Computation of Multiparticle Amplitudes in QCD: MHV and Beyond

In the previous lecture, I introduced the Parke-Taylor <u>MHV</u> amplitudes. These are color-ordered amplitudes that take simple forms to all orders in perturbation theory. For all-gluon amplitudes, we proved that amplitudes in which external gluons have <u>all</u> + or one − vanish. The same argument shows that amplitudes with all − or one + vanish. The simplest nonvanishing tree amplitudes are then the amplitudes with two − or two +, called Maximal Helicity Violating (MHV) and anti-MHV. These are:

$$i\mathcal{M}(g_1^+ \cdots g_i^- \cdots g_j^- \cdots g_N^+) = i g^{N-2} \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle N1\rangle}$$

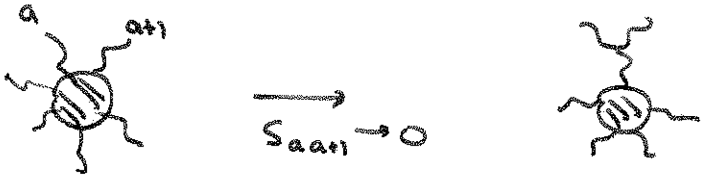
$$i\mathcal{M}(g_1^- \cdots g_i^+ \cdots g_j^+ \cdots g_N^-) = i(-1)^N g^{N-2} \frac{[ij]^4}{[12][23][34]\cdots[N1]}$$

Similarly, for $q\bar{q}$ + gluon tree amplitudes, the amplitudes with all + gluons or all − gluons vanish, and the MHV amplitudes are

$$i\mathcal{M}(q_1^- g_2^+ \cdots g_i^- \cdots g_{N-1}^+ \bar{q}_N^+) = i g^{N-2} \frac{\langle 1i\rangle^3\langle Ni\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle N1\rangle}$$

$$i\mathcal{M}(q_1^- g_2^- \cdots g_i^+ \cdots g_{N-1}^- \bar{q}_N^+) = -i(-1)^N \frac{[1i][Ni]^3}{[12][23][34]\cdots[N1]}$$

These are very beautiful formulae. The collinear singularities are explicitly displayed – and these appear only for the <u>nearest</u> <u>neighbors</u> in color order, corresponding to singularities



In the previous lecture, I proved only the case $N=4$. In this lecture, I will prove the general formula and, using the technology we develop, discuss some other applications.

In deriving the general formulae, a crucial building block is the 3 point on-shell amplitude. At first sight it seems that this cannot make sense, since if $1,2,3$ are lightlike vectors and

$$1+2+3=0,$$

then

$$2(2\cdot 3) = (2+3)^2 = 1^2 = 0$$

so there are no invariants for this vertex to depend on.

However, more carefully

$$2\cdot(2\cdot 3) = \langle 23\rangle[32]$$

so it is possible that $\langle 23\rangle$ might be <u>zero</u> while the amplitude still depends on $[32]$. For <u>real-valued</u> lightlike vectors, $[32] = \langle 23\rangle^*$. However, if we allow ourselves to consider general complex lightlike vectors, this is not true and we gain more freedom.

With this in mind, let's compute the on-shell 3-point amplitudes of color-ordered QCD. Begin with the $qg\bar{q}$ vertex:

$$= \frac{ig}{\sqrt{2}} \langle 1\gamma^\mu 3] \left(-\frac{1}{\sqrt{2}} \frac{[r\gamma_\mu 2\rangle}{[r2]}\right)$$

using an arbitrary reference vector r for the gluon polarization

$$= -ig \frac{\langle 12\rangle [r3]}{[r2]} \cdot \frac{\langle 12\rangle}{\langle 12\rangle}$$

$$[r2]\langle 12\rangle = -[r2]\langle 21\rangle = +[r3]\langle 31\rangle \quad \text{using} \quad 1+2+3=0$$

$$= -ig \frac{\langle 12\rangle^2}{\langle 31\rangle}$$

$$= ig \frac{\langle 12\rangle^3 \langle 32\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$ which is the MHV form!

A similar calculation gives the $qg^+\bar{q}$ vertex in the anti-MHV form.

Next,

$$= \frac{ig}{\sqrt{2}} \left[\varepsilon_1\cdot\varepsilon_2\, (1-2)\cdot\varepsilon_3 + \varepsilon_2\cdot\varepsilon_3\, (2-3)\cdot\varepsilon_1 + \varepsilon_3\cdot\varepsilon_1\, (3-1)\cdot\varepsilon_2\right]$$

choose $\quad \varepsilon_1^\mu = -\frac{1}{\sqrt{2}} \frac{[r\gamma^\mu 1\rangle}{[r1]} \qquad \varepsilon_2 = -\frac{1}{\sqrt{2}} \frac{[r\gamma^\mu 2\rangle}{[r2]} \qquad \varepsilon_3 = \frac{1}{\sqrt{2}} \frac{\langle s\gamma^\mu 3]}{\langle s3\rangle}$

so that $\varepsilon_1\cdot\varepsilon_2 = 0$. Then

$$\varepsilon_2\cdot\varepsilon_3 = -\frac{[r3]\langle s2\rangle}{[r2]\langle s3\rangle} \qquad \varepsilon_3\cdot\varepsilon_1 = -\frac{[r3]\langle s1\rangle}{[r1]\langle s3\rangle}$$

and the vertex becomes

$$\left(\frac{ig}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{[r1][r2]\langle s3\rangle}\Big[\,0 - [r3]\langle s2\rangle\,[r(2-3)1\rangle - [r3]\langle s1\rangle\,[r(3-1)2\rangle\,\Big]$$

$$= \frac{ig}{2}\,\frac{1}{[r1][r2]\langle s3\rangle}(-2)[r3]\Big[\langle s2\rangle[r(3)1\rangle - \langle s1\rangle[r(3)2\rangle\Big]$$

$$= ig\,\frac{[r3]}{[r1][r2]\langle s3\rangle}\Big([r3]\langle 31\rangle\langle 2s\rangle + [r3]\langle 32\rangle\langle s1\rangle\Big)$$

$$= ig\,\frac{[r3]^2}{[r1][r2]\langle s3\rangle}\big(-\langle 3s\rangle\langle 12\rangle\big)$$ using the Schouten identity

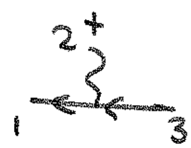
$$= ig\,\frac{\langle 12\rangle[r3]^2}{[r1][r2]}\,\frac{\langle 12\rangle^2}{\langle 12\rangle^2}$$

$$[r1]\langle 12\rangle = -[r3]\langle 32\rangle = [r3]\langle 23\rangle$$
$$[r2]\langle 12\rangle = -[r3]\langle 13\rangle = [r3]\langle 31\rangle$$

$$= ig\,\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} = ig\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

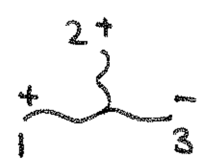
again, the MHV form. In all, we find the following elementary vertices:

$$= ig\,\frac{\langle 12\rangle^3\langle 32\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$



$$= +ig\,\frac{[12][32]^3}{[12][23][31]}$$

$$= ig\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

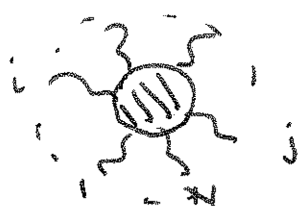


$$= -ig\,\frac{[12]^4}{[12][23][31]}$$

Note that the 3-point vertex with <u>one</u> $-$ is <u>nonvanishing</u>, and, with two $+$'s, has the anti-MHV form.

Now we would like to use these vertices as building-blocks for general QCD tree amplitudes. There is an obvious problem that the vertices on p. 4 take a simple form ~~because they~~ are <u>on-shell</u> vertices, while to build Feynman diagrams we need <u>off</u>-shell vertices. ~~However,~~ <u>Britto</u>, <u>Cachazo</u>, and <u>Feng</u> discovered a very beautiful method that circumvents this difficulty.

We already are working with complex vectors for which the identity $\langle pq\rangle^* = [qp]$ does not hold. So let us go one step further along this line. Consider a color-ordered amplitude

$i\mathcal{M} =$ 

and apply a <u>complex</u> shift to two of the legs. $i \to \hat{i}$ $\quad j \to \hat{j}$

$$\hat{i}\rangle = i\rangle \qquad \hat{i}] = i] + z\, j]$$

$$\hat{j}\rangle = j\rangle - z\, i\rangle \qquad \hat{j}] = j]$$

The momenta $\hat{i}$ and $\hat{j}$ are then

$$\hat{i} = i\rangle[i + z\, i\rangle[j \qquad \hat{j} = j\rangle[j - z\, i\rangle[j$$

so $\hat{i} + \hat{j} = i + j$ and momentum remains conserved. Call the amplitude evaluated at the shifted point $i\mathcal{M}(z)$.

Now consider

$$\oint \frac{dz}{2\pi i} \, \frac{1}{z} \, i\mathcal{M}(z)$$

around a large circle in the complex plane. If

$$i\mathcal{M}(z) \to 0 \qquad \text{as} \quad z \to \infty$$

then this integral vanishes. In that case, we can use this integral as a useful tool for calculating the original amplitude

$i\mathcal{M} = i\mathcal{M}(z=0)$.

For an N-gluon amplitude, $\mathcal{M}(z)$ does not vanish at ∞ in all cases. However, if the helicities of the gluons are

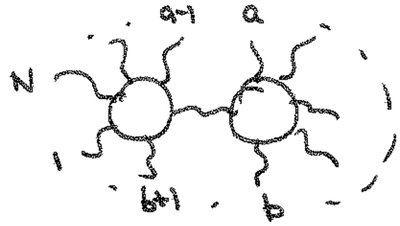
$$\begin{aligned} i &= - & j &= + \\ i &= - & j &= - \\ i &= + & j &= + \end{aligned}$$

(but not $i = +$, $j = -$), $\mathcal{M} \to 0$ as $z \to \infty$. I will show this later in the lecture. The same rules hold for external quarks.

Let's assume ~~that~~ we have chosen i and j so that one of these cases applies. Then we can evaluate the integral from the poles of the integrand and apply the constraint that the total integral is zero. There is an obvious pole at $z=0$ whose residue is

$$i\mathcal{M}(0)$$

the original amplitude to be evaluated. There are additional poles where denominators of the amplitude vanish. Since we are dealing with tree amplitudes, a denominator corresponds to a factorization



$$\sim \quad \frac{i}{s_{ab}(z)}$$

If i and j are on the same side of the propagator chosen, $s_{ab}(z) = \left(\sum_a^b p_k\right)^2$ is independent of z and this denominator does not give a pole in z. If i is on the left and j is on the right

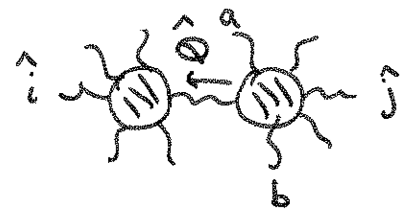
$$s_{ab}(z) = \left(\sum_a^b p_k + z\, i\rangle [j\right)^2$$

$$= s_{ab}(0) + z \langle i \left(\sum_a^b p_k\right) j]$$

and there is a pole at

$$z_* = - \frac{s_{ab}(0)}{\langle i \sum_a^b p_k\, j]}$$

the residue of this pole is



$$\hat{Q} = \sum_a^b p_k + z\, |i\rangle [j|$$

$$iM(g_{b+1} \cdots g_{\hat{i}} \cdots g_{a-1}\, g_{-\hat{Q}}) \; \frac{i}{\langle i\, |\hat{Q}|\, j]} \; iM(g_{\hat{Q}}\, g_a \cdots g_{\hat{j}} \cdots g_b)$$

$$\times \; \frac{2\pi i}{2\pi i\, z_*}$$

$$= \; (-1) \cdot \; iM(g_{b+1} \cdots g_{-\hat{Q}}) \; \frac{i}{s_{ab}} \; iM(g_{\hat{Q}} \cdots g_b)$$

If the sum of the pole vanishes :

$$iM \; = \; \sum_{\text{cuts.}} \; iM(g_{b+1} \cdots g_{\hat{i}} \cdots g_{-\hat{Q}}) \frac{i}{s_{ab}} \; iM(g_{\hat{Q}} \cdots g_{\hat{j}} \cdots g_b)$$

This is a recursion formula that allows us to evaluate iM in terms of lower-point amplitudes. Notice that, since the poles appear at $s_{ab}(z_*) = 0$, the amplitudes on the right are evaluated at on-shell but complex-valued momenta.

There are two subtleties needed in applying this formula. First, the recursion formula involves an amplitude evaluated at $\hat{Q}$ and an amplitude evaluated at $-\hat{Q}$. Thus, we need the spinors for $-Q$. These are given by $-Q\rangle = (-1)^{\frac{1}{2}} Q\rangle$. For a cut through a boson line, the $-Q$ spinors come in pairs, and it does not matter which square root we take as long as we are consistent. For definiteness, I will use

$$-Q\rangle = i\, Q\rangle \qquad -Q] = i\, Q]$$

A cut through a fermion line involves a fermion propagator, which is

$$i\,\frac{Q]\langle Q}{Q^2} \qquad \text{or} \qquad i\,\frac{Q\rangle[Q}{Q^2}$$

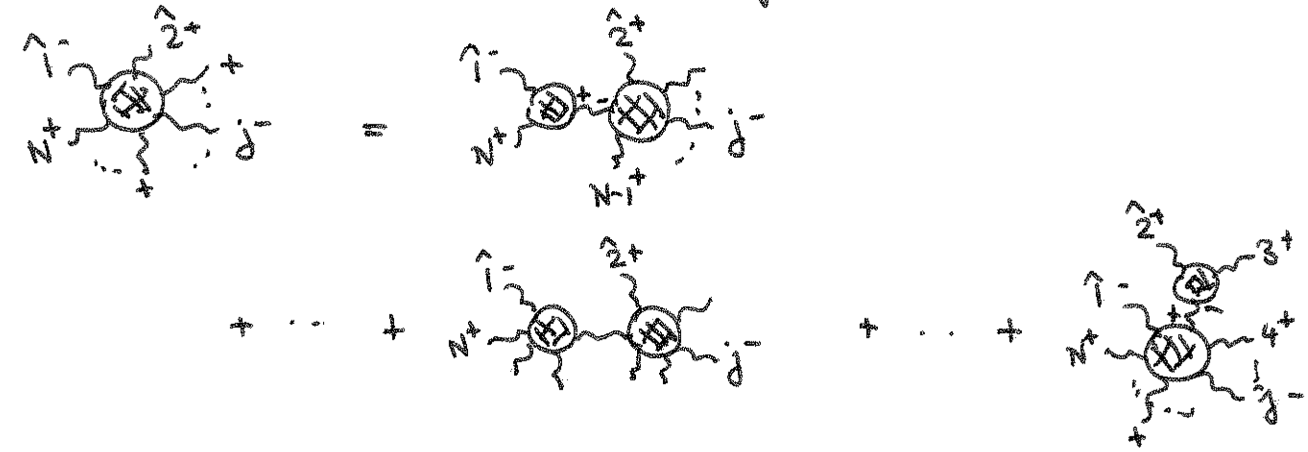
In either case, we have a $Q]$ not a $-Q]$ on the left. So, if we use the rule above to compute the amplitude on the left, we need to multiply by $(-i)$ for a fermion cut.

Now we are ready to apply the Britto Cachazo Feng (BCF) recursion formula. As a first application, I will prove the MHV formula for gluon amplitudes for general N. The analysis is due to Risager. The other three formulae on p. 1 can be proved by the same method.

The proof will be by induction. We have already verified the formula explicitly for N=3,4. Now assume the formula for N-1, prove it for N. To make the analysis, choose the gluon 1 to be one of the two gluons with $-$ helicity. Shift this gluon in the square bracket and its next neighbor in the angle bracket:

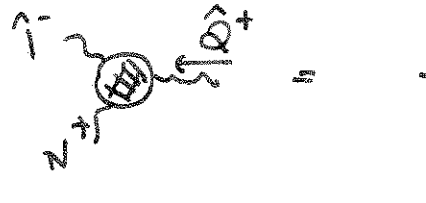
$$\hat{1}] = 1] + z\, 2] \qquad \hat{2}\rangle = 2\rangle - z\, 1\rangle$$

Then the BCF recursion formula implies



In each cut diagram, we need to consider both possible assignments of helicity to the cut line. The helicities are outgoing, so the possible assignments are $+\sim -$ or $-\sim +$. In the first diagram, the assignment with + on the right gives an on-shell amplitude with only one $-$; this is zero. In the last diagram, the assignment with + on the right gives a 3-point vertex with all + ; this is zero. For all other diagrams, either assignment gives an on-shell diagram with only one $-$ on one side or the other. Thus, all of these diagrams are zero, leaving over only the first and the last.

The first diagram involves the structure



$$= -ig\,\frac{[-\hat{Q}N]^4}{[N\hat{1}][\hat{1}-\hat{Q}][-\hat{Q}N]} = +ig\,\frac{[\hat{Q}N]^3}{[N\hat{1}][\hat{1}\hat{Q}]}$$

Now $\hat{Q} = 1\rangle[1 + N\rangle[N + z\,1\rangle[2$

This diagram is expected to have a pole at

$$z_* = -\frac{s_{1N}}{\langle 1(1+N)2]} = -\frac{\langle 1N\rangle[N1]}{\langle 1N\rangle[N2]} = -\frac{[N1]}{[N2]}$$

But, compute the various factors.

$$[N\hat{1}] = [N\,1] + z\,[N2] = [N2]\,(z - z_*)$$

To analyze the other factors, multiply by $\langle \hat{Q} a\rangle$, where a is a random lightlike vector

$$[\hat{1}\hat{Q}]\langle \hat{Q} a\rangle = ([1 + z[2)\,(\ 1]\langle 1 + N]\langle N + z\ 2]\langle 1\ )\ a\rangle$$

$$= ([1N] + z\,[2N])\langle Na\rangle + z\,[21]\langle 1a\rangle + z\,[12]\langle 1a\rangle$$

$$= (z - z_*)\,[2N]\langle Na\rangle$$

$$[\hat{Q} N]\langle \hat{Q} a\rangle = -\,[N\hat{Q}]\langle \hat{Q} a\rangle = -\,[N\,(1]\langle 1 + N]\langle N + z\,2]\langle 1)\ a$$

$$= -([N1] + z\,[N2])\langle 1a\rangle = [2N]\langle 1a\rangle(z - z_*)$$

In all, the vertex behaves as

$$\frac{(z - z_*)^3}{(z - z_*)\,(z - z_*)} \sim (z - z_*)^1$$

and this cancels the pole from the propagator.

The only nonzero contribution, then, comes from the last diagram. This is (using the induction hypothesis)

$$\left(i g^{n-3} \frac{\langle \hat{1} j\rangle^4}{\langle \hat{1}\,{-\hat{Q}}\rangle\langle -\hat{Q}\,4\rangle\langle 45\rangle\cdots\langle N\hat{1}\rangle} \right) \frac{i}{s_{23}} \left(-ig \frac{[\hat{2}3]^4}{[\hat{Q}\hat{2}][\hat{2}3][3\hat{Q}]} \right)$$

But $\hat{1}\rangle = 1\rangle$ $\quad \hat{2}] = 2]$ so this simplifies to

$$-ig^{n-2} \frac{\langle 1j\rangle^4\ [23]^4}{\langle 45\rangle\cdots\langle N1\rangle\ \ \langle 1\hat{Q}\rangle[\hat{Q}3]\ \ \langle 4\hat{Q}\rangle[\hat{Q}2]\,[23]\,\langle 23\rangle\,[23]\,(-1)} \frac{1}{\ }$$

with $\qquad \hat{Q} = -\,2\rangle[2 \;-\; 3\rangle[3 \;+\; z\,1\rangle[2$

now
$$\langle 1\hat{Q}\rangle[\hat{Q}3] = -\langle 12\rangle[23]$$
$$\langle 4\hat{Q}\rangle[\hat{Q}2] = -\langle 43\rangle[32] = -\langle 34\rangle[23]$$

and we find

$$= ig^{n-2}\,\frac{\langle 1j\rangle^4\,[\cancel{23}]^4}{\langle 45\rangle\cdots\langle N1\rangle\,\langle 12\rangle[\cancel{23}]\,\langle 34\rangle[\cancel{23}][\cancel{23}]\langle 23\rangle[\cancel{23}]}$$

$$= ig^{n-2}\,\frac{\langle 1j\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\cdots\langle N1\rangle}$$

This proves the induction step, and we are done!

We still need to pick up one loose end in our derivation of the BCF formula — the derivation of the rules on p. 6 for the allowed shifts. We would like to find the condition under which, for the shift

$i \to \hat{i} = i\rangle([i + z[j) \qquad j \to \hat{j} = (j\rangle$ [redacted] $z\,i\rangle)[j$

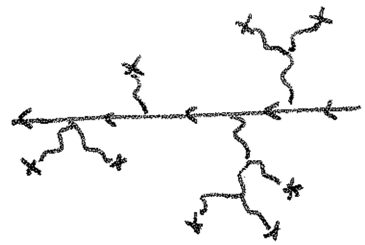
the shifted amplitude $\mathcal{M}(z)$ vanishes as $z \to \infty$. Kaplan and Arkani-Hamed suggested analyzing this in the following way: As $z \to \infty$, $\hat{i}$ and $\hat{j}$ approach the large lightlike vectors

$$\pm z\;\; i\rangle[j$$

We should imagine a particle carrying this large lightlike momentum through the diagram and interacting with quanta carrying finite momenta. Those particles can be described as a background field. Thus, quantize the high-momentum particle in background-field Feynman gauge. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\, A^a_\mu \left(-\underset{\sim}{D}^2\right) A^{\mu a} + g f^{abc} A^b_\mu \underset{\sim}{F}^{\mu\nu b} A^c_\nu$$

where $\underset{\sim}{D}_\mu$ and $\underset{\sim}{F}^{\mu\nu}$ are evaluated with the background field. If we use only the first term, for the momentum, we obtain diagrams of the form



where the large momentum $q = z\, i\rangle[j$ moves along the solid line. Each propagator gives $\frac{1}{z}$, each vertex gives at most one z from one factor of momentum; thus these diagrams are $\mathcal{O}(z)$. These diagrams also conserve the internal Lorentz index of A_μ. To violate this conservation, we need to use the $F^{\mu\nu}$ vertex, which has no powers of z. Thus.

$$i\mathcal{M}^{\mu\nu} = \quad \mu \; \text{———} \; \nu$$

$$\underset{z\to\infty}{\longrightarrow} \left(\begin{matrix} c\, z \\ + \cdots \end{matrix}\right) g^{\mu\nu} + A^{\mu\nu} + \frac{1}{z} B^{\mu\nu} + \cdots$$

where the $A^{\mu\nu}$ term comes from making one $F^{\mu\nu}$ insertion and is therefore antisymmetric in $[\mu\nu]$

Now choose polarization vectors. We can take j and i to be the reference vectors for $\hat{i}$ and $\hat{j}$, respectively. Then

$$\varepsilon_+^\mu(\hat{i}) = \frac{1}{\sqrt{2}} \frac{\langle j\gamma^\mu \hat{i}]}{\langle j\hat{i}\rangle} = \frac{1}{\sqrt{2}}\left(\frac{\langle j\gamma^\mu i]}{\langle ji\rangle} + z\frac{\langle j\gamma^\mu j]}{\langle ji\rangle}\right)$$

$$\varepsilon_-^\mu(\hat{i}) = -\frac{1}{\sqrt{2}} \frac{[j\gamma^\mu \hat{i}\rangle}{[j\hat{i}]} = -\frac{1}{\sqrt{2}} \frac{[j\gamma^\mu i\rangle}{[ji]}$$

$$\varepsilon_+^\mu(\hat{j}) = \frac{1}{\sqrt{2}} \frac{\langle i\gamma^\mu \hat{j}]}{\langle i\hat{j}\rangle} = \frac{1}{\sqrt{2}} \frac{\langle i\gamma^\mu j]}{\langle ij\rangle}$$

$$\varepsilon_-^\mu(\hat{j}) = -\frac{1}{\sqrt{2}} \frac{[i\gamma^\mu \hat{j}\rangle}{[i\hat{j}]} = -\frac{1}{\sqrt{2}} \frac{[i\gamma^\mu j\rangle - z[i\gamma^\mu i\rangle}{[ij]}$$

With $\quad q = i\rangle[j \qquad \bar{q} = j\rangle[i \qquad k_i = i\rangle[i \qquad k_j = j\rangle[j$

$$\varepsilon_+(\hat{i}) \sim (\bar{q} + zk_j) \qquad \varepsilon_-(\hat{i}) \sim q \qquad \varepsilon_+(\hat{j}) \sim q \qquad \varepsilon_-(\hat{j}) \sim (\bar{q} - zk_i)$$

with all factors of z now made explicit. Note that

$$q^2 = \bar{q}^2 = k_i^2 = k_j^2 = 0 \qquad k_i\cdot q = k_j \cdot q = 0 \qquad k_i\cdot\bar{q} = k_j\cdot\bar{q} = 0$$

Now evaluate $\mathcal{M}(z)$ for the various cases. For $i = -\quad j = +$

$$i\mathcal{M}_{-+}(z) \sim q_\mu\left[(cz+\dots)g^{\mu\nu} + A^{\mu\nu} + \frac{1}{z}B^{\mu\nu} + \dots\right]q_\nu$$

$$\sim (cz+\dots)q^2 + A^{\mu\nu}q_\mu q_\nu + \mathcal{O}(\tfrac{1}{z})$$

since A is antisymmetric, $\quad i\mathcal{M}_{-+}(z) = \mathcal{O}(\tfrac{1}{z})$.

For $i=-$ $j=-$ it will be useful to have the Ward identity for $\mathcal{M}$:

$$k_{\hat{i}\mu}\mathcal{M}^{\mu\nu}\varepsilon_\nu(\hat{j}) = 0$$

$$(k_i + zq)_\mu \mathcal{M}^{\mu\nu}\varepsilon_\nu(\hat{j}) = 0$$

or
$$q_\mu \mathcal{M}^{\mu\nu}\varepsilon_\nu(\hat{j}) = -\frac{1}{z} k_{i\mu}\mathcal{M}^{\mu\nu}\varepsilon_\nu(\hat{j})$$

Then for $i=-$ $j=-$

$$i\mathcal{M}_{--}(z) \sim q_\mu \mathcal{M}^{\mu\nu}(z)\ (\bar{q} - z k_i)_\nu$$

$$= -\frac{1}{z} k_{i\mu}\ \left[(cz+\ldots) g^{\mu\nu} + A^{\mu\nu} + \frac{1}{z}B^{\mu\nu} + \ldots\right](\bar{q} - z k_i)_\nu$$

$$= -\frac{1}{z}(cz+\ldots)(k_i\bar{q} - z k_i^2) + A^{\mu\nu} k_{i\mu} k_{i\nu} + \mathcal{O}\left(\frac{1}{z}\right)$$

$$= \mathcal{O}\left(\frac{1}{z}\right)$$

A similar argument gives $i\mathcal{M}_{++}(z) = \mathcal{O}\left(\frac{1}{z}\right)$. For the final case

$$i\mathcal{M}_{+-} = \mathcal{O}(z^3)$$

and there is no BCF recursion formula for this shift.

Up to this point, we have only discussed spinor methods for massless particles. However, in discussions of QCD for hadron colliders, we also need to deal with massive particles — W, Z, and t. Let me now briefly discuss spinor methods for these particles.

For W and Z, the simplest method is to realize that the W is unstable and decays to a pair of massless fermions, for example $W^+ \to \ell^+ \nu$. We can include the decay explicitly in a Feynman diagram for W production

$$\mathcal{M}^\mu \quad \to \quad \mathcal{M}^\mu \, \frac{i}{q^2 - m_W^2} \, \langle 1 \gamma_\mu 2]$$

In fact, we can use this structure to represent the W polarization vector. If we integrate over the decay angles in the W^+ rest frame

$$\int \frac{d\Omega}{4\pi} \, \langle 1 \gamma_\mu 2] \langle 2 \gamma^\nu 1]$$

$$= \int \frac{d\Omega}{4\pi} \, \mathrm{tr}\,[\not{k}_1 \gamma_\mu \not{k}_2 \gamma_\nu]$$

This tensor is obviously orthogonal to the W^+ 4 vector q and otherwise has no preferred vector, so

$$\int \frac{d\Omega}{4\pi} \, \langle 1 \gamma_\mu 2] \langle 2 \gamma_\nu 1] \;=\; A\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}\right)$$

To evaluate A, contract on $\mu\nu$:

$$\int \frac{d\Omega}{4\pi} \; 2 \, \langle 12\rangle [12] \;=\; 3A$$

$$= \; -2\, m_W^2$$

The structure on the right-hand side is just the sum of W polarization vectors, so

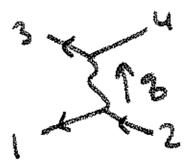
$$\frac{3}{2m_W^2}\int\frac{d\Omega}{4\pi}\,\langle 1\gamma^\mu 2]\,\langle 2\gamma^\nu 1] = \sum_{\text{pols.}} \varepsilon^\mu(q)\,\varepsilon^{\nu}(q)^*$$

We can use this identity to evaluate amplitudes for W production in the following way:

1. Write the QCD amplitude in terms of spinors, and replace the W polarization vector by $\langle 1\gamma^\mu 2]$. Evaluate.
2. Integrate the vectors 1,2 over the decay angles in the W^+ rest frame
3. Multiply the result by $\frac{3}{2m_W^2}$.

Typically, we will evaluate the integral over the decay angles by Monte Carlo. Then the selected points will actually be distributed according to the correct polarized decay distribution for $W^+\to \ell^+\nu$. This is a nice bonus of this technique.

W production amplitudes computed in this way have their own MHV formulae. Consider, for example, the amplitude for the simple Drell-Yan process:



$$i\mathcal{M} = \left(\frac{ig_W}{\sqrt{2}}\right)^2 \frac{-i}{s_{12}}\,\langle 1\gamma^\mu 2]\,\langle 3\gamma_\mu 4]$$

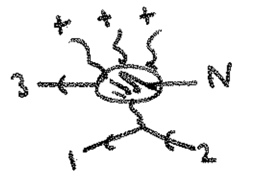
$$= ig_W^2\,\frac{\langle 13\rangle[42]}{\langle 12\rangle[21]}\cdot\frac{\langle 13\rangle}{\langle 13\rangle}$$

$$= ig_W^2\,\frac{\langle 13\rangle^2\,[42]}{\langle 12\rangle\left(-[24]\langle 43\rangle\right)}$$

so that

$$i\mathcal{M} = -ig_w^2 \frac{\langle 13\rangle^2}{\langle 12\rangle\langle 34\rangle}$$

This appears to be of an MHV form. In fact, the color-ordered amplitude with all +-helicity gluons has a similar simple form



$$= -ig_w^2\, g^{N-4} \frac{\langle 13\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle\cdots\langle N-1\,N\rangle}$$

and

$$= -i(-1)^N g_w^2\, g^{N-4} \frac{[2N]^2}{[12][34][45]\cdots[N-1\,N]}$$

For a massive colored particle that can emit many gluons, we need a more sophisticated treatment of the spin states. Schwinn and Weinzierl have provided a very effective formalism, based on introducing a massless reference vector q. Let p be the on-shell momentum of a massive fermion. Then let

$$p^\flat = p - \frac{m^2}{2p\cdot q}\, q \qquad q^2 = 0$$

$p^\flat$ is a lightlike vector. Now we can generalize:

		massless	$\longrightarrow$ massive	
outgoing fermion	$+$	$[p$	$\dfrac{\langle q(\not{p}+m)}{\langle q p^\flat\rangle}$	right-handed fermion
	$-$	$\langle p$	$\dfrac{[q(\not{p}+m)}{[q p^\flat]}$	left-handed fermion

outgoing antifermion

$$+ \qquad p] \quad \to \quad \frac{(\not{p}-m)\,q\rangle}{\langle p^b q\rangle} \qquad \text{right-handed antifermion}$$

$$- \qquad p\rangle \quad \to \quad \frac{(\not{p}-m)\,q]}{[p^b q]} \qquad \text{left-handed antifermion}$$

As $m \to 0$ $\quad \langle q(\not{p}+m) \to \langle q \not{p} = \langle q p^b\rangle [p^b$

and we go back to the previous forms. These expressions are correct if they give the correct polarization sum

$$\sum_h u_h(p)\,\bar{u}_h(p) = \not{p} + m$$

In our notation for spinor calculations, this reads:

$$(-i)\left[v_+(-p)\,\bar{u}_-(p) + v_-(-p)\,\bar{u}_+(p)\right]$$

[The $(-i)$ is the one discussed on p. 9.]

$$= \frac{(\not{p}+m)\,q\rangle [q\,(\not{p}+m)}{\langle p^b q\rangle\,[q p^b]} + \frac{(\not{p}+m)\,q]\langle q\,(\not{p}+m)}{[p^b q]\langle q p^b\rangle}$$

The sum of terms is

$$\frac{\not{p}\not{q}\not{p} + m(\not{q}\not{p} + \not{p}\not{q}) + m^2 \not{q}}{2p^b \cdot q}$$

$$= \frac{\not{p}\,2p\cdot q - \cancel{m^2 \not{q}} + m\{\not{q},\not{p}\} + \cancel{m^2 \not{q}}}{2p\cdot q} = (\not{p}+m)$$

It is interesting to use this formalism to compute the cross section for $e^+e^- \to t\bar{t}$.

$1 \to e^+_R \quad 2 \to e^-_L \quad 3 \to t \quad 4 \to \bar{t}$

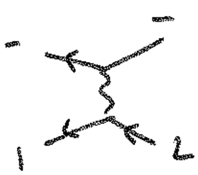
For convenience, set $q = 1$. Then

$$= (-ie)(+i\tfrac{2}{3}e)\cdot \langle 1\gamma^\mu 2] \frac{-i}{s_{12}} \frac{\langle 1(3+m)\gamma_\mu(4-m)1\rangle}{\langle 13^b\rangle\langle 4^b 1\rangle}$$

$$= -i\tfrac{2}{3}e^2 \frac{1}{s_{12}} \frac{m(\langle 1\gamma_\mu 4 1\rangle - \langle 1 3\gamma_\mu 1\rangle)}{\langle 13^b\rangle\langle 4^b 1\rangle} \langle 1\gamma^\mu 2]$$

$$= -i\tfrac{2}{3}e^2 \frac{1}{s_{12}} m \frac{\langle 11\rangle[241\rangle - \langle 132]\langle 11\rangle}{\langle 13^b\rangle\langle 4^b 1\rangle}$$

$$= 0$$



$$= -i\tfrac{2}{3}e^2 \frac{1}{s_{12}} \frac{m}{[13^b][4^b 1]} \frac{([1\gamma_\mu 4 1] - [13\gamma_\mu 1])}{} \langle 1\gamma^\mu 2]$$

$$= -i\tfrac{2}{3}e^2 \frac{2}{s_{12}} \frac{m}{[13^b][4^b 1]} ([12]\langle 141] - [131\rangle[21])$$

$$= -i\tfrac{4}{3}e^2 \frac{1}{s_{12}} \frac{m\,[12]}{[13^b][4^b 1]} \langle 1\,\underbrace{(3+4)}_{= -1-2}\,1]$$

$$= -i\tfrac{4}{3}e^2 \frac{1}{s_{12}} \frac{m\,[12]}{[13^b][4^b 1]} \langle 12\rangle[21] \times(-1)$$

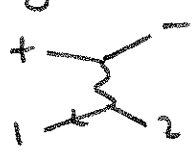
$$= i\tfrac{4}{3}e^2 \frac{m\,[12]}{[13^b][4^b 1]}$$

$$= -i\frac{2}{3}e^2 \frac{\langle 1\gamma^\mu 2]}{s_{12}} \frac{[1(3\gamma_\mu 4 - m^2\gamma_\mu)1\rangle}{[13^b]\langle 4^b 1\rangle}$$

$$= -i\frac{2}{3}e^2 \frac{2}{s_{12}} \frac{[[13\,1\rangle[24\,1\rangle + m^2\cdot 0]}{[13^b]\langle 4^b 1\rangle}$$

$$= -i\frac{4}{3}e^2 \frac{\langle 3^b 1\rangle [24^b]}{s_{12}}$$

similarly



$$= -i\frac{4}{3}e^2 \frac{[3^b 2]\langle 14^b\rangle}{s_{12}}$$

The sum over polarizations is then proportional to

$$\left|\frac{m[12]}{[13^b][4^b 1]}\right|^2 + \left|\frac{\langle 3^b 1\rangle[24^b]}{s_{12}}\right|^2 + \left|\frac{[3^b 2]\langle 14^b\rangle}{s_{12}}\right|^2$$

And $|[13^b]|^2 = 2\,1\cdot 3^b = 2\,1\cdot 3$

set $1 = (-E, 0, 0, E)$ $\quad 2 = (-E, 0, 0, -E)$ $\quad s = \sin\theta$

$3 = (E, ps, 0\ pc)$ $\quad 4 = (E, -ps\ 0\ -pc)$ $\quad c = \cos\theta$

then this evaluates to:

$$\frac{m^2}{(E+pc)(E-pc)} + \frac{(E+pc)(E+pc - \frac{m^2}{E-pc})}{4E^2} + \frac{(E-pc)(E-pc - \frac{m^2}{E+pc})}{4E^2}$$

$$= \frac{1}{2E^2}\left(E^2 + (pc)^2 + m^2\right)$$

$$= \frac{1}{2}\left(\left(1+\frac{m^2}{E^2}\right) + \left(1-\frac{m^2}{E^2}\right)c^2\right)$$

Supplying the various needed factors

$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to t\bar{t}) = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{p}{E}\cdot 4\left(\tfrac{2}{3}e^2\right)^2 \cdot \underset{\text{color}}{3} \cdot \overbrace{\tfrac{1}{4}\cdot 2}^{e^-e^+\text{ pols.}}$$

$$\times \quad \frac{1}{2}\left(\left(1+\frac{m^2}{E^2}\right) + \left(1-\frac{m^2}{E^2}\right)\cos^2\theta\right)$$

$$= \frac{\pi\alpha^2}{2s}\left(\frac{2}{3}\right)^2\cdot 3\cdot \frac{p}{E}\left(\left(1+\frac{m^2}{E^2}\right)+\left(1-\frac{m^2}{E^2}\right)\cos^2\theta\right)$$

which is correct!

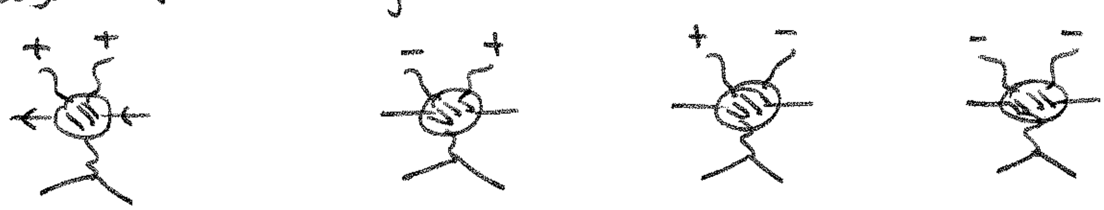
To conclude this lecture, I would like to give a practical example of a QCD computation that can be done with the BCF recursion formula. We have already studied W+1 jet production. The amplitudes for that process are various crosses of

$$0 \to q\bar{q}g + W$$

What if we want to study W + 2 jet production? We need to compute

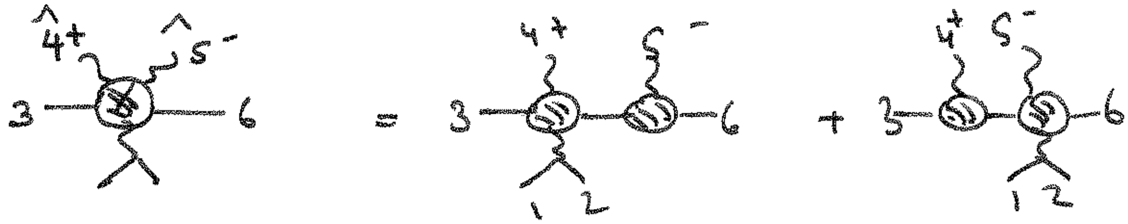
$$0 \to q g g \bar{q} + W$$

amplitudes. There are four cases:



The first and last are MHV and are computed by the simple formulae on p. 18. Let's compute one of the other cases by BCF.

With a shift of the two gluon legs.



with $\hat{4}\rangle = 4\rangle + z\,5\rangle \qquad \hat{5}] = 5] - z\,4]$

the first diagram is

$$-i g_w^2 g \frac{\langle 13\rangle^2}{\langle 12\rangle\langle 3\hat{4}\rangle\langle \hat{4}\hat{Q}\rangle} \overset{\text{fermion cut}}{(-i)} \frac{i}{s_{56}}\, ig \frac{\langle \hat{Q}5\rangle^3 \langle 65\rangle}{\langle \hat{Q}5\rangle\langle 56\rangle\langle 6\hat{Q}\rangle} .$$

$$= g_w^2 g^2 \frac{1}{i} \frac{\langle 13\rangle^2 \langle \hat{Q}5\rangle^2 (-1)}{\langle 12\rangle\langle 3\hat{4}\rangle\langle 4Q\rangle\langle 6\hat{Q}\rangle s_{56}}$$

$$\hat{Q} = -5\rangle[5 - 6\rangle[6 + z\,5\rangle[4$$

so $$z_* = \frac{s_{56}}{\langle 5(5+6)4]} = \frac{\langle 56\rangle[65]}{\langle 56\rangle[64]} = \frac{[65]}{[64]}$$

$$\langle 3\hat{4}\rangle = \langle 34\rangle + z\langle 35\rangle = \frac{\langle 34\rangle[64] + \langle 35\rangle[65]}{[64]}$$

$$= \frac{1}{[46]}\langle 3(4+5)6]$$

$$= i g_w^2 g^2 \frac{\langle 13\rangle^2 [46]}{\langle 12\rangle s_{56} \langle 3(4+5)6]} \frac{\langle 5\hat{Q}\rangle^2}{\langle 4\hat{Q}\rangle\langle 6\hat{Q}\rangle} \cdot \left(\frac{[\hat{Q}4]}{[\hat{Q}4]}\right)^2$$

$$\langle 5\hat{Q}\rangle[\hat{Q}4] = -\langle 56\rangle[64]$$

$$\langle 6\hat{Q}\rangle[\hat{Q}4] = -\langle 65\rangle[54] = -\langle 56\rangle[45]$$

$$\langle \hat{4}\hat{Q}\rangle[\hat{Q}4] = -\langle 4(5+6)4] - z\langle 5(5+6)4]$$

$$= -s_{45} - s_{46} - \frac{[65]}{[64]}\langle 56\rangle[64]$$

$$= -s_{45} - s_{46} - s_{56} = -s_{456} = -(4+5+6)^2$$

Assembly the pieces:

$$= i g_w^2 g^2 \frac{\langle 13\rangle^3 [46] \langle \cancel{5}6\rangle^2 [46]^2}{\langle 12\rangle\langle \cancel{5}6\rangle[65]\langle \cancel{5}6\rangle[45] s_{456}\langle 3(4+5)6]}$$

$$= -i g_w^2 g^2 \frac{\langle 13\rangle^2 [46]^3}{\langle 12\rangle [45][56] s_{456}\langle 3(4+5)6]}$$

the second diagram is

$$i g \frac{[3\hat{4}][-\hat{Q}\hat{4}]^3}{[3\hat{Q}][\hat{4}\,-\hat{Q}][-\hat{Q}3]} (-i)\frac{i}{s_{34}} \; i g_w^2 g \frac{[26]^2}{[12][\hat{Q}\hat{5}][\hat{5}6]}$$

$$= -g_w^2 g^2 \cdot i \cdot (-1) \frac{[\hat{4}\hat{Q}]^2 [26]^2}{[3\hat{Q}][12][\hat{5}\hat{Q}][\hat{5}6]} \frac{1}{s_{34}}$$

$$= i g_w^2 g^2 \frac{[26]^2 [4\hat{Q}]^2}{[12][34]\langle 34\rangle(-1)[\hat{5}6][\hat{5}\hat{Q}][3\hat{Q}]}$$

$$Q = 3\rangle[3 + 4\rangle[4 + z5\rangle[4$$

$$z_* = -\frac{s_{34}}{\langle 5(3+4)4]} = -\frac{\langle 43\rangle[34]}{\langle 53\rangle[34]} = -\frac{\langle 34\rangle}{\langle 35\rangle}$$

$$[\hat{5}6] = [56] - z[46] = [56] + \frac{\langle 34\rangle[46]}{\langle 35\rangle}$$

$$= \frac{1}{\langle 35\rangle}\langle 3(4+5)6]$$

$$= -ig_w^2 g^2 \frac{[26]^2[4\hat{Q}]^2\langle 35\rangle}{[12][34]\langle 34\rangle\langle 3(4+5)6][5\hat{Q}][\hat{5}\hat{Q}]}\cdot\left(\frac{\langle\hat{Q}5\rangle}{\langle\hat{Q}5\rangle}\right)^2$$

$$[4\hat{Q}]\langle\hat{Q}5\rangle = [43]\langle 35\rangle = -[34]\langle 35\rangle$$

$$[3\hat{Q}]\langle\hat{Q}5\rangle = [34]\langle 45\rangle$$

$$[5\hat{Q}][\hat{Q}\hat{5}] = s_{345}$$

$$= -ig_w^2g^2\frac{[26]^2\langle 35\rangle^3\cancel{[34]^2}}{[12]\langle 34\rangle\cancel{[34]^2}\langle 45\rangle s_{345}\langle 3(4+5)6]}$$

in all

$$= -ig_w^2g^2\left[\frac{\langle 13\rangle^2[46]^3}{\langle 12\rangle[45][56]s_{456}\langle 3(4+5)6]} + \frac{[26]^2\langle 35\rangle^3}{[12]\langle 34\rangle\langle 45\rangle s_{345}\langle 3(4+5)6]}\right]$$

This is a quite simple expression for what would have been a quite involved Feynman diagram computation!