The Parton Model

Now we begin our discussion of parton-parton hard scattering processes in hadron-hadron collisions. I will base the discussion on quantum chromodynamics (QCD), the description of the strong interactions as a Yang-Mills gauge theory of quarks and gluons. If we ignore the masses of the quarks — as I will do throughout most of this course — the Lagrangian of QCD is extremely simple

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \sum_{f=1}^{n_f} \bar{q}_f \, i\not{D} \, q_f$$

n_f is the number of quark flavors; usually we will have $n_f = 5$ for u,d,s,c,b. This simple Lagrangian contains a lot of physics!

In particular, there are two important properties of QCD that I hope you are already familiar with:

<u>Quark Confinement</u>: The finite-energy bound states of QCD are singlets of the gauge group SU(3). Allowed states are, for example:

mesons: $q\bar{q}$ baryons qqq glueballs gg

When we talk about quark and gluon reactions, we must realize that quarks and gluons produced in large momentum transfer reactions pull other quarks and gluons out of the vacuum and eventually organize themselves into color-singlet systems. The final aspects of this process involve non-perturbative aspects of QCD. The early stages of the process are part of the theory of hard scattering, and we will discuss them later in the course.

Asymptotic Freedom: For large momentum transfer, the coupling constant g_s of QCD tends to zero. Thus, though the theory contains strong interactions for which nonperturbative effects are essential, hard-scattering reactions are described by weak-coupling perturbation theory.

Asymptotic freedom and quark confinement are independent concepts. It is possible for a gauge theory to have either one without the other. In QCD, with the gauge group $SU(3)$ and $n_f = 5$, we have both.

Asymptotic freedom can be derived from perturbation theory (see, for example, Peskin + Schroeder, Ch. 16). Here are the basic formulae:

g_s evolves according to the renormalization group (RG) equation:

$$\frac{d}{d\log Q} g_s(Q) = \beta(g_s(Q))$$

where

$$\beta(g_s) = -\frac{b_0 g_s^3}{(4\pi)^2} - \frac{b_1 g_s^5}{(4\pi)^4} - \cdots$$

$$b_0 = 11 - \tfrac{2}{3} n_f \qquad b_1 = 102 - \tfrac{38}{3} n_f$$

b_0 was calculated by Gross, Politzer, and Wilczek. The recognition that $\beta(g) < 0$, so that $g_s \to 0$ as $\log Q \to \infty$, won them the Nobel Prize. b_1 was calculated by Belavin, Caswell, and Jones; this was also a landmark in the development of QCD.

The solution of the RG equation is

$$\alpha_s(Q) = \frac{g_s^2(Q)}{4\pi} = \frac{4\pi}{b_0 \log Q^2/\Lambda^2} - \frac{4\pi b_1 \log\log(Q^2/\Lambda^2)}{b_0^3 \log^2(Q^2/\Lambda^2)} + \cdots$$

Thus α_s decreases very slowly, only as a $\log^{-1} Q$. The point where the right-hand side formally diverges is called Λ or Λ_{QCD}. QCD with massless quarks has precisely one parameter, the value of the coupling constant. We can specify this by giving the value of Λ or by giving the value of α_s at a fixed value of Q.

Because quarks and gluons are not observable in isolation, it is not possible to measure α_s directly, as, for example, we measure α in QED from the strength of the Coulomb interaction. Typically, we determine α_s by comparing some observable of QCD hard scattering processes:

$$\left(\begin{matrix}\text{observable in scattering} \\ \text{w. mom. transfer } Q\end{matrix}\right) = f(\alpha_s(Q))$$

and then solving for $\alpha_s(Q)$. The function $f(\alpha_s)$ is obtained from perturbation theory and depends on technical details of the calculation, e.g. how UV and IR divergencies are regulated. Thus, a value of α_s depends on these details, called generally the "scheme" of calculation. A value of α_s determined in one scheme can be used to make predictions for another observable computed in the same scheme, or it can be converted in a well-defined way to a value of α_s in another scheme. In this course, I will consistently use the $\overline{MS}$ ("em-ess-bar") scheme. All schemes agree at the leading order in α_s, so I will define $\overline{MS}$ when we begin to discuss higher-order corrections.

The current value of α_s quoted by the Particle Data Group (2008) is:

$$\alpha_s^{\overline{MS}}(Q = m_Z) = 0.1176 \pm .002$$

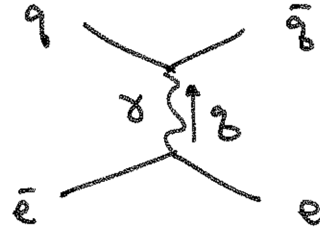
corresponding to $\Lambda_{QCD} \sim 200$ MeV. Fig. p.2 shows various determinations of α_s, each plotted at the value of Q at which the measurement was made. The values come from many sources, including the total cross section and event shape variables in e^+e^- annihilation to hadrons, sum rules in inelastic ep scattering, rates of hard scattering in $p\bar{p}$ collisions, and the hadronic branching ratio of the τ lepton. The very precise point near $Q=10$ comes from lattice gauge theory calculations of the Υ spectrum; the bare coupling constant used in the best fit can be converted to a value of $\alpha_s^{\overline{MS}}$. The black curve is the prediction of the RG equation for $\alpha_s(Q)$. There is strong evidence that this solution is correct. Fig. p. 3 shows the PDG compilation of — in their opinion — the most reliable determinations of α_s, converted to values of $\alpha_s^{\overline{MS}}(m_Z)$. We will discuss the QCD theory behind some of these determinations later in the course.

You should keep in mind the value $\alpha_s \sim 0.1 - 0.2$ for $10 < Q < 1000$. This is not so small, so we will need to compute beyond the leading order in QCD to obtain accurate predictions. Also, note that the α_s expansion is also an expansion in $1/\log Q$. If the diagrams contributing to an order α_s correction are enhanced by a factor $\log Q$, the effective expansion parameter is

$$\alpha_s \log Q \sim \mathcal{O}(1)$$

When this happens, we need to sum all orders in perturbation theory to obtain an accurate answer.

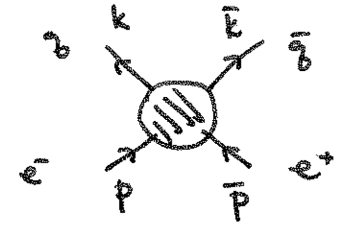
The simplest QCD process is the reaction $e^+e^- \to$ hadrons. Here there are no quarks or gluons in the initial state. The leading contribution from Feynman diagrams involves zero powers of α_s; it is the pure QED diagram:



(I assume $q^2 \ll m_Z^2$ and ignore the weak-interaction contribution to this process.)

Let's compute the cross section that follows from this diagram. Here and throughout this course, I will encourage you to compute scattering amplitudes between <u>external states</u> of definite <u>helicity</u> rather than computing spin-averaged matrix element squares. We will see the advantage already in this process.

Notate the momenta as



$$s = 2p\cdot\bar p = 2k\cdot\bar k$$
$$t = -2p\cdot k = -2\bar p\cdot\bar k$$
$$u = -2p\cdot\bar k = -2\bar p\cdot k$$

ignoring all masses.

I will treat the external fermions as <u>massless</u>; then the external states are most conveniently described as right-handed (R, helicity $h=+\frac{1}{2}$) or left-handed (L, helicity $h=-\frac{1}{2}$) fermions. For massless fermions in QED and QCD, helicity is conserved (and, in fact, the L and R fermions obey completely separate equations of motion).

So there are only four possible processes:

$$e^-_L e^+_R \to q_L \bar q_R \qquad\qquad e^-_L e^+_R \to q_R \bar q_L$$
$$e^-_R e^+_L \to q_L \bar q_R \qquad\qquad e^-_R e^+_L \to q_R \bar q_L$$

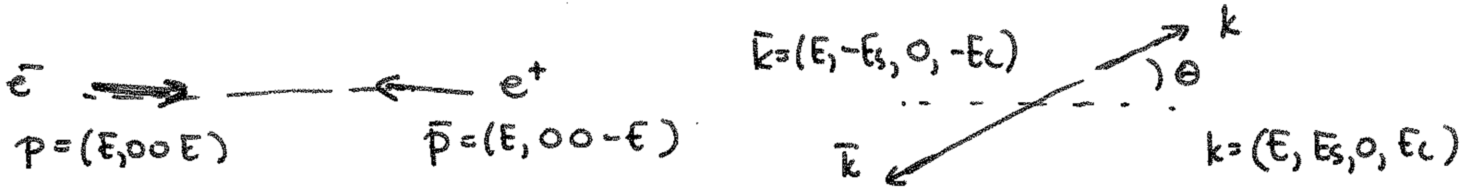
The scattering amplitude is $\qquad Q_f$ = quark electric charge

$$i\mathcal{M}(e^-e^+\to q\bar q) = (-ie)(+iQ_f e)\,\bar v(\bar p)\gamma^\mu u(p)\,\frac{-i}{q^2}\,\bar u(k)\gamma_\mu v(\bar k)$$

to evaluate this, choose the representation of the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix} \qquad \sigma^\mu = (1, \vec\sigma)^\mu \qquad \bar\sigma^\mu = (1, -\vec\sigma)^\mu$$

for $e^-_L e^+_R \to q_L \bar q_R$, the Dirac spinors are:



$s = \sin\theta \qquad c = \cos\theta$

$s_2 = \sin\theta/2 \qquad c_2 = \cos\theta/2$

$$u_L(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ \hline 0 \\ 0 \end{pmatrix} \qquad u_L(k) = \sqrt{2E}\begin{pmatrix} -s_2 \\ c_2 \\ \hline 0 \\ 0 \end{pmatrix}$$

$$v_R(\bar p) = \sqrt{2E}\begin{pmatrix} -1 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix} \qquad v_R(\bar k) = \sqrt{2E}\begin{pmatrix} -c_2 \\ -s_2 \\ \hline 0 \\ 0 \end{pmatrix}$$

then

$$\bar v(\bar p)\gamma^\mu u(p) = 2E\,(-1\,0)\,(1, -\vec\sigma)^\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2E\,(0\,1, -i, 0)$$

$$\bar u(k)\gamma^\mu v(\bar k) = 2E\,(-s_2 c_2)\,(1, -\vec\sigma)^\mu \begin{pmatrix} -c_2 \\ -s_2 \end{pmatrix} = 2E\,(0, c_2^2 - s_2^2, i, -2c_2 s_2)$$

$$= 2E\,(0, \cos\theta, +i, -\sin\theta)$$

this last vector is just the rotation of

$$[2E\,(0, 1, -i, 0)]^* \quad \text{by the angle } \theta$$

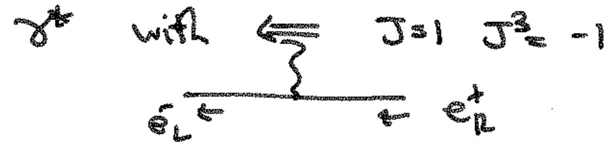
Write

$$\varepsilon_-^\mu(p) = \frac{1}{\sqrt2}(0\,1, -i, 0)$$

This is the polarization vector with $J=1$, $J^3=-1$ about the $\hat 3$ or p axis. We have:

$$i\mathcal{M} = \frac{-ie^2 Q_f}{q^2}\, 2\cdot(2E)^2\, \varepsilon_-^\mu(p)\cdot\varepsilon^*_{-\mu}(k)$$

This directly reflects the angular momentum flow in the process:



More explicitly:

$$i\mathcal{M} = -ie^2 Q_f \frac{(2E)^2}{(2E)^2} (0\,1\,-i\,0)^\mu (0, \cos\theta, i, -\sin\theta)_\mu$$

$$= ie^2 Q_f (1+\cos\theta)$$

Compare this to $\quad t = -2E^2(1-\cos\theta) \quad u = -2E^2(1+\cos\theta) \quad s = 4E^2$

$$i\mathcal{M}(e^-_L e^+_R \to q_L \bar{q}_R) = -2ie^2 Q_f \frac{u}{s}$$

Switching from $e^-_L e^+_R$ to $e^-_R e^+_L$ or from $q_L \bar{q}_R$ to $q_R \bar{q}_L$, we switch $\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0\,1, -i, 0)$ to $\varepsilon^\mu_+ = \frac{1}{\sqrt{2}}(0, 1, i, 0)$. This switches $(1+\cos\theta)$ to $(1-\cos\theta)$. Thus, the four helicity amplitudes are given by the simple values:

$$i\mathcal{M}(e^-_L e^+_R \to q_L \bar{q}_R) = i\mathcal{M}(e^-_R e^+_L \to q_R \bar{q}_L) = -2ie^2 Q_f \frac{u}{s}$$

$$i\mathcal{M}(e^-_L e^+_R \to q_R \bar{q}_L) = i\mathcal{M}(e^-_R e^+_L \to q_L \bar{q}_R) = -2ie^2 Q_f \frac{t}{s}$$

From these expressions, we can construct the unpolarized differential cross section:

$$\frac{d\sigma}{d\cos\theta_{cm}} = \frac{1}{2s}\frac{1}{16\pi} \frac{1}{4}\sum |\mathcal{M}|^2$$

$$= \frac{1}{2s}\frac{1}{16\pi} \cdot e^4 Q_f^2 \cdot \frac{2}{4}\left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right] \cdot 3$$

The final factor 3 is the sum over quark colors in the final state.

This gives

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}(1+\cos^2\theta)\cdot(3Q_f^2)$$

It is useful to compare this formula to the cross section for the QED process $e^+e^- \to \mu^+\mu^-$. All factors are the same except for the last one. Then we predict

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_f 3Q_f^2$$

Now

$$\sum_f 3Q_f^2 = \begin{cases} 2 & uds \\ 3\frac{1}{3} & udsc \\ 3\frac{2}{3} & udscb \end{cases}$$

This prediction works amazingly well. Figs. p. 4 shows the data. The prediction above is shown in green. The ratio R is constant between the quark thresholds (marked by the positions of quarkonium resonances) to a very good approximation. The horizontal red line shows the prediction including QCD corrections (to be discussed later in the course); this works even better.

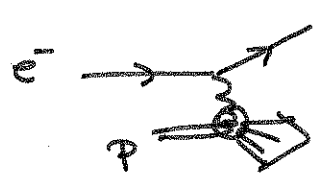
The prediction is even more remarkable when we look at the form of the final states. Figs. p. 5, 6 show two typical events of $e^+e^- \to$ hadrons at $E_{cm} = m_Z = 91$ GeV. The final state particles, which are mainly pions, form two collimated streams — the same jets that we saw in $p\bar{p}$ hard scattering events. The angular distribution of the jets is shown in Figs. p. 7 (from ALEPH measurements at 91 GeV). It is well described by a $(1+\cos^2\theta)$ distribution. I should note that the prediction

$$\frac{d\sigma}{d\cos\theta} \sim (1+\cos^2\theta)$$

is characteristic for e^+e^- annihilation to a pair of spin-½ particles. All of the final state particles are pions. However, the distribution of jets follows this spin-½ distribution. The jets know that they originate from quarks.

To analyze hadron-hadron scattering, we will need to discuss QCD reactions with hadrons in the initial state. The simplest process of this type is inelastic electron-proton scattering. In particular, I will concentrate on the hard scattering of an electron from a proton, so-called "deep inelastic scattering".

Deep inelastic scattering was famously studied as one of the original experiments at SLAC in the 1960's, using the apparatus shown in Figs. p. 8. The design of the experiment (SLAC-MIT collaboration) was very simple. A transport line of magnets selected scattered electrons at a fixed energy. The spectrometer could be rotated to different angles. The experiment could then measure the cross section for an electron to break up the proton

(

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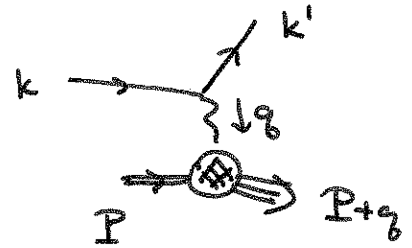
and scatter to a given final energy and momentum. The final state results from the disruptions of the proton was ignored; this was the genius of the experiment. Some data from the experiment for different spectrometer settings is shown in Figs. p.9. At low values of the energy W transferred to the proton, we see clear hadronic resonances. At large W, the inclusive inelastic scattering cross section goes over to a smooth behavior.

Let's first set up the kinematics of the experiment



The initial e^- and p momenta are k and P. The final momentum 4-vector of the electron, k', is measured. Then we know

$$q = k - k'$$

Convenient invariants are

$$Q^2 = -q^2 \qquad x = \frac{Q^2}{2P\cdot q} \qquad y = \frac{2P\cdot q}{2P\cdot k}$$

y is the fraction of the electron's energy, in the lab frame $P = (m, \vec{0})$, that is transferred to the proton. Thus $0 \le y \le 1$.

If we ignore the mass of the proton, then also $0 \le x \le 1$:

$$0 < (P+q)^2 = 2P\cdot q - Q^2 \quad \text{so} \quad 2P\cdot q > Q^2 > 0$$

All three invariants are known when the final electron momentum is measured. If $k^0 \gg m_{proton}$,

$$S = (P+k)^2 = 2P\cdot k$$

and so

$$Q^2 = xyS$$

I will now compute the cross section for inelastic scattering, as a function of x and y, using a model in which the proton is a bound state of quarks and gluons. It is useful to think about the process in the CM frame where the proton is highly boosted

The proton contains many quarks and gluons (collectively, "partons"). The partons interact, but they cannot exchange large transverse momenta, except through hard scatterings suppressed by powers of α_s. Thus it makes sense to model the proton as a collection of partons travelling approximately parallel to P. Ignoring all masses, we can write the momentum of each parton as

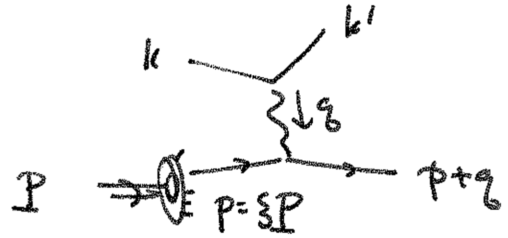
$$p^\mu = \xi P^\mu \qquad 0<\xi<1$$

ξ is called the "longitudinal fraction". I will describe the proton wavefunction by giving the probability distribution

$$d\xi \, f_f(\xi)$$

that a quark or gluon of flavor f has longitudinal fraction ξ.

The simplest process contributing to deep inelastic electron scattering is the scattering of the electron from one quark:



There is a wonderful observation, due to Feynman: In this process, ξ is observable. Ignoring masses,

$$(p+q)^2 = 0 \qquad \text{for the final quark}$$

then $$0 = 2pq + q^2 = 2\xi P\cdot q - Q^2$$

or $$\xi = \frac{Q^2}{2P\cdot q} = x$$

So the measurement of the final electron momentum picks out the momentum of the quark from which it scattered.

Now we are ready to construct the cross section. There are four nonzero helicity amplitudes:

$$e^-_L q_L \to e^-_L q_L \qquad\qquad e^-_L q_R \to e^-_L q_R$$

$$e^-_R q_L \to e^-_R q_L \qquad\qquad e^-_R q_R \to e^-_R q_R$$

The amplitudes for these processes can be obtained from the amplitudes computed on p. 8 by using crossing symmetry: Cross the e^+ into the final state and the $\bar{q}$ into the initial state. This gives

$$i\mathcal{M}(e^-_L q_L \to e^-_L q_L) = i\mathcal{M}(e^-_R q_R \to e^-_R q_R) = -2ie^2 Q_f \frac{s}{t}$$

$$i\mathcal{M}(e^-_L q_R \to e^-_L q_R) = i\mathcal{M}(e^-_R q_L \to e^-_R q_L) = -2ie^2 Q_f \frac{u}{t}$$

where s, t, u are the invariants of the eq scatt'g process

$$s = (k+p)^2 = 2p\cdot k = 2\xi P\cdot k = x \cdot s(\bar{e}p)$$

$$t = q^2 = -Q^2$$

$$u = -s-t = -x\, s(ep) - Q^2 = -x s(\bar{e}p) - xy\, s(\bar{e}p)$$

$$= -(1-y)\, x\, s(\bar{e}p)$$

Then

$$\frac{d\sigma}{d\cos\theta_{cm}}(\bar{e}q) = \frac{1}{2(x s_{\bar{e}p})} \frac{1}{16\pi} \frac{2}{4}\, 4e^4 Q_f^2 \left(\frac{x s_{\bar{e}p}}{Q^2}\right)^2 [1+(1-y)^2]$$

$$= \pi\alpha^2 Q_f^2 \frac{1}{Q^4}\, x s_{\bar{e}p}\, [1+(1-y)^2]$$

Now $dt = dQ^2 = -\frac{1}{2} s\, d\cos\theta_{cm} = -\frac{1}{2} x s_{\bar{e}p}\, d\cos\theta_{cm}$

so
$$\frac{d\sigma}{dQ^2}(\bar{e}q) = \frac{2\pi\alpha^2 Q_f^2}{Q^4}[1+(1-y)^2]$$

This is the electron-quark cross section. To write the electron-proton cross section, sum over quark flavors and integrate over longitudinal fractions with the corresponding probabilities

$$\sigma(e^-p \to e^- + (\text{hadrons})) = \int dQ^2 \int d\xi \sum_f f_f(\xi) \, \frac{2\pi\alpha^2 Q_f^2}{Q^4} \left[1 + (1-y)^2\right]$$

Now $\xi = x$, $Q^2 = xys$, so

$$d\xi \, dQ^2 = dx \, dQ^2 = dx \, dy \left| \frac{\partial(x, Q^2)}{\partial(x,y)} \right| = dx \, dy \begin{vmatrix} 1 & ys \\ 0 & xs \end{vmatrix}$$

$$= dx \, dy \; xs$$

so, finally,

$$\frac{d\sigma}{dx\,dy}(\underbrace{e^-p \to e^- + X}_{X\text{: any hadrons}}) = \Big[\underbrace{\sum_f Q_f^2 \, x \, f_f(x)}_{\text{call this } F_2(x)}\Big] \cdot \left(\frac{2\pi\alpha^2 s}{Q^4} \left[1 + (1-y)^2\right] \right)$$

This is a remarkable expression. The deep inelastic cross section factorizes into a pure QED piece that can be evaluated from the kinematics of the final electron and a function $F_2(x)$ that encodes the proton structure. This factorization is called "Bjorken scaling". Bjorken encouraged the SLAC-MIT experimenters to make the test

$$\left[d\sigma / dx\,dy \right] \Big/ \frac{2\pi\alpha^2 s}{Q^4} (1 + (1-y)^2) \stackrel{?}{=} F_2(x) \quad \text{independent of } Q^2 \; !$$

The left-hand side of this equation (for data points with $Q^2 > 1 \text{ GeV}^2$) is plotted from the SLAC-MIT data in

Figs p. 10. It really works! The function $F_2(x)$ gives a model of the proton structure. According to the figure, the typical momentum fraction of a quark in the proton is about

$$\xi \sim 0.2$$

as we might expect if the proton contains 3 quarks and some gluons.

The functions $f_f(x)$ are called "parton distribution functions" or pdf's. If we could compute the proton wavefunction in QCD, we could compute the pdf's and apply them to specific processes. This requires nonperturbative QCD and is very difficult. Alternatively, we can measure the pdf's in electron scattering and other processes and use the results to evaluate cross sections in hadron-hadron scattering.

We will discuss the determination of pdf's in more detail later in the course. For the moment, I would like to point out that there have been many more lepton-hadron deep inelastic scattering experiments, using μ and ν probes in addition to e. The largest Q^2 data currently has been obtained at the $e^{\pm}p$ collider HERA at DESY. This facility collides 28 GeV e^-'s or e^+'s with a beam of 920 GeV protons.

This allows us to observe deep inelastic scattering with $Q^2 > (100\ \text{GeV})^2$.

Figs. p. 11 shows a deep inelastic scattg event from HERA. The proton beam is coming in from the right. An e^+ coming from the left has been scattered backwards. The hadronic final state consists of a single jet, associated with the struck quark, plus proton fragments that go down the beampipe to the left.

Figs. p. 12 shows a measurmt of $F_2(x)$ from HERA. Note the dramatic extension of the range to very small values of x.