Introduction to

Supersymmetry

and

Supergravity

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outline:

1.) Representations of SUSY in 4 dimensions

2.) SUSY actions in 4 dimensions

3.) Extended and higher-dimensional SUSY

4.) Supergravity in 4 and higher dimensions

5.) Coupling of matter fields to supergravity

6.) Realistic models of particle physics with SUSY

1. Poincaré group and spinors in 4-dimensions

This is a course on supersymmetry, the possible symmetry between fermions and bosons in relativistic quantum theory. Supersymmetry (henceforth, SUSY) is likely to be a property of the real world, but I would like to postpone a discussion of the reasons for that until later in the course. For most of this course, I will discuss SUSY as a mathematical property, an extension of the usual symmetries of quantum field theory.

Even just as a mathematical construct, SUSY has some amazing and unexpected properties:

① It is not possible to build relativistic quantum theory in which two fields, say ϕ and ψ, transform under SUSY and the other fields are ignorant of the symmetry. Either everything transforms under SUSY, or nothing does.

② SUSY implies the cancellation of the vacuum zero-point energy. This could well be the first step in making sense of this puzzling quantity.

③ SUSY implies other divergence cancellations and nonrenormalization theorems. The more SUSY, the more finite.

④ Extended (non-minimal) supersymmetry is naturally connected to formalisms for quantum theory in higher dimensions.

⑤ SUSY + gravity = supergravity, a nontrivial modification of space-time theory. This theory incorporates the "Rarita - Schwinger gauge invariance" of the spin-$3/2$ field and is, in fact, the only way to define spin-$3/2$ consistently.

⑥ Extended supergravity or supergravity in higher dimensions makes essential use of p-form gauge fields.

Many of the themes I have just touched on — all or nothing symmetry, higher dimensions, p-forms — have a natural place in string theory. In fact, SUSY has a close relation to string theory. On one hand, SUSY was originally discovered as the world sheet symmetry of the Neveu - Schwarz - Ramond string, and extended SUSY theories were discovered as the massless sectors of string theories. On the other hand, many properties of string theories — some even in nonsupersymmetric ones — actually follow from SUSY.

So, let's try to build up a theory of fermion - boson symmetries mathematically.

A supersymmetry is a symmetry that carries bosonic into fermionic states and viceversa:

$$Q\,|f\rangle = |b\rangle \qquad\qquad Q\,|b\rangle = |f\rangle$$

Since bosonic states have integer spin and fermionic states have half-integer spin, Q must carry spin-½. In a relativistic quantum field theory, we must have:

$$[Q, H] = 0$$

$$[Q, M^{ab}] = \text{correct for a spin-½-charge.}$$

and we must have several Q's, which fill out a spin-½ representation. Q should act on fields, something like

$$[Q, \phi(x)] = \psi(x)$$

$$[Q, \psi(x)] = \text{something with } \phi$$

to make all of this precise, we have to know how big spin-½ representations are and how to form invariants from their indices.

So we have to study the Lorentz and Poincaré algebras. Let me begin by writing the basic formulae for these algebras.

In general, in this course:

a,b, m,n Latin letters denote Minkowski indices 0123 or 012…

i j denote spatial indices 123 (we won't use these often)

α, β Greek letters denote spinor indices

Minkowski metric: $$\eta^{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

antisymmetric symbol: ε^{abcd}

$$\varepsilon^{0123} = +1 \qquad \varepsilon_{0123} = -1$$

Acting on fields $\phi(x)$, an infinitesimal Lorentz transformation is generated by the (anti-Hermitian) operator

$$M^{ab} = x^a \partial^b - x^b \partial^a$$

these operators obey the algebra

$$[M^{ab}, M^{cd}] = M^{ad}\eta^{bc} - M^{ac}\eta^{bd} - M^{bd}\eta^{ac} + M^{bc}\eta^{ad}$$

Let's begin by working out the representations of this algebra — the Lorentz algebra — in 4-dimensions. (Later in the course, we will discuss the representations in d-dimensions.)

Define $\quad J^i = \frac{i}{2}\epsilon^{ijk}M^{jk} \qquad$ ie. $\quad J^3 = iM^{12}$

(J + K are Hermitian) $\quad K^i = iM^{0i} \qquad$ ie $\quad K^1 = iM^{01}$

$$[M^{12}, M^{23}] = -M^{13} = M^{31}$$

so $\quad [J^3, J^1] = iJ^2 \quad$ and $\quad [J^i, J^j] = i\epsilon^{ijk}J^k$

$$[M^{01}, M^{12}] = -M^{02}$$

so $\quad [K^1, J^3] = -iK^2 \quad$ and $\quad [K^i, J^j] = i\epsilon^{ijk}K^k$

$$[M^{01}, M^{02}] = -M^{12}$$

so $\quad [K^1, K^2] = -iJ^3 \quad$ and $\quad [K^i, K^j] = -i\epsilon^{ijk}J^k$

to find the representation of this algebra, write

$$J_+^i = \tfrac{1}{2}(J^i + iK^i) \qquad J_-^i = \tfrac{1}{2}(J^i - iK^i)$$

$$[J_+^i, J_+^j] = i\epsilon^{ijk}\,\tfrac{1}{4}[J^k + 2iK^k - (i)^2J^k]$$

$$= i\epsilon^{ijk}J_+^k$$

$$[J_+^i, J_-^k] = i\epsilon^{ijk}\tfrac{1}{4}[J^k + iK^k - iK^k - (i)(-i)J^k]$$
$$= 0$$

so J_+^i, J_-^i commute and satisfy

$$[J_+^i, J_+^j] = i\epsilon^{ijk}J_+^k \qquad [J_-^i, J_-^j] = i\epsilon^{ijk}J_-^k$$

and $(J_+^i)^+ = J_-^i$ & vice versa.

the finite-dimensional representations of the J_+, J_- algebras are spin-j $j = 0, \frac{1}{2}, 1, \ldots$. A given finite-dimensional representation is labelled

$$(j_+, j_-)$$

with $(j_+, j_-)^* = (j_-, j_+)$

The simplest representations are:

$(0,0) \longrightarrow$ scalar

$(\frac{1}{2}, 0)$ $\qquad$ $(0, \frac{1}{2})$

$(\frac{1}{2}, \frac{1}{2})$ $\qquad$ $(1,0)$ $\qquad$ $(0,1)$

Let's work these out explicitly.

in general $\quad J^i = J_+^i + J_-^i \quad$ addition of angular momenta

$$K^i = -i\,(J_+^i - J_-^i)$$

an infinitesimal rotation is $\quad (1 - i\,\theta^i J^i)$

we can write this more generally as

$$(1 + \tfrac{1}{2}\,\omega_{ab} M^{ab})$$

with $\quad \omega^{12} = \theta^3 \qquad M^{12} = -i\,J^3 \quad$ etc.

boosts can also be include by

$$\omega^{01} = -\eta^1 \qquad M^{01} = -i\,K^1 \quad \text{etc.}$$

$$(1 + \tfrac{1}{2}\,\omega_{ab} M^{ab}) = (1 - i\,\vec{\theta}\cdot\vec{J} - i\,\vec{\eta}\cdot\vec{K})$$

Now, the representation $(\tfrac{1}{2}, 0)$ has

$$J_+^i = \sigma^i/2 \qquad J_-^i = 0 \Rightarrow J^i = \sigma^i/2 \qquad K^i = i\,\sigma^i/2$$

$$(1 + \tfrac{1}{2}\,\omega_{ab} M^{ab}) = (1 - i\,\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\eta}\cdot\vec{\sigma}/2)$$

this is the transformation law of a left-handed Weyl fermion

$$e^{-\vec{\eta}\cdot\vec{\sigma}/2}\,\sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} = \sqrt{2E'}\begin{pmatrix}0\\1\end{pmatrix}$$

Similarly, the representation $(0,\frac{1}{2})$ has

$$J_+^i = 0 \qquad J_-^i = \vec{\sigma}^i/2 \quad \Rightarrow \quad J^i = \sigma^i/2 \qquad K^i = +i\,\sigma^i/2$$

$$1 + \tfrac{1}{2}\omega_{ab}M^{ab} \;=\; 1 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\eta}\cdot\vec{\sigma}/2$$

this is the transformation law of a right handed Weyl fermion.

I'd like to show explicitly that $(\frac{1}{2},0)^* = (0,\frac{1}{2})$

Let ξ_α transform under $(\frac{1}{2},0)$

$$\xi^* \longrightarrow \left(1 + i\vec{\theta}\cdot\vec{\sigma}^T/2 - \vec{\eta}\cdot\vec{\sigma}^T/2\right)\xi^*$$

Let $\quad C = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$C^{-1} = -C \qquad C^T = -C \qquad C^* = C$$

$$C\xi^* \rightarrow C\left(1 + i\vec{\theta}\cdot\vec{\sigma}^T/2 - \vec{\eta}\cdot\vec{\sigma}^T/2\right)\xi^*$$

now C obeys.

$$C(\vec{\sigma})^T = -(\vec{\sigma})C$$

so $\quad C\xi^* \rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\eta}\cdot\vec{\sigma}/2\right)(C\xi^*)$

similarly, if η transforms as $(0,\frac{1}{2})$, $C\eta^*$ transforms as $(\frac{1}{2},0)$

The transformation of $(\frac{1}{2},\frac{1}{2})$ can be written, up to a c, as follows. A $(\frac{1}{2},\frac{1}{2})$ is a 2×2 matrix $Z_{\alpha\beta}$

$\alpha \in (\frac{1}{2},0)$ $\quad \beta \in (0,\frac{1}{2})\cdot c$

$$Z \to \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\eta}\cdot\vec{\sigma}/2\right) Z \left(1 + i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\eta}\cdot\vec{\sigma}/2\right)$$

$$= Z - i\vec{\theta}\cdot[\vec{\sigma}/2, Z] - \vec{\eta}\,\{\vec{\sigma}/2, Z\}$$

write $\quad Z = Z^0 - Z^i\sigma^i$

$$= Z^0 - Z^i\sigma^i - i\,\theta^i\,\epsilon^{ijk}(-Z^j)\,\sigma^k - \vec{\eta}\,Z^0 + \vec{\eta}\cdot\vec{Z}$$

so under a rotation

$$\delta Z^i = \epsilon^{ijk}\,\theta^j Z^k \qquad \delta Z^0 = 0$$

under a boost

$$\delta Z^i = \eta^i Z^0 \qquad \delta Z^0 = \eta^i Z^i$$

This is just the transformation of a 4-vector. So

$$(\tfrac{1}{2},\tfrac{1}{2}) \cong 4\text{-vector}.$$

The transformation law of a 4-vector can also be written in another way, one that generalizes to any d. Write the representation

$$(M^{ab})_{mn} = \delta^a_m \delta^b_n - \delta^a_n \delta^b_m$$

to multiply M's raise & lower w. η^{mp}. This indeed satisfies

$$([M^{ab}, M^{cd}])_{mp}$$

$$= (M^{ad}\eta^{bc} - M^{ac}\eta^{bd} - M^{bd}\eta^{ac} + M^{bc}\eta^{ad})_{mp}$$

acting on a 4-vector Z^m

$$Z^m \longrightarrow (1 + \tfrac{1}{2}\omega_{ab}M^{ab})^m{}_n Z^n$$

$$= Z^m + \tfrac{1}{2}\omega_{ab}\,[\eta^{am}\eta^b{}_n - (a\leftrightarrow b)]\, Z^n$$

$$= Z^m + \omega^m{}_n Z^n$$

rotation : $\omega^i{}_j = - \epsilon^{ijk}\theta^k$

$$\delta Z^i = \epsilon^{ijk}\theta^j Z^k \qquad \delta Z^0 = 0$$

boost $\qquad \omega^{0i} = -\eta^i \qquad \omega^0{}_i = +\eta^i = \omega^i{}_0$

$$\delta Z^i = \eta^i Z^0 \qquad \delta Z^0 = \eta^i Z^i$$

The relation between the two representation of the 4-vector is given by a set of invariants — constant matrices that are invariant under transformations on all indices. Let

$$\left(\sigma^{\overset{\text{4-vector}}{a}}\right)_{\underset{(\frac12,0)}{\alpha}\,\underset{(0\frac12)c}{\beta}} = (1, \vec{\sigma})^a_{\alpha\beta}$$

then

$$\sigma^a = (\delta^a_b + \omega^a{}_b)\left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\eta}\cdot\vec{\sigma}/2\right)\sigma^b\left(1 + i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\eta}\cdot\vec{\sigma}/2\right)$$

so $\sigma^a_{\alpha\beta}$ is a change of basis from $(\frac12,\frac12)$ to the usual 4-vector. Similarly

$$\bar{\sigma}^a = (1, -\vec{\sigma})^a_{\alpha\beta}$$

has this property w. the $(0,\frac12)$ transformation. The two sets of matrices are related by,

$$C\,\sigma^a = (\bar{\sigma}^a)^T C$$

$$C\,\bar{\sigma}^a = (\sigma^a)^T C$$

Now let's discuss $(1,0)$ and $(0,1)$. I'll do the whole discussion in terms of invariant tensors. Start w. $(1,0)$

$$(\tfrac12,0) \times (\tfrac12,0) = \underset{\text{1 state}}{(0,0)} + \underset{\text{3 states}}{(1,0)}$$

we can make an object w. $(\frac{1}{2},0)\times(\frac{1}{2},0)$ indices by

$$[\sigma^a \bar{\sigma}^b C]_{\alpha\beta}$$

now
$$\sigma^a \bar{\sigma}^b = (1,\vec{\sigma})^a (1,-\vec{\sigma})^b$$
$$= \eta^{ab} + 2\sigma^{ab}$$

where
$$\sigma^{ab} = \frac{1}{4}(\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a)$$

the η^{ab} part is
$$\eta^{ab} C_{\alpha\beta}$$

the antisymmetric combination of $(\frac{1}{2},0)\times(\frac{1}{2},0) = (0,0)$

The σ^{ab} part is an antisymmetric tensor (6 states)

By explicit calculation, one can show:

$$(\sigma^{ab}C)_{\alpha\beta} \text{ is } \underline{\text{symmetric}} \text{ under } (\alpha \leftrightarrow \beta)$$

so this projects onto $[(\frac{1}{2},0)\times(\frac{1}{2},0)]_{\text{symm.}} = (1,0)$

$$\frac{1}{2}\epsilon^{abcd}(\sigma_{cd}C) = +i(\sigma^{ab}C)$$

"self-dual antisymmetric tensor". So there are 3-states, not 6, in this combination

Similarly
$$\bar{\sigma}^{ab} = \frac{1}{4}(\bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a)$$

satisfies

$$(\bar{\sigma}^{ab}C)_{\alpha\beta} \text{ is symmetric in } \alpha \leftrightarrow \beta$$

$$\tfrac{1}{2}\epsilon^{abcd}\,\bar{\sigma}_{cd} = -i\,\bar{\sigma}^{ab} \quad \text{"anti-self-dual"}$$

so

$$(1,0) = \text{self-dual antisymmetric tensor}$$

$$(0,1) = \text{anti-self-dual antisymmetric tensor}$$

$$(1,0) + (0,1) = \text{general antisymmetric tensor}$$

$$= \text{representation of } F_{ab}$$

The matrices σ^a, σ^{ab}, and $\bar{\sigma}^{ab}$ obey some useful identities. You should verify them

$$\sigma^a \bar{\sigma}^b = \eta^{ab} + 2\,\sigma^{ab}$$

$$\bar{\sigma}^a \sigma^a = \eta^{ab} + 2\,\bar{\sigma}^{ab}$$

$$\sigma^a \bar{\sigma}^b \sigma^c = \eta^{ab}\sigma^c - \eta^{ac}\sigma^b + \eta^{bc}\sigma^a + i\,\epsilon^{abcd}\sigma_d$$

$$\bar{\sigma}^a \sigma^b \bar{\sigma}^c = \eta^{ab}\bar{\sigma}^c - \eta^{ac}\bar{\sigma}^b + \eta^{bc}\bar{\sigma}^a - i\,\epsilon^{abcd}\bar{\sigma}_d$$

$$\sigma^a \bar{\sigma}^{bc} - \sigma^{bc} \sigma^a = \eta^{ab} \sigma^c - \eta^{ac} \sigma^b$$
$$\bar{\sigma}^a \sigma^{bc} - \bar{\sigma}^{bc} \bar{\sigma}^a = \eta^{ab} \bar{\sigma}^c - \eta^{ac} \bar{\sigma}^b$$

When we construct field theories, we will choose fields in particular finite-dimensional representation of the Lorentz group. For the first part of the course, we will be concerned with spin-0, spin-$\frac{1}{2}$, and spin-1

scalars $\quad \phi(x)$

spinors $\quad \psi_\alpha(x) \quad \chi_\beta(x) \quad$ in $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$

vectors $\quad A_a(x)$

In this course, I will mainly deal with these fields classically. But even here there is a subtlety: classical fermions are represented by anticommuting numbers. I will choose rules for manipulating these based on making the simplest correspondence to the behavior of operators. [Think of classical fields as living under a fermionic functional integral.]

$$\psi_1 \psi_2 = - \psi_2 \psi_1$$
$$(\psi_1 \psi_2)^* = \psi_2^* \psi_1^* = - \psi_1^* \psi_2^*$$

then, for example, for ψ a $(\frac{1}{2},0)$ = left-hand Weyl

$$\int d^4x \; \psi^\dagger i \bar{\sigma}^a \partial_a \psi$$

is a Lorentz-invariant action. It's conjugate is

$$\left(\int d^4x \; \psi^* i \bar{\sigma}^a \partial_a \psi \right)^*$$

$$= \int d^4x \; (\partial_a \psi)^* (-i \bar{\sigma}^a)^\dagger \psi$$

$$= \int d^4x \; (-\partial_a \psi^*) \, i \bar{\sigma}^a \psi$$

$$= \int d^4x \; \psi^* \, i \bar{\sigma}^a \partial_a \psi$$

Note also that, if χ is $(0,\frac{1}{2})$ = right-handed Weyl

$$\int d^4x \; \chi^\dagger i \sigma^a \partial_a \chi$$ is Lorentz invariant but

if we set $\chi = c\,\psi^*$

$$\chi^\dagger = \psi^T c^T = -\psi^T c$$

$$\int d^4x \; \chi^\dagger i \sigma^a \partial_a \chi = \int d^4x \left(-\psi^T c \, (i\sigma^a) \, \partial_a \, c \, \psi^* \right)$$

$$= \int d^4x \; \psi^T (i \bar{\sigma}^a)^T \partial_a \psi^* \qquad c^2 = -1$$

$$= \int d^4x \, \left(-\partial_a \psi^T (i\bar{\sigma}^a)^T \psi^*\right) \quad \text{integration by parts}$$

$$= \int d^4x \, \psi^* \, i\bar{\sigma}^a \partial_a \psi \quad \text{interchange of } \psi^* \text{ and } \psi$$

so

$$\int d^4x \, \chi^\dagger i\sigma^a \partial_a \chi = \int d^4x \, \psi^* \, i\bar{\sigma}^a \partial_a \psi$$

that is, we can freely replace $(0,\frac{1}{2})$ fermion fields by $(\frac{1}{2},0)$ fermion fields. The new fields create the antiparticles of the old and live in the complex conjugate representations.

From now on, I will write all 4-dimensional fermion systems in terms of $(\frac{1}{2},0)$ or left-hand Weyl fermions.

Notation for fermions is quite subtle; be careful with it. Wess & Bagger write $(0,\frac{1}{2})$ indices as dotted

$$\psi_\alpha \qquad \chi_{\dot{\alpha}}$$

However, they also represent c as raising and lowering of indices

$$(\psi_\alpha)^* = \chi_{\dot{\alpha}}$$

but my $\chi = \chi^{\dot{\alpha}}$

contractions $\psi_1^\alpha \psi_{2\alpha}$ and $\chi_{1\dot\alpha} \chi_2^{\dot\alpha}$ are Lorentz invariant.
I will write these as:

$$\psi_1^T c\, \psi_2 \qquad \text{and} \qquad \psi_1^\dagger c\, \psi_2^*$$

note that these objects are symmetric under $1 \leftrightarrow 2$

$$\psi_1^T c \psi_2 = - \psi_2^T c^T \psi_1 = + \psi_2^T c \psi_1$$

There is one more bit of fermion technology that will be useful in our study. Often we will encounter fermion bilinears and we will need to rearrange the indices.
For example, consider η, ξ $(\frac{1}{2},0)$ Weyl fermions

$$\eta_\alpha \xi^\dagger_\beta$$

This is a 4-component object w. indices in $(\frac{1}{2},0) \times (\frac{1}{2},0)^*$, so it must be equivalent to something of the form

$$(\sigma^a)_{\alpha\beta} V_a$$

we can compute V_a by tracing with $\bar\sigma^b$

$$\mathrm{tr}\ \bar\sigma^b \sigma^a = 2\eta^{ab}$$

$$\operatorname{tr}\ \bar\sigma^b (\eta\, \xi^+) = - \xi^+ \bar\sigma^b \eta$$

so

$$\eta_a \xi^+_\beta = -\tfrac{1}{2} (\sigma^a)_{\alpha\beta} (\xi^+ \bar\sigma_a \eta)$$

Similarly, using the fact that $C_{\alpha\beta}$, $(\sigma^{ab}C)_{\alpha\beta}$ are a complete set of matrices w. indices $(\tfrac{1}{2},0)\times(\tfrac{1}{2},0)$, you can prove

$$\eta_\alpha \xi^T_\beta = \tfrac{1}{2} C_{\alpha\beta} (\xi^T C \eta) - \tfrac{1}{2} (\sigma^{ab} C)_{\alpha\beta} (\xi^T C \sigma_{ab} \eta)$$

$$\eta^*_\alpha \xi^+_\beta = \tfrac{1}{2} C_{\alpha\beta} (\xi^+ C \eta^*) - \tfrac{1}{2} (C\bar\sigma^{ab})_{\alpha\beta} (\xi^+ \bar\sigma_{ab} C \eta^*)$$

(It is also instructive to prove that

(1) the first equation here is the conjugate of the second

(2) $\xi^T C \eta$ is <u>symmetric</u> under $\xi \leftrightarrow \eta$

(3) $\xi^T C \sigma_{ab} \eta$ is <u>antisymmetric</u> under $\xi \leftrightarrow \eta$.)

Fermion index rearrangements such as those shown here are called Fierz <u>identities</u>. These play an important role in the technology of SUSY.