10. Formalism of Gravity and Spin - 3/2

We have now come to the point in the course where we must leave renormalizable theories with spins ≤ 1 and go on to higher spin. Actually, there are three separate questions that it is appropriate to ask at this point:

① The universe includes gravity and, by our earlier analysis, we must make every field in the universe supersymmetric. So, how do we supersymmetrize gravity?

② By the same token, can we supersymmetrize spin - 3/2?

③ So far, we have discussed only global or rigid SUSY transformations. Can we make SUSY a local symmetry with an appropriate gauge field?

All of these questions are answered by a common theory — supergravity. In this theory, the supersymmetric particle multiplet is a massless spin - 2 particle, the graviton, plus a massless spin - 3/2 particle, the gravitino. In gravity, local translations are part of the gauge group. Since SUSY transformations (anti-) commute to translations, we need also local SUSY. The gauge particle of local SUSY must have the quantum numbers of the

SUSY current $J_{m\alpha}$ and so it must be a spin-3/2 field. So all the pieces fall into place.

In order to write the Lagrangian and verify the SUSY invariance, we need first to clarify some elements of the formalism. You are probably familiar with Einstein's theory of gravity. But it is less familiar how to couple this theory to spinors, which we will have to do to include SUSY. Also, I would like to explain some properties of the Rarita - Schwinger field equation that describes spin 3/2.

First of all, I would like to write a formalism for general relativity that can incorporate spinor fields, or fields with any Lorentz indices. To describe the parallel transport of such a field, we need a covariant derivative that can act on any spin. Such a derivative can be build from a gauge field whose gauge group is the Lorentz group. So write

$$D_m = \left(\partial_m + \tfrac{1}{2}\omega_m^{\ ab} M_{ab}\right)$$

where $\omega_m^{\ ab}$ is a gauge field of $SO(3,1)$ ($SO(d-1,1)$) and M_{ab} is a finite-dimensional representation of the Lorentz algebra. D_m can be applied to

vector: $$D_m V^a = \partial_m V^a + \omega_m{}^a{}_b V^b$$

spinor: $$D_m \psi = \partial_m \psi + \tfrac{1}{2}\omega_m^{\ ab}\sigma_{ab}\psi$$

$$D_m \psi^\dagger = \partial_m \psi^\dagger - \tfrac{1}{2}\omega_m^{\ ab}\psi^\dagger\bar{\sigma}_{ab}$$

$\omega_m{}^{ab}$ is called the "spin connection".

The spin connection is somehow connected to the metric on curved space. As an intermediate step, I would like to introduce an object representing a local orthonormal frame erected at each point in space.

Label coordinate indices ("world indices") as $m\,n\,p\,\ldots$

Label indices in the orthonormal frame ("tangent space indices") as $a, b, c, \ldots$

Let the change of basis from one to the other be

$$e_m^{\;a}(x)$$

In 4-d, $a, m = 0123$, so this object is called a "vierbein". In $d > 4$, it is a "vielbein". The element of path length is

$$ds^2 = dx^a \eta_{ab}\, dx^b = dx^m g_{mn}(x)\, dx^n$$

so
$$e_m^{\;a} \eta_{ab}\, e_n^{\;b} = g_{mn}(x)$$

Define the inverse of $e_m^{\;a}$ as $e_a^{\;m}$; then

$$e_m^{\;a} e_b^{\;m} = \delta_b^a \quad \text{or} \quad e_m^{\;a} e^{mb} = \eta^{ab}$$

I will define the matrix M_{ab} in D_m to act only on tangent-space indices. To differentiate world indices, we use the covariant derivative of general relativity

$$\nabla_m V^n = \partial_m V^n + \Gamma^n_{mp} V^p$$

$$\nabla_m V_n = \partial_m V_n - \Gamma^p_{mn} V_p$$

where Γ^p_{mn} is the "affine connection", defined to obey

$$\nabla_m g_{np} = 0$$

With v-beins, this is insured by writing

$$\partial_m e^a_n - \Gamma^p_{mn} e^a_p + \omega_m{}^a{}_b e^b_n = 0$$

and this equation allows us to solve for Γ^p_{mn} in terms of ω_m^{ab}.

I hope you also recall from general relativity ~~that~~ antisymmetric derivatives

$$\partial_m V_n - \partial_n V_m$$

already transform like covariant tensors and do not need Γ. I will use this observation to omit Γ^p_{mn} whenever possible, for example, in the electromagnetic field strength

$$F_{mn} = \partial_m A_n - \partial_n A_m$$

We now need to write the equation that relates ω_m^{ab} to e_m^a.

Consider the torsion, defined by

$$T_{mn}{}^a = D_m e_n^a - D_n e_m^a$$

$$= \partial_m e_n^a - \partial_n e_m^a + \omega_m{}^a{}_b e_n^b - \omega_n{}^a{}_b e_m^b$$

This is a tensor built of e_n^a and ω_m^{ab}. Notice that it has dimension $(\text{mass})^1$, whereas the curvature tensor R has dimension $(\text{mass})^2$. In general relativity books, there is a page in which Einstein is riding in an elevator, and the argument is given that the first visible effects of gravity should enter proportional to the local curvature, i.e. proportional to $(\Delta x)^2 R$. This requires

$$T_{mn}{}^a = 0$$

Since $T_{mn}{}^a$ is a tensor, this is a sensible covariant condition. I'll come back later to the question of whether this is the right condition to place on ω and e.

In any case, if $T_{mn}{}^a$ has an explicit value, we can use this value to solve for ω in terms of e: Write

$$T_{abc} = e_a^m e_b^n T_{mn}{}^c$$

$$C_{abc} = e_a^m e_b^n (\partial_m e_n^c - \partial_n e_m^c)$$

Then the equation for T becomes:

(with 3 permutations of indices)

$$C_{abc} - T_{abc} + \omega_{acb} - \omega_{bca} = 0$$
$$-C_{bca} + T_{bca} - \omega_{bac} + \omega_{cab} = 0$$
$$C_{cab} - T_{cab} + \omega_{cba} - \omega_{abc} = 0$$

add the equations, using $\omega_{abc} = -\omega_{acb}$

$$(C_{abc} - T_{abc}) - (C_{bca} - T_{bca}) + (C_{cab} - T_{cab}) - 2\omega_{abc} = 0$$

or

$$\omega_m{}^{bc} = \frac{1}{2} e_m{}^a \left((C-T)_{abc} - (C-T)_{acb} - (C-T)_{bca} \right)$$

Be careful with the structure here: $\omega_m{}^{bc}$ is antisymmetric on the last 2 indices, but C and T are antisymmetric on the first 2 indices.

You can check (it is somewhat lengthy) that setting $T = 0$ and inserting the resulting expression for ω into

$$\Gamma_{mn}^p = e_a^p \left(\partial_m e_n^a + \omega_m{}^a{}_b e_n^b \right)$$

gives

$$\Gamma_{mn}^p = \frac{1}{2} g^{pq} \left(\partial_m g_{nq} + \partial_n g_{mq} - \partial_q g_{mn} \right)$$
$$= \left\{ {p \atop mn} \right\}$$

the usual expression in general relativity. When $T_{mn}{}^a \neq 0$, there are extra terms in $\Gamma_{mn}{}^{p}$ and, in particular

$$\Gamma_{mn}{}^{p} - \Gamma_{nm}{}^{p} = T_{mn}{}^{p}$$

i.e. $\Gamma_{mn}{}^{p}$ is no longer symmetric in $\{mn\}$.

The curvature tensor is conveniently defined in terms of $\omega_m{}^{ab}$. Curvature is the rotation obtained by parallel-transporting about a small square. So we should define the curvature tensor by

$$[D_m, D_n] = \frac{1}{2} R_{mn}{}^{ab} M_{ab}$$

this gives

$$R_{mn}{}^{ab} = \partial_m \omega_n{}^{ab} - \partial_n \omega_m{}^{ab} + \omega_m{}^{a}{}_{c}\,\omega_n{}^{cb} - \omega_n{}^{a}{}_{c}\,\omega_m{}^{cb}$$

In this way, $R_{mn}{}^{ab}$ is the <u>field</u> strength associated with the spin connection gauge field. You could also insert

$$\omega_m{}^{ab} = e^{nb}\left(\Gamma_{mn}{}^{p}\, e_p{}^{a} - \partial_m e_n{}^{a}\right)$$

and find that

$$R_{mn}{}^{ab} = e_p^{a}\, e^{qb}\left(\partial_m \Gamma_n{}^{p}{}_{q} - \partial_n \Gamma_m{}^{p}{}_{q} + \Gamma_m{}^{p}{}_{r}\Gamma_n{}^{r}{}_{q} - \Gamma_n{}^{p}{}_{r}\Gamma_m{}^{r}{}_{q}\right)$$

the usual expression in general relativity.

Now we have to write an action for gravity. In principle, we could write a dimension - 4 term:

$$\mathcal{L} \propto \left(R_{mn}{}^{ab}\right)^2$$

as in Yang-Mills theory. However, the lowest-dimension possibility is the Hilbert action

$$\mathcal{L}_H = -\frac{1}{2K^2}\, e\, R$$

where $e = \det e_m^a = \sqrt{-\det g_{mn}} = \sqrt{-g}$

and

$$R = e_a^m e_b^n R_{mn}{}^{ab} \qquad \text{the curvature scalar}$$

Let's define also $\quad R_{mp} = e_{pa} e_b^n R_{mn}{}^{ab}$

the "Ricci tensor".

I would like to think of this action as a function of the two fields e_m^a, $\omega_m{}^{ab}$, with $\omega_m{}^{ab}$ not yet subject to a constraint on the torsion. Let's vary $\mathcal{L}_H$ with respect to these fields and see what happens.

Varying with respect to ω:

$$\delta\mathcal{L}_H = -\frac{1}{2K^2}\, e\, e_a^m e_b^n \left(D_m \delta\omega_n{}^{ab} - D_n \delta\omega_m{}^{ab}\right)$$

To simplify this, we

$$-4!\, e = \epsilon^{klmn} \epsilon_{abcd}\, e_k^a e_l^b e_m^c e_n^d$$

or

$$-2e\,(e_a^m e_b^n - e_b^m e_a^n) = \epsilon^{klmn} \epsilon_{abcd}\, e_k^c e_l^d$$

so

$$\delta \mathcal{L}_H = \frac{1}{4K^2}\, \epsilon^{klmn} \epsilon_{abcd}\, e_k^c e_l^d\, D_m \delta\omega_n{}^{ab}$$

$$= \frac{1}{4K^2}(-2)\, \epsilon^{klmn} \epsilon_{abcd}\, e_k^c\, (D_m e_l^d)\, \delta\omega_n^{ab}$$

this vanishes only if $\quad D_m e_l^d - D_l e_m^d = T_{ml}{}^d = 0$

so we recover the torsion constraint of Riemannian geometry from $\delta \mathcal{L}_H / \delta\omega = 0$.

Now vary $\mathcal{L}_H$ w. respect to e_m^a:

$$\delta e = e\, e_a^m\, \delta e_m^a \qquad \delta e_a^m = -e_b^m\, \delta e_n^b\, e_a^n$$

so

$$\delta \mathcal{L}_H = -\frac{e}{2K^2}\, \delta e_n^b\, [\, e_b^n R - 2\, e_b^m e_c^n\, R_m{}^c\,]$$

so $\quad \delta \mathcal{L}_H = 0 \quad \Rightarrow \quad R_{mn} - \frac{1}{2} g_{mn} R = 0$

the vacuum Einstein equation! To identify κ^2, couple in a scalar field

$$\mathcal{L}_{scalar} = \frac{1}{2} e\; e_a^m e^{na}\, \partial_m \phi\, \partial_n \phi$$

$$\delta\mathcal{L}_{scalar} = \frac{e}{2}\,\delta e_n^{\ b}\left(e_b^{\ n}\, g^{pq}\partial_p\phi\,\partial_q\phi - 2\, e_b^{\ m} e_c^{\ n} e^{pc}\,\partial_m\phi\,\partial_p\phi\right)$$

$$= \frac{e}{2}\,\delta e_n^{\ b}\,(-2)\left(e_b^{\ m}\, g^{np}\,\partial_m\phi\,\partial_p\phi - \tfrac{1}{2} e_b^{\ n}\, g^{pq}\partial_p\phi\,\partial_q\phi\right)$$

so

$$\delta\left(\mathcal{L}_H + \mathcal{L}_{scalar}\right) = 0$$

$$\Rightarrow \quad R_{mp} - \tfrac{1}{2} g_{mp} R = \kappa^2\, \Pi_{mp}$$

where $\quad \Pi_{mp} = \partial_m\phi\,\partial_p\phi - \tfrac{1}{2} g_{mp}(\partial\phi)^2$

we can then identify $\underline{\kappa^2 = 8\pi G_N}$

I would like to discuss one more aspect of the gravitational field theory, the solution of the linearized field equation. After eliminating $\omega_m^{\ ab}$ w. the torsion constraint, we have as our field $e_m^{\ a}$, 16 real components. Acting on this are 10 gauge degrees of freedom, local Lorentz invariance (6) and local translations (4). Near flat space

$$e_m^{\ a} \cong \delta_m^{\ a} + \tfrac{1}{2} h_m^{\ a}$$

and we can use local Lorentz transformation to make $h_m^{\ a}$ symmetric:

$$h_{mn} = + h_{nm} \qquad\qquad g_{mn} \cong \eta_{mn} + h_{mn}$$

Now put this linearized form into the vacuum Einstein equations.

After some calculation, we find

$$\Box_{kl;mn} h_{mn} = 0$$

where

$$(\Box)_{kl;mn} = -\partial^2(\eta_{km}\eta_{ln} + \eta_{kn}\eta_{lm})$$
$$+ (\partial_k\partial_m\eta_{ln} + \partial_k\partial_n\eta_{lm} + \partial_l\partial_m\eta_{kn} + \partial_l\partial_n\eta_{km})$$
$$- 2(\partial_k\partial_l\eta_{mn} + \eta_{kl}\partial_m\partial_n)$$
$$+ 2\eta_{kl}\eta_{mn}\partial^2$$

This operator obeys symm. under $k \leftrightarrow l$, $m \leftrightarrow n$, $kl \leftrightarrow mn$ and

$$\partial^k \Box_{kl;mn} = 0$$

reflecting linearized local translation invariance:

$$\delta h_{mn} = \partial_m v_n + \partial_n v_m$$

We can fix this gauge invariance with the gauge condition (de Donder gauge):

$$\partial^k h_{kl} = 0$$

Subject to this constraint

$$\Box_{kl;mn} h_{mn} = -2\partial^2 h_{mn} - 2\partial_k\partial_l h_m{}^m + 2\eta_{kl}\partial^2 h_m{}^m$$

$$= 0$$

if $\quad -\partial^2 h_{mn} = 0, \quad h_m{}^m = 0$

the solutions of $\partial^2 h_{mn} = 0$ are plane waves:

$$h_{mn} = (v^1_m v^2_n + v^2_m v^1_n)\, e^{-ip\cdot x}, \quad p^2 = 0$$

let $p = (p,0,0,p)$; then we can build v^i out of the basis:

$$\begin{aligned} p &= (p,0,0,p) \\ \tilde{p} &= (p,0,0,-p) \\ \epsilon_+ &= \tfrac{1}{\sqrt{2}}(0,1,i,0) \\ \epsilon_- &= \tfrac{1}{\sqrt{2}}(0,1,-i,0) \end{aligned}$$

note that $p^2 = 0$, $\tilde{p}^2 = 0$, $\epsilon_+^2 = 0$, $\epsilon_-^2 = 0$

but

$$p\cdot\tilde{p} = 2p^2$$
$$\epsilon_+\cdot\epsilon_- = 1$$

Now,

if v^1 or $v^2 = p$, this is a gauge motion; ignore it

if v^1 or $v^2 = \tilde{p}$, the solution will violate

$$\partial^k h_{k\ell} = -ip^k h_{k\ell} = 0$$

so we are left with three possibilities:

$$h_{mn} = \quad \epsilon_+^m\epsilon_+^n e^{-ip\cdot x} \qquad \epsilon_-^m\epsilon_-^n e^{-ip\cdot x} \qquad \underbrace{(\epsilon_+^m\epsilon_-^n + \epsilon_-^m\epsilon_+^n)}_{\text{this has } h_m{}^m \neq 0} e^{-ip\cdot x}$$

so we find two solutions to the linearized field equation:

$$\begin{cases} h^{mn} = \epsilon_+^m \epsilon_+^n e^{-ip\cdot x} & \text{helicity} = +2 \\ h^{mn} = \epsilon_-^m \epsilon_-^n e^{-ip\cdot x} & \text{helicity} = -2 \end{cases}$$

so Einstein's gravity really is the massless spin-2 field theory!

Having now discussed the formalism of spin-2, what about spin-3/2. The spin 3/2 field is a vector-spinor

$$\psi_{m\alpha}$$

In 4-dimensions, I will take this to be a Weyl spinor. By analogy to the Dirac equation, we should expect to find a 1st order field equation. Also, since ψ_m has an $m=0$ component with negative metric, this equation should have a gauge invariance. An appropriate equation was written down by Rarita and Schwinger

$$-\epsilon^{klmn}\bar{\sigma}_l \partial_m \psi_n = 0$$

(We go back to flat space for a moment.)

This equation has a gauge invariance:

$$\delta\psi_n = \partial_n \chi$$

where $\chi_\alpha(x)$ is a spinor parameter. I would now like to show that the solutions of this equation are massless helicity $\pm 3/2$ waves. To do this, let's first write a few more forms of the equation:

$$-\epsilon_{klmn}\,\epsilon^{kpqr}\,\bar\sigma_p\,\partial_q\psi_r = 0$$

$$\hookrightarrow \quad \bar\sigma_p\psi_{qr} + \bar\sigma_q\psi_{rp} + \bar\sigma_r\psi_{pq} = 0$$

where $\psi_{qr} = \partial_q\psi_r - \partial_r\psi_q$ is the Rarita Schwinger field strength.

$$0 = -\sigma_k\,\epsilon^{kpqr}\,\bar\sigma_p\,\partial_q\psi_r$$

$$= -\epsilon^{kpqr}\,\sigma_{kp}\,\psi_{qr}$$

$$= -2i\,\sigma^{qr}\psi_{qr} \quad\longrightarrow\quad \sigma^{qr}\psi_{qr} = 0$$

$$0 = -\bar\sigma_k\sigma_l\,\epsilon^{kpqr}\,\bar\sigma_p\,\partial_q\psi_r$$

$$= -\frac{1}{2}\left[\bar\sigma_k\sigma_l\bar\sigma_p\right]\epsilon^{kpqr}\psi_{qr}$$

$$= -\frac{1}{2}\Big[\underbrace{\eta_{kl}\bar\sigma_p + \eta_{lp}\bar\sigma_k}_{\substack{=0 \text{ by} \\ k\leftrightarrow p \\ \text{antisymmetry}}} - \underbrace{\eta_{kp}\bar\sigma_l}_{=0} - i\,\epsilon_{klpm}\bar\sigma^m\Big]\epsilon^{kpqr}\psi_{qr}$$

$$= \frac{-i}{2}\,\epsilon_{kplm}\,\epsilon^{kpqr}\,\bar\sigma^m\psi_{qr}$$

$$= i\left(\bar\sigma^m\psi_{lm} - \bar\sigma^m\psi_{lm}\right)$$

$$= 2i\,\bar\sigma^m\psi_{lm} \quad\longrightarrow\quad \bar\sigma^m\psi_{lm} = 0$$

Now let's solve these equations. To do this, impose a gauge condition

$$\bar\sigma^m \psi_m = 0$$

Subject to this condition, the last equation on p. 14 becomes

$$\bar\sigma^m \partial_\ell \psi_m - \bar\sigma^m \partial_m \psi_\ell = 0$$

$$\partial_\ell (\underbrace{\bar\sigma^m \psi_m}_{0}) - (\bar\sigma \cdot \partial)\, \psi_\ell = 0$$

so ψ_ℓ obeys the Dirac equation, or rather, the massless Weyl equation. In 4-dimensions, this equation has only one solution. For $\vec p \parallel \hat 3$ & positive energy

$$\bar\sigma \cdot \partial\, \chi = 0 \quad \Rightarrow \quad \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ip\cdot x}$$

So the positive energy solutions of the Rarita-Schwinger equation with $\vec p \parallel \hat 3$ must be of the form

$$\psi_m = v_m \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ip\cdot x}$$

There are four choices for v_m, the vectors on p. 12. Again, $v_m = p_m$ is a gauge variation and should be ignored. For $v_m = \tilde p_m$, check the Rarita-Schwinger equation:

$$i\, \epsilon^{k\ell mn}\, \bar\sigma_\ell\, p_m v_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

for $k =$: since $v_n = \tilde p_n$ has only 0 and 3 components of the equation becomes

$$\left[i\epsilon^{1203}\sigma^2 p\cdot p + i\epsilon^{1230}\sigma^2(-p)(p)\right]\binom{0}{1}$$

$$= 2ip^2\,\sigma^2\binom{0}{1} \neq 0$$

Next, try $v_m = \mathcal{E}_{+m}$. For $k=0$

$$\left[i\epsilon^{0132}\bar{\sigma}_1 p_3(\epsilon_+)_2 + i\epsilon^{0231}\bar{\sigma}_2 p_3(\epsilon_+)_1\right]\binom{0}{1}$$

$$= \left[i\epsilon^{0132}\sigma^1(-p)\left(\frac{-i}{\sqrt{2}}\right) + i\epsilon^{0231}\sigma^2(-p)\left(\frac{-1}{\sqrt{2}}\right)\right]\binom{0}{1}$$

$$= p\left(\frac{\sigma^1+i\sigma^2}{\sqrt{2}}\right)\binom{0}{1}$$

$$= p\sqrt{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}\binom{0}{1} \neq 0$$

Finally, for $v_m = (\mathcal{E}_-)_m$, for $k=0$ we have

$$\left[i\epsilon^{0132}\sigma^1(-p)\left(\frac{+i}{\sqrt{2}}\right) + i\epsilon^{0231}\sigma^2(-p)\left(\frac{-1}{\sqrt{2}}\right)\right]\binom{0}{1}$$

$$= p\frac{1}{\sqrt{2}}\left(-(\sigma^1-i\sigma^2)\right)\binom{0}{1} = 0$$

similarly, for $k=1$

$$\left[i\epsilon^{1032}(1)(-p)\left(\frac{+i}{\sqrt{2}}\right) + i\epsilon^{1302}(\sigma^3)\,p\left(\frac{+i}{\sqrt{2}}\right)\right]\binom{0}{1}$$

$$= \frac{p}{\sqrt{2}}(1+\sigma^3)\binom{0}{1} = 0$$

and $k=2,3$ check similarly.

so for $$-\epsilon^{klmn}\bar{\sigma}_l \partial_m \psi_n = 0$$

we find one positive energy wave solution:

$$\psi_n = (\epsilon_-)_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ip\cdot x}$$

with helicity $= -3/2$. The conjugate equation

$$\partial_m \psi_k^\dagger \epsilon^{klmn} \bar{\sigma}_l = 0$$

has a positive energy antiparticle solution with helicity $= +3/2$. Thus, the Rarita-Schwinger equation is an appropriate massless spin $3/2$ wave equation.

Finally, I would like to write the Lagrangian of the Rarita-Schwinger system in curved space. This can be done as follows:

$$\mathcal{L}_{RS} = -\frac{1}{2}\left[\psi_k^\dagger \epsilon^{klmn}\bar{\sigma}_l D_m \psi_n - D_m \psi_k^\dagger \epsilon^{klmn} \bar{\sigma}_l \psi_n\right]$$

Note that this Lagrangian is Hermitian with the signs given.

I define
$$D_m \psi_n = \partial_m \psi_n + \tfrac{1}{2}\omega_m{}^{ab}\sigma_{ab}\psi_n$$
$$D_m \psi_k^\dagger = \partial_m \psi_k^\dagger - \tfrac{1}{2}\omega_m{}^{ab}\psi_k^\dagger \bar{\sigma}_{ab}$$
$$\bar{\sigma}_l = e_l^{\ b}\bar{\sigma}_b \quad \text{(world} \rightarrow \text{tangent space indices)}$$

The variational equation for ψ is:

$$-\epsilon^{klmn}\bar{\sigma}_l D_m\psi_n \quad -\tfrac{1}{2}\epsilon^{klmn} D_m(e_l^b)\,\bar{\sigma}_b\psi_n = 0$$

$$-\epsilon^{klmn}\bar{\sigma}_l D_m\psi_n - \tfrac{1}{4}\epsilon^{klmn}\underbrace{T_{ml}{}^b}_{\text{torsion.}}\bar{\sigma}_b\psi_n = 0$$

I claim that the second term will vanish for the "appropriate" torsion constraint, to be derived in the next lecture. Then the field equation is

$$-\epsilon^{klmn}\bar{\sigma}_l D_m\psi_n = 0$$

or in term of $\qquad \psi_{mn} = D_m\psi_n - D_n\psi_m$

$$-\epsilon^{klmn}\bar{\sigma}_l\psi_{mn} = 0$$

$$\bar{\sigma}_l\psi_{mn} + \bar{\sigma}_m\psi_{nl} + \bar{\sigma}_n\psi_{lm} = 0$$

$$\sigma^{mn}\psi_{mn} = 0$$

$$\bar{\sigma}^m\psi_{mn} = 0$$

as in flat space