# 11. Supergravity

In the previous lecture, I introduced the formulation of gravity in terms of vierbein and spin connection and the Rarita - Schwinger equation for a spin-3/2 field. Ferrara, Freedman, and van Nieuwenhuizen put the pieces together and showed that the resulting theory is locally supersymmetric. This is supergravity.

$$\mathcal{L} = \mathcal{L}_H + \mathcal{L}_{RS}$$

$$= -\frac{1}{2\kappa^2} e R - \frac{1}{2}\left[ \psi^+_k \epsilon^{klmn} \sigma_l D_m \psi_n - D_m \psi^+_k \epsilon^{klmn} \sigma_l \psi_n \right]$$

viewed as a function of $e_m^{\ a}$, $\omega_m^{\ ab}$, $\psi_{m\alpha}$. In this lecture, I will demonstrate that this theory has local $N=1$ SUSY.

To begin, let me determine $\omega_m^{\ ab}$ in this theory. As in the previous example, I will treat $\omega_m^{\ ab}$ as an independent field in $\mathcal{L}$ and determine its equation by setting

$$\frac{\delta \mathcal{L}}{\delta \omega_m^{\ ab}} = 0 .$$

This will be a constraint equation that allows us to solve algebraically for $\omega_m^{ab}$. To study the variation w respect to $\omega$, I would like to again rewrite the Hilbert action as

$$\mathcal{L}_H = -\frac{1}{2K^2}\, e\, e_a^m e_b^n R_{mn}{}^{ab}$$

$$= +\frac{1}{8K^2}\, \epsilon^{klmn} \epsilon_{abcd}\, e_k^c e_l^d R_{mn}{}^{ab}$$

then

$$\delta\mathcal{L} = \frac{1}{\cancel{8}K^2}\, \epsilon^{klmn} \epsilon_{abcd}\, e_k^c e_l^d D_m \delta\omega_n{}^{ab} \cdot 2$$

$$- \frac{1}{2}\epsilon^{klmn}\left\{ \psi_k^+ \bar\sigma_l \tfrac{1}{2}\delta\omega_m^{ab}\sigma_{ab}\psi_n + \tfrac{1}{2}\delta\omega_m^{ab}\psi_k^+\bar\sigma_{ab}\bar\sigma_l\psi_n \right\}$$

$$= -\frac{1}{2K^2}\, \epsilon^{klmn} \epsilon_{abcd} (D_m e_k^c)\, e_l^d\, \delta\omega_n{}^{ab}$$

$$- \frac{1}{\cancel{2}4}\epsilon^{klmn}\, \delta\omega_m^{ab}\, e_l^d\, \psi_k^+ (\bar\sigma_d \sigma_{ab} + \bar\sigma_{ab}\bar\sigma_d)\psi_n$$

$$= \frac{1}{2K^2}\, \epsilon^{klmn}\, \delta\omega_m^{ab}\, e_l^d \left\{ \epsilon_{abcd} D_n e_k^c \right.$$

$$\left. - \frac{K^2}{2}\psi_k^+ (\bar\sigma_d\sigma_{ab} + \bar\sigma_{ab}\bar\sigma_d)\psi_n \right\}$$

now $\quad \bar\sigma_{ab}\bar\sigma_d + \bar\sigma_d\sigma_{ab} = -i\,\epsilon_{abdc}\,\bar\sigma^c$

$$= \frac{1}{2K^2}\, \epsilon^{klmn} \epsilon_{abcd}\, \delta\omega_m^{ab}\, e_l^d$$

$$\cdot \left\{ -D_k e_n^c - \frac{iK^2}{2}\psi_k^+ \bar\sigma^c \psi_n \right\}$$

so $$T_{mn}{}^c = D_m e_n^c - D_n e_m^c = -i\frac{k^2}{2}\left(\psi_m^+ \bar\sigma^c \psi_n - \psi_n^+ \bar\sigma^c \psi_m\right)$$

(more conveniently: $D_{[m} e_{n]}^c = -i\, k^2/2\ \psi_{[m}^+ \bar\sigma^c \psi_{n]}$ )

The torsion is fixed to be nonzero and a natural bilinear in the gravitino fields. Nevertheless, we can solve explicitly for $\omega_m{}^{ab}$ in terms of $e_m^a$ and $\psi_m$ as indicated in the previous lecture.

Looking back at the gravitino field equation from the previous lecture:

$$\epsilon^{klmn} T_{ml}{}^b \bar\sigma_b \psi_n = -ik^2 \epsilon^{klmn} \bar\sigma_b \psi_n \ \psi_m^+ \bar\sigma^b \psi_l$$

$$\text{Fierz id.} \quad = -ik^2 \epsilon^{klmn} (-2)\, c\, \psi_m^* \left(\psi_l^T c\, \psi_n\right)$$

$$= 0$$

since $\psi_l^T c\, \psi_n$ is symmetric under $(l \leftrightarrow n)$

so the gravitino field equation is

$$-\epsilon^{klmn} \bar\sigma_l D_m \psi_n = 0$$

as promised.

Now I would like to show that $\mathcal{L}$ is supersymmetric. In fact, we will consider local supersymmetry, with a susy parameter $\xi(x)$ that depends on $x$. The transformation laws are:

$$\delta_\xi e_m^a = \kappa\left(\xi^\dagger \bar\sigma^a \psi_m + \psi_m^\dagger \bar\sigma^a \xi\right)$$

$$\delta_\xi \psi_m = -2\frac{i}{\kappa} D_m \xi$$

The transformation of $e_m^a$ is similar to that of $A_m$ in an N=1 SUSY gauge theory. However, now $\xi$ is a function of $x$. The transformation of $\psi_m$ includes dependence on $\partial_m\xi$, so $\psi_m$ is the gauge field of local SUSY. Notice ~~that~~ the SUSY invariance will generate the gauge invariance of the Rarita-Schwinger equation.

What is the transformation law for $\omega_m^{ab}$? This does not have to be postulated independently. Rather, we solve for $\omega_m^{ab}$ in terms of $e_m^a$ and $\psi_m$ using ~~the~~ torsion constraint and then plug in the above expressions. After a lengthy computation, this gives

$$\delta_\xi \omega_m^{bc} = \frac{\kappa}{2} e_m^a \left\{ \xi^\dagger \bar\sigma_c \psi_{ab} + \xi^\dagger \bar\sigma_b \psi_{ca} - \xi^\dagger \bar\sigma_a \psi_{bc} \right.$$
$$\left. + \psi_{ab}^\dagger \bar\sigma_c \xi + \psi_{ca}^\dagger \bar\sigma_b \xi - \psi_{bc}^\dagger \bar\sigma_a \xi \right\}$$

where $\psi_{ab} = e_a^m e_b^n (D_m\psi_n - D_n\psi_m)$. The minus signs look odd, but they are correct.

Now I would like to show that

(1) $$[\delta_\xi, \delta_\eta] = 2i(\xi^\dagger\bar\sigma^m\eta - \eta^\dagger\bar\sigma^m\xi)\,\partial_m + \text{(possible gauge terms)}$$

(2) $$\delta_\xi \mathcal{L} = 0$$

As before, (2) is easier and I'll do that first.

In computing $\delta_\xi\mathcal{L}$, we have three terms:

$$\delta_\xi\mathcal{L} = \frac{\delta\mathcal{L}}{\delta e_m^a}\,\delta_\xi e_m^a + \frac{\delta\mathcal{L}}{\delta\omega_m^{ab}}\,\delta_\xi\omega_m^{ab} + \frac{\delta\mathcal{L}}{\delta\Psi_m}\,\delta_\xi\Psi_m$$

However, $\omega_m^{ab}$ has been determined by the condition $\frac{\delta\mathcal{L}}{\delta\omega_m{}^{ab}} = 0$, so the second term vanishes. We need only compute the terms w. explicit variation of $e_m^a$ or $\Psi_m$.

Then

R is a function of $\underline{\omega}$ only.

$$\delta_\xi\mathcal{L}_H = \frac{1}{4K^2}\,\epsilon^{klmn}\,\epsilon_{abcd}\;\delta e_k^c\;e_l^d\;R_{mn}{}^{ab}$$

$$= \frac{1}{4K^2}\,\epsilon^{klmn}\,\epsilon_{abcd}\cdot K\cdot(\xi^\dagger\bar\sigma^c\Psi_k + \Psi_k^\dagger\bar\sigma^c\xi)\,e_l^d\,R_{mn}{}^{ab}$$

The variation of $\mathcal{L}_{RS}$ is somewhat more complicated. We have to vary $\Psi_k^\dagger$, $\Psi_n$, and also the $e_l^d$ inside $\bar\sigma_l = e_l^d\bar\sigma_d$. I would like to do this in two steps: first ignore the presence of this $e_l^d$ and then correct for it. Call these pieces (a) and (b)

$$\delta_\xi \mathcal{L}_{RS}(a) = -\tfrac{1}{2}\Big[ \psi_k^+ \epsilon^{klmn} \bar\sigma_l D_m \left(-\tfrac{2i}{\kappa} D_n \xi\right)$$
$$+ \left(\tfrac{2i}{\kappa} D_k \xi^+\right) \epsilon^{klmn} \bar\sigma_l D_m \psi_n$$
$$- D_m \psi_k^+ \epsilon^{klmn} \bar\sigma_l \left(-\tfrac{2i}{\kappa} D_n \xi\right)$$
$$- \left(\tfrac{2i}{\kappa} D_m D_k \xi^+\right) \epsilon^{klmn} \bar\sigma_l \psi_n \Big]$$

It will be useful to ~~integrate~~ by parts in the middle two terms. To do this, we have to pass $D_k$ or $D_n$ through $e_l^d$. Put these terms into $\delta_\xi \mathcal{L}_{RS}(b)$. With this revision

$$\delta_\xi \mathcal{L}_{RS}(a) = \left(-\tfrac{1}{2}\right)\left(-\tfrac{2i}{\kappa}\right) \epsilon^{klmn}$$
$$\{ \psi_k^+ \bar\sigma_l D_m D_n \xi + \xi^+ \bar\sigma_l D_k D_m \psi_n$$
$$+ (D_n D_m \psi_k^+) \bar\sigma_l \xi + D_m D_k \xi^+ \bar\sigma_l \psi_n \}$$

now replace

$$D_{[m} D_{n]} \xi = \tfrac{1}{2} [D_m, D_n] \xi = \tfrac{1}{2} \cdot \tfrac{1}{2} R_{mn}{}^{ab} \sigma_{ab} \xi$$

$$= \frac{i}{\kappa} \epsilon^{klmn} e_l^d \{ \psi_k^+ \bar\sigma_d \tfrac{1}{4} R_{mn}{}^{ab} \sigma_{ab} \xi + \xi^+ \bar\sigma_d \tfrac{1}{4} R_{km}{}^{ab} \sigma_{ab} \psi_n$$
$$- \tfrac{1}{4} R_{nm}{}^{ab} \psi_k^+ \bar\sigma_{ab} \bar\sigma_d \xi - \tfrac{1}{4} R_{mk}{}^{ab} \xi^+ \bar\sigma_{ab} \bar\sigma_d \psi_n \}$$

$$\delta_\xi \mathcal{L}_{RS}(a) = \frac{i}{4K} \epsilon^{klmn} e_l^{\ d} R_{mn}^{\ \ ab}$$

$$\{ \psi_k^+ \bar\sigma_d \sigma_{ab} \xi + \psi_k^+ \bar\sigma_{ab} \bar\sigma_d \xi$$

$$+ \xi^+ \bar\sigma_d \sigma_{ab} \psi_k + \xi^+ \bar\sigma_{ab} \bar\sigma_d \psi_k \}$$

how use $\quad \bar\sigma_{ab} \bar\sigma_d + \bar\sigma_d \sigma_{ab} = -i \, \epsilon_{abdc} \bar\sigma^c$

$$= +i \, \epsilon_{abcd} \bar\sigma^c$$

$$= -\frac{1}{4K} \epsilon^{klmn} e_l^{\ d} R_{mn}^{\ \ ab} \epsilon_{abcd}$$

$$\{ \psi_k^+ \bar\sigma^c \xi + \xi^+ \bar\sigma^c \psi_k \}$$

so this precisely cancels $\delta_\xi \mathcal{L}_H$. Now we just have to clean up

$$\delta_\xi \mathcal{L}_{RS}(b) = (\text{terms from integration by parts})$$

$$+ (\text{terms from } \delta e_l^{\ d} \text{ in } \mathcal{L}_{RS})$$

$$= \left(-\frac{1}{2}\right)\left(\frac{-2i}{K}\right) \epsilon^{klmn} \xi^+ (D_k e_l^{\ d}) \bar\sigma_d D_m \psi_n$$

$$+ \left(-\frac{1}{2}\right)\left(\frac{-2i}{K}\right) \epsilon^{klmn} (D_m \psi_k^+)(D_n e_l^{\ d}) \bar\sigma_d \xi$$

] Int. by parts.

$$-\frac{1}{2} \epsilon^{klmn} (\psi_k^+ \bar\sigma_d D_m \psi_n - D_m \psi_k^+ \bar\sigma_d \psi_n) \, K (\xi^+ \bar\sigma^d \psi_l + \psi_l^+ \bar\sigma^d \xi)$$

$$\delta_\xi \mathcal{L}_{RS}(b) = \frac{i}{K}\epsilon^{klmn}(\xi^\dagger \bar\sigma_d D_m \psi_n)\,(D_k e_l^d)$$
$$+ \frac{i}{k}\epsilon^{klmn}(D_m\psi_k^\dagger \bar\sigma_d \xi)\,(D_m e_l^d)$$
$$-\frac{K}{2}\epsilon^{klmn}(\psi_k^\dagger\bar\sigma_d D_m\psi_n - D_m\psi_k^\dagger\bar\sigma_d\psi_n)\,(\xi^\dagger\bar\sigma^d\psi_l + \psi_l^\dagger\bar\sigma^d\xi)$$

In the first two terms, replace

$$D_{[k}e_{l]}^d = -i\frac{K^2}{2}\,\psi_{[k}^\dagger\bar\sigma^c\psi_{l]}$$

In the last term, Fierz! Note that

$$\psi_k^\dagger\bar\sigma_d D_m\psi_n\;\psi_l^\dagger\bar\sigma^d\xi = -2\,\psi_k^\dagger c\,\psi_l^*\;\xi^T c\,D_m\psi_n$$
$$= 0$$

since $\psi_k^\dagger c\,\psi_l^*$ is symmetric under $(k\leftrightarrow l)$

Similarly, the term with $\bar\sigma_d\psi_n \cdots \bar\sigma^d\psi_l$ vanishes. What remains is

$$\delta_\xi \mathcal{L}_{RS}(b) = \frac{i}{K}\epsilon^{klmn}(\xi^\dagger\bar\sigma_d D_m\psi_n)\left(-i\frac{K^2}{2}\right)(\psi_k^\dagger\bar\sigma^d\psi_l)$$
$$+ \frac{i}{k}\epsilon^{klmn}(D_m\psi_n^\dagger\bar\sigma_d\xi)\left(-i\frac{K^2}{2}\right)(\psi_n^\dagger\bar\sigma^d\psi_l)$$
$$-\frac{K}{2}\epsilon^{klmn}\left\{\psi_k^\dagger\bar\sigma^d\psi_l\;\xi^\dagger\bar\sigma_d D_m\psi_n - D_m\psi_n^\dagger\bar\sigma_d\xi\;\psi_l^\dagger\bar\sigma^d\psi_n\right\}$$
$$= \frac{K}{2}\epsilon^{klmn}\left\{(\xi^\dagger\bar\sigma_d D_m\psi_n)(\psi_k^\dagger\bar\sigma^d\psi_l) - (\xi^\dagger\bar\sigma_d D_m\psi_n)(\psi_k^\dagger\bar\sigma^d\psi_l)\right.$$
$$\left. + (D_m\psi_n^\dagger\bar\sigma_d\xi)(\psi_n^\dagger\bar\sigma^d\psi_l) + (D_m\psi_n^\dagger\bar\sigma_d\xi)(\psi_l^\dagger\bar\sigma^d\psi_n)\right\}$$

and everything cancels! Thus, finally

$$\delta_\xi \mathcal{L} = 0$$

Now study the closure of the SUSY commutators. Let's compute

$$[\delta_\xi, \delta_\eta]\, e_m{}^a$$
$$= \delta_\xi\ \kappa(\eta^\dagger \bar\sigma^a \psi_m + \psi_m^\dagger \bar\sigma^a \eta) \quad - \quad (\xi \leftrightarrow \eta)$$
$$= \kappa\{ \eta^\dagger \bar\sigma^a (-\tfrac{2i}{\kappa}) D_m \xi \ + \ \tfrac{2i}{\kappa} D_m \xi^\dagger \bar\sigma^a \eta \} \ - (\xi \leftrightarrow \eta)$$
$$= 2i \left[ (D_m \xi^\dagger) \bar\sigma^a \eta \ + \ \xi^\dagger \bar\sigma^a (D_m \eta) \right] \ - (\xi \leftrightarrow \eta)$$

I would like to manipulate this expression in a somewhat non-obvious way.

$$= 2i\ \partial_m (\xi^\dagger \bar\sigma^a \eta) \ + \ 2i\ \tfrac{1}{2} \omega_m{}^{cd}\ \xi^\dagger (\bar\sigma^a \sigma_{cd} - \bar\sigma_{cd} \bar\sigma^a) \eta$$
$$- (\xi \leftrightarrow \eta)$$

now $\quad \bar\sigma^a \sigma_{cd} - \bar\sigma_{cd} \bar\sigma^a = \eta_{ac} \bar\sigma_d - \eta_{ad} \bar\sigma_c$

$$= 2i\ \partial_m (\xi^\dagger \bar\sigma^n \eta\ e_n{}^a) \ + \ 2i\ \omega_m{}^a{}_b\ \xi^\dagger \bar\sigma^b \eta \ - (\xi \leftrightarrow \eta)$$
$$= 2i\ \partial_m (\xi^\dagger \bar\sigma^n \eta)\ e_n{}^a \ + \ 2i\ \xi^\dagger \bar\sigma^n \eta\ (\partial_m e_n{}^a)$$
$$+ \ 2i\ \omega_m{}^a{}_b\ \xi^\dagger \bar\sigma^b \eta \ - \ (\xi \leftrightarrow \eta)$$

$$= \partial_m (2i\,\xi^\dagger\bar\sigma^n\eta)\, e_n{}^a + 2i(\xi^\dagger\bar\sigma^n\eta)\,\partial_n e_m{}^a$$

$$+\ 2i(\xi^\dagger\bar\sigma^n\eta)(\partial_m e_n{}^a - \partial_n e_m{}^a)$$

$$+\ 2i(\xi^\dagger\bar\sigma^n\eta)\,\omega_m{}^{ab} e_n{}^b - 2i\,\xi^\dagger\bar\sigma^n\eta\,\omega_n{}^{ab} e_m{}^b$$

$\sim$ = term added + subtracted

$$+\ 2i\,\xi^\dagger\bar\sigma^n\eta\,\omega_n{}^{ab} e_m{}^b$$

$$-(\xi\leftrightarrow\eta)$$

$$= \partial_m (2i\,\xi^\dagger\bar\sigma^n\eta)\, e_n{}^a + (2i\,\xi^\dagger\bar\sigma^n\eta)\,\partial_n e_m{}^a$$

$$+\ 2i(\xi^\dagger\bar\sigma^n\eta)(D_m e_n{}^a - D_n e_m{}^a)$$

$$+\ 2i\,\xi^\dagger\bar\sigma^n\eta\,\omega_n{}^{ab} e_m{}^b - (\xi\leftrightarrow\eta)$$

& substitute for the torsion

$$= \partial_m (2i\,\xi^\dagger\bar\sigma^n\eta)\, e_n{}^a + (2i\,\xi^\dagger\bar\sigma^n\eta)\,\partial_n e_m{}^a$$

$$+\ 2i(\xi^\dagger\bar\sigma^n\eta)\left(-\frac{i\kappa^2}{2}\right)(\psi_m^\dagger\bar\sigma^a\psi_n - \psi_n^\dagger\bar\sigma^a\psi_m)$$

$$+\ (2i\,\xi^\dagger\bar\sigma^n\eta)\,\omega_n{}^{ab} e_{mb} - (\xi\leftrightarrow\eta)$$

this expression has now come into the following form:

$$[\delta_\xi, \delta_\eta]\, e_m{}^a = \partial_m(a^n)\, e_n{}^a + a^n \partial_n e_m{}^a$$
$$+ \kappa \left( \Xi^\dagger \bar\sigma^a \psi_m + \psi_m^\dagger \bar\sigma^a \Xi \right)$$
$$+ \Omega^{ab} e_{mb}$$

The first line is a local translation with parameter

$$a^n(x) = 2i \left( \xi^\dagger \bar\sigma^n \eta - \eta^\dagger \bar\sigma^n \xi \right) \qquad \text{as \underline{required}}$$

The second line is a local SUSY transformation with parameter

$$\Xi(x) = \kappa \left( \xi^\dagger \bar\sigma^n \eta - \eta^\dagger \bar\sigma^n \xi \right) \psi_n$$

similar to the gauge term in $[\delta_\xi, \delta_\eta] A_m$ in the SUSY action on a gauge field. The third term is a local Lorentz rotation with parameter $\Omega^{ab}(x) = 2i\left(\xi^\dagger \bar\sigma^n \eta - \eta^\dagger \bar\sigma^n \xi\right) \omega_n{}^{ab}$

So the commutator of local SUSY transformations closes onto a combination of local <u>translations</u>, local SUSY, and local Lorentz, at least on $e_m^a$. To complete the proof of closure, we need to show:

$$[\delta_\xi, \delta_\eta] \psi_m = \partial_m(a^n)\psi_n + a^n \partial_n \psi_m$$
$$- \frac{2i}{\kappa} D_m \Xi + \frac{1}{2} \Omega^{ab} \sigma_{ab} \psi_m$$

with the same parameters as above.

In fact, we will not be able to show this without using the gravitino equation of motion. In the system as we have described it, the SUSY commutators close only on shell. However, assume the gravitino equation and proceed.

$$[\delta_\xi, \delta_\eta]\, \psi_m = \delta_\xi\left(-\frac{2i}{\kappa} D_m \eta\right) - (\xi \leftrightarrow \eta)$$

$$= \left(-\frac{2i}{\kappa}\right) \frac{1}{2}(\delta_\xi \omega_m{}^{bc})\, \sigma_{bc}\, \eta - (\xi \leftrightarrow \eta)$$

$$= -\frac{i}{\kappa}(\sigma_{bc}\eta)(\delta_\xi \omega_m{}^{bc}) - (\xi \leftrightarrow \eta)$$

Now, I gave $\delta_\xi \omega_m{}^{bc}$ on p. 4. To make progress, it will be useful to simplify that expression using the field equation in the form

$$\bar\sigma_c \psi_{ab} + \bar\sigma_b \psi_{ca} \overset{!}{+} \bar\sigma_a \psi_{bc} = 0$$

so, with the field equation

$$\delta_\xi \omega_m{}^{bc} = \kappa\, e_m^a \left\{ \xi^\dagger \bar\sigma_c \psi_{ab} + \xi^\dagger \bar\sigma_b \psi_{ca} + \psi_{ab}^\dagger \bar\sigma_c \xi + \psi_{ca}^\dagger \bar\sigma_b \right\}$$

since this is contracted with a structure antisymmetric made $(b \leftrightarrow c)$

$$[\delta_\xi, \delta_\eta] \psi_m = -2i \, (\sigma_{bc} \eta) \, e_m^a \{ \xi^\dagger \bar\sigma_c \psi_{ab} + \psi_{ab}^\dagger \bar\sigma_c \xi \} - (\xi \leftrightarrow \eta)$$

next rewrite

$$\sigma_{bc} \eta = \tfrac{1}{4} [\sigma_b \bar\sigma_c - \sigma_c \bar\sigma_b] \eta$$
$$= \tfrac{1}{2} \sigma_b \bar\sigma_c \eta - \tfrac{1}{2} \eta_{bc} \eta$$

and use the field equation again in the form $\bar\sigma^c \psi_{ac} = 0$

$$= -i \, \sigma_b \bar\sigma_c \eta \, (\xi^\dagger \bar\sigma_c \psi_{ab} + \psi_{ab}^\dagger \bar\sigma_c \xi) \, e_m^a - (\xi \leftrightarrow \eta)$$

now Fierz; the term with

$$\bar\sigma_c \eta \; \psi_{ab}^\dagger \bar\sigma_c \xi = -2 \, c \, \psi_{ab}^\alpha \; \xi^T c \eta$$

$\to 0$ after antisymmetrizing on $(\xi \leftrightarrow \eta)$

the other term gives

$$= -i \, (\xi^\dagger \bar\sigma_c \eta) \, (\sigma^b \bar\sigma_c \psi_{ab}) \, e_m^a - (\xi \leftrightarrow \eta)$$

Finally, replace $\sigma^b \bar\sigma_c \psi_{ab} = 2 \eta^b{}_c \, \psi_{ab} - \sigma_c \bar\sigma^b \psi_{ab}$

and use the field equation to set $\bar\sigma^b \psi_{ab} = 0$.

Then

$$[\delta_\xi, \delta_\eta]\, \psi_m = -2i\,(\xi^\dagger \bar\sigma^c \eta - \eta^\dagger \bar\sigma^c \xi)\; e_m^a\, \psi_{ac}$$

$$= -2i\,(\xi^\dagger \bar\sigma^n \eta - \eta^\dagger \bar\sigma^n \xi)\; \psi_{mn}$$

$$= 2i\,(\xi^\dagger \bar\sigma^n \eta - \eta^\dagger \bar\sigma^n \xi)\,(D_n \psi_m - D_m \psi_n)$$

$$= 2i\,(\xi^\dagger \bar\sigma^n \eta - \eta^\dagger \bar\sigma^n \xi)\; \partial_n \psi_m$$

$$+\; 2i\,(\xi^\dagger \bar\sigma^n \eta - \eta^\dagger \bar\sigma^n \xi)\; \tfrac{1}{2}\omega_n^{ab}\, \sigma_{ab} \psi_m$$

$$-\,2i\, D_m \left( (\xi^\dagger \bar\sigma^n \eta - \eta^\dagger \bar\sigma^n \xi)\, \psi_n \right)$$

$$+\; 2i\, \partial_m (\xi^\dagger \bar\sigma^n \eta - \eta^\dagger \sigma^n \xi)\; \psi_n$$

this has now taken the correct form.

$$[\delta_\xi, \delta_\eta]\, \psi_m = a^n \partial_n \psi_m + \tfrac{1}{2}\Omega^{ab}\sigma_{ab}\psi_m$$

$$-\, \frac{2i}{K} D_m \Xi \;+\; \partial_m (a^n)\, \psi_n$$

and our proof is complete.

To couple supergravity to matter, it will be very useful to have a system that realizes local SUSY off-shell. For this, we must add some auxiliary fields to the physical supergravity fields $e_m^a$, $\psi_m$. It is not so hard to count the off-shell degrees of freedom needed:

$$e_m^a = 16 \text{ components} - (\underset{\text{local Lorentz}}{6} + \underset{\text{local translation}}{4} \text{ gauge degrees of freedom}) = 6$$

$$\psi_m = 4 \times 2 \text{ complex} = 16 \text{ real components}$$

$$- \begin{pmatrix} 2 \text{ complex gauge d. of freedom} \\ = 4 \text{ real} \end{pmatrix} = 12$$

Rarita Schwinger

So we need 6 more bosonic auxiliary fields. A possible choice is a <u>vector</u> plus a <u>complex</u> scalar:

$$e_m^a \quad \psi_m \quad b_m \quad M$$

I will describe the theory of this system in a future lecture.