LHS} = -\tfrac{1}{2}\gamma^{bc}\gamma_c\psi_{mb} + \tfrac{1}{4}\gamma^{bc}\gamma_m\psi_{bc} - \tfrac{1}{4}\gamma^{bc}{}_m\psi_{bc} - \gamma^b\psi_{bm}$$

$$= -\tfrac{1}{2}(+4)\gamma^b\psi_{mb} + \tfrac{1}{4}\gamma^{bcm}\psi_{bc} + \tfrac{1}{2}\gamma^b\psi_{bc}e^c_m - 0 - 0$$

$$= 0 + 0 + 0$$

$$= 0 \checkmark$$

for $\eta\bar{\xi} = \gamma^d$ $\quad$ (replace $\gamma^{bc}$ in the first term by $\gamma^b\gamma^c - \eta^{bc}$, $\eta^{bc}(2\gamma_c\psi_{mb} - \gamma_m\psi_{bc}) = 0$

$$\text{RHS} = -2\,\mathrm{tr}[\gamma^d\gamma^c]\psi_{cm} = -8\psi_{dm}$$

$$\text{LHS} = -\tfrac{1}{2}\gamma^b\gamma^c\gamma^d\gamma_c\psi_{mb} + \tfrac{1}{4}\gamma^{bc}\gamma^d\gamma_m\psi_{bc}$$

$$-\tfrac{1}{4}\gamma^{bc}{}_m\gamma^d\psi_{bc} - \gamma^b\gamma^d\psi_{bm}$$

$$= -\tfrac{1}{2}\gamma^b[-(5-2)\gamma^d]\psi_{mb} + \tfrac{1}{4}\left(2\gamma^{bcd}{}_m + 2(\gamma^b{}_m\eta^{cd} - \gamma^c{}_m\eta^{bd})\right)\psi_{bc}$$

$$+\left[\tfrac{1}{4}(-\gamma^{bd}\eta^c{}_m + \gamma^{cd}\eta^b{}_m) + \tfrac{1}{4}(\eta^b{}_m\eta^{cd} - \eta^{bd}\eta^c{}_m)\right]\psi_{bc}$$

$$-2\psi_{dm}$$

$$= -\frac{3}{2}\cdot 2\,\psi_{dm} + \frac{1}{2}\gamma^{bcd}{}_m\psi_{bc} + \underbrace{\gamma_m{}^c\psi_{dc}}_{= -\psi_{dm}}$$

$$-\frac{1}{2}\psi_{dm} - \frac{1}{2}\psi_{dm} - 2\psi_{dm}$$

now $\gamma^{bcd}{}_m\psi_{bc} = \gamma^d{}_m{}^{bc}\psi_{bc} = \gamma^d{}_m\psi_{bb} - \gamma^{db}\psi_{mb} + \gamma_m{}^b\psi_{db}$

$$= -2\psi_{dm}$$

so

$$\text{RHS} = +\psi_{dm}\left(-3 - 1 - 1 - \tfrac{1}{2} - \tfrac{1}{2} - 2\right) = -8\,\psi_{dm} \checkmark$$

for $\eta\bar{\xi} = \gamma^{ef}$

LHS $\propto$ tr $\gamma^{ef}\gamma^c = 0$

to evaluate the RHS, commute $\gamma^{ef}$ to the left

$$\gamma^b\gamma^{ef} = 2\eta^{be}\gamma^f - (e\leftrightarrow f)$$

$$\text{RHS} = -\frac{1}{2}\gamma^b\underbrace{\gamma^c\gamma^{ef}\gamma_c}_{=\gamma^{ef}\cdot(5-4)}\psi_{mb} + \frac{1}{4}\gamma^{bc}\gamma^{ef}\gamma_m\psi_{bc}$$

$$-\frac{1}{4}\gamma^{bc}{}_m\gamma^{ef}\psi_{bc} - \gamma^b\gamma^{ef}\psi_{bm}$$

$$= -\frac{1}{2}\gamma^{ef}\underbrace{\gamma^b\psi_{mb}}_{0} - \frac{1}{2}\left[2\eta^{be}\gamma^f\psi_{mb} - (e\leftrightarrow f)\right]$$

$$+\frac{1}{4}\gamma^{ef}\underbrace{\gamma^{bc}\gamma_m\psi_{bc}}_{0} + \frac{1}{4}\left[(\gamma^{bf}2\eta^{ec} + 2\eta^{be}\gamma^{fc})\gamma_m\psi_{bc} - (e\leftrightarrow f)\right]$$

$$-\frac{1}{4}\left[\gamma^{ef}\underbrace{\gamma^{bc}{}_m\psi_{bc}}_{0}\right] - \frac{1}{4}\left[(\gamma^{bcf}2e^e_m + \gamma^{bf}{}_m2\eta^{ec} + \gamma^{fc}{}_m2\eta^{eb})\psi_{bc}\right.$$

$$\left. - (e\leftrightarrow f)\right]$$

$$-\gamma^{ef}\underbrace{\gamma^b\psi_{bm}}_{0} - \left[2\eta^{eb}\gamma^f\psi_{bm} - (e\leftrightarrow f)\right]$$

$$= -\gamma^f \psi_{me} + \gamma^{bf}\gamma_m \psi_{be} - \tfrac{1}{2}\underbrace{\gamma^{bcf}\psi_{bc}}_{0} e_m^e - \gamma^{bf}{}_m \psi_{be}$$

$$- 2\gamma^f \psi_{em} \qquad - (e \leftrightarrow f)$$

now

$$\gamma^{bf}{}_m \psi_{be} = \gamma^f{}_m{}^b \psi_{be} = -\gamma^f \psi_{me} + \gamma_m \psi_{fe}$$

$$\gamma^{bf}\gamma_m \psi_{be} = 2e_m^f \gamma^b \psi_{be} - 2e_m^b \gamma^f \psi_{be} + \gamma_m \gamma^{bf}\psi_{be}$$

$$= 0 - 2\gamma^f \psi_{me} + \gamma_m \psi_{fe}$$

so

$$= -\gamma^f \psi_{me} - 2\gamma^f \psi_{me} + \cancel{\gamma_m \psi_{fe}} + \gamma^f \psi_{me} - \cancel{\gamma_m \psi_{fe}} + 2\gamma^f \psi_{me}$$

$$- (e \leftrightarrow f)$$

$$= 0 \quad \checkmark$$

so indeed the Fierz identity is satisfied and the SUSY commutator closes correctly on $\psi_m$.

You might think ~~that~~ this Fierz identity is pure technicalia, but actually it is the crucial step. It is the step that checks that the bosonic content ($e_m^a + A_m$) is what is needed to be supersymmetric with $\psi_m$.

The 5-d supergravity has one more notable surprise. The nonlinear terms include a Chern-Simons term:

$$\epsilon^{abcde} F_{ab} F_{cd} A_e$$

which is necessary for $\delta_\xi \mathcal{L} = 0$ at the nonlinear level.

There is one more case in which the content ($e_m^a + \psi_m$) has a relatively simple completion to a supersymmetric theory. This is in the case of 11 dimensions, the supergravity with maximal supersymmetry. In that case we can add a 3-index antisymmetric tensor gauge field $C_{mnp}$. Such a field has particles corresponding to 3-index antisymmetric tensors transverse to $p, \tilde{p}$. The number of such particles is:

$$\frac{(d-2)(d-3)(d-4)}{3!}$$

In 11-d:

$$e_m^a : 44 \qquad C_{mnp} : 84 \qquad \stackrel{!}{=} \qquad \psi_m \quad 128$$

The associated supergravity theory was constructed in a landmark paper of Cremmer, Scherk, and Julia. The supergravity Lagrangian begins with

$$\mathcal{L} = -\frac{1}{2\kappa^2} eR - i\frac{e}{4}\left(\bar{\psi}_k \gamma^{klm} D_l \psi_m - D_l \bar{\psi}_k \gamma^{klm} \psi_m\right)$$

$$- \frac{e}{48}\left(G_{klmn}\right)^2 + \cdots$$

where $G_{klmn} = \partial_k C_{lmn} - \partial_l C_{kmn} - \partial_m C_{lkn} - \partial_n C_{lmk}$ is the gauge-invariant field strength of $C_{lmn}$. The rest of the structure is quite similar to the 5-d case, including the presence of a Chern-Simons term:

$$\epsilon^{m_1 \cdots m_{11}} G_{m_1 \cdots m_4} G_{m_5 \cdots m_8} C_{m_9 m_{10} m_{11}}.$$

Cremmer Scherk and Julia actually discovered the field content of the 11-d supergravity by thinking about string theory, and, indeed, there is a close connection. The Type IIA string theory in 10-d has the following particle content: $i\,j$ etc $= 1-8$ transverse indices.

NS × NS sector : $\quad \phi \quad B_{ij} \quad g_{ij}$

R × R sector : $\quad A_i \quad C_{ijk}$

NS × R sector : $\quad \psi_i\,,\ \chi$

R × NS sector : $\quad \psi'_i\,, \chi'$ $\quad$ } spinors of opposite chirality

Cremmer Julia & Scherk noticed that these states assemble into particle multiplets in 11-d: $\quad I\,J = 1-9$

$$\phi \quad A_i \quad g_{ij} \quad \longrightarrow \quad G_{IJ}$$

$$B_{ij} \quad C_{ijk} \quad \longrightarrow \quad C_{IJK}$$

$$\psi_i \ \chi \quad \psi'_i \ \chi' \quad \longrightarrow \quad \psi_I$$

The last line notes the doubling of fermion states from 10 to 11 dimensions. After Witten's discovery that the string-coupling limit of the Type IIA theory is an 11-dimensional theory — M theory — it has been expected that 11-d supergravity is the zero-mass sector of this theory.

Supergravity theories in other dimensions require additional particles in both the boson and fermion sectors. Two important examples are:

6 - dimensions:

bosons: $e_m^a$ : 9 $\quad\phi$ : 1 "dilaton" $\quad B_{mn}$ : 6

fermions: $\psi_m$ : 12 $\quad\chi$ : 4

giving a thy with 16 + 16 particles

10 - dimensions:

bosons: $e_m^a$ : 35 $\quad\phi$ : 1 dilaton $\quad B_{mn}$ : 28

fermions: $\psi_m$ : 56 $\quad\chi$ : 8

giving a thy with 64 + 64 particles.

This 10-d thy is obtained in superstring thy by making an "orientifold" projection by the symmetry that interchanges the two sectors of the Type IIA thy. Similarly the classical 10-d supergravity can be obtained by dimensionally reducing the 11-d supergravity and then applying a $Z_2$ projection.