Computation of Multiparticle Amplitudes in QCD: color-ordered amplitudes

In the previous lecture, I introduced a spinor-product technology for computing amplitudes in QED. I would now like to extend this discussion to QCD. Here there is another set of simplifications that works together with the spinor-product simplifications.

To begin, analyze the process $q_L \bar{q}_R \to gg$. As before, we treat all particles as outgoing, so we analyze this as $0 \to q_L g g \bar{q}_R$ (1 2 3 4).

There are 3 Feynman diagrams at tree level:

then

$$i\mathcal{M} = (ig)^2 \Big[\bar{u}(1)\, \gamma\cdot\varepsilon(2)\, \frac{i\,(1+2)}{s_{12}}\, \gamma\cdot\varepsilon(3)\, u(4)\; t^a t^b$$

$$+\ \bar{u}(1)\, \gamma\cdot\varepsilon(3)\, \frac{i\,(1+3)}{s_{13}}\, \gamma\cdot\varepsilon(2)\, u(4)\; t^b t^a$$

$$+\ i f^{abc} \left(\frac{-i}{s_{23}}\right) \bar{u}(1)\gamma^\lambda u(4)\; t^c$$

$$\cdot \big[\ \varepsilon(2)\cdot\varepsilon(3)\,(2-3)_\lambda + \varepsilon_\lambda(3)\,(2\cdot 3+2)\cdot\varepsilon(2) + \varepsilon_\lambda(2)\,(-2\cdot 2-3)\cdot\varepsilon(3)\ \big]\ \Big]$$

Color is an issue here. The first two terms have different color structures. The third term can be brought into a combination of the first two: $\quad if^{abc}t^c = [t^a, t^b] = t^a t^b - t^b t^a$

Then the complete amplitude can be written as

$$i\mathcal{M} = i\mathbb{M}(1234)\cdot(2t^a t^b) + i\,\mathbb{M}(1324)\,(2t^b t^a)$$

where

$$i\mathbb{M}(1234) = \frac{1}{2}(ig)^2\Big[\bar{u}(1)\,\gamma\cdot\epsilon(2)\,\frac{i\,(1+2)}{s_{12}}\,\gamma\cdot\epsilon(3)\,u(4)$$

$$-\frac{i}{s_{23}}\,\bar{u}(1)\gamma^\lambda u(4)\,\big[\,\epsilon(2)\cdot\epsilon(3)\,(2-3)_\lambda + \epsilon_\lambda(3)\;2\;3\cdot\epsilon(2) - 2\,\epsilon_\lambda(2)\;2\cdot\epsilon(3)\,\big]\Big]$$

The elements $i\mathbb{M}$ are called color-ordered amplitudes. For later convenience, I will rescale the color matrices

$$T^a = \sqrt{2}\,t^a \qquad \text{so that} \qquad \mathrm{tr}\, T^a T^b = \delta^{ab}$$

As SU(N) generators, the T^a obey

$$T^a_{ij}\,T^a_{k\ell} = \delta_{i\ell}\,\delta_{kj} - \frac{1}{N}\,\delta_{ij}\delta_{k\ell} \qquad a = 1 - N^2-1$$

Later in the lecture, we will work with U(N). For this we add an additional generator

$$T^0 = \frac{1}{\sqrt{N}}\,\mathbf{1}_n$$

Summing over $a = 0, 1, \ldots N^2-1$ gives

$$T^a_{ij}\,T^a_{k\ell} = \delta_{i\ell}\,\delta_{kj}$$

Let's now compute the above color-ordered amplitude explicitly. As in the QED case, there are four possible assignments of the gluon helicity. For $g_+(2)\, g_+(3)$, choose the polarization vectors

$$\varepsilon^\mu(2) = \frac{1}{\sqrt{2}} \frac{\langle 1 \gamma^\mu 2]}{\langle 12\rangle} \qquad \varepsilon^\mu(3) = \frac{1}{\sqrt{2}} \frac{\langle 1 \gamma^\mu 3]}{\langle 13\rangle}$$

In the first line of $\mathcal{M}$ we have

$$\bar{u}(1)\, \gamma\cdot\varepsilon(2) \cdots \sim \langle 1 \gamma^\mu \cdots \,\cdot\, \langle 1 \gamma_\mu 2] \sim \langle 11\rangle [2 \cdots = 0$$

For the second line, we have

$$\varepsilon(2)\cdot\varepsilon(3) \sim \langle 11\rangle = 0$$

$$\bar{u}(1)\gamma^\mu u(4) \cdot \varepsilon_\mu(2) \sim \langle 1\gamma^\mu 4]\, \langle 1 \gamma_\mu 2] \sim \langle 11\rangle \cdots = 0$$

and similarly for $\varepsilon_\mu(3)$. Then

$$i\mathcal{M}(\bar{q}_1^- g_2^+ g_3^+ q_4^+) = 0$$

Similarly, choosing

$$\varepsilon^\mu(2) = -\frac{1}{\sqrt{2}} \frac{[4 \gamma^\mu 2\rangle}{[42]} \qquad \varepsilon^\mu(3) = -\frac{1}{\sqrt{2}} \frac{[4 \gamma^\mu 3\rangle}{[43]}$$

we see that

$$i\mathcal{M}(\bar{q}_1^- g_2^- g_3^- q_4^+) = 0$$

The remaining two cases are independent color-ordered amplitudes and must both be computed. For $g_2^+ g_3^-$, choose

$$\varepsilon^\mu(2) = \frac{1}{\sqrt{2}} \frac{\langle 1 \gamma^\mu 2]}{\langle 12\rangle} \qquad \varepsilon^\mu(3) = -\frac{1}{\sqrt{2}} \frac{[4 \gamma^\mu 3\rangle}{[43]}$$

All terms vanish except for the term involving $\varepsilon(2)\cdot\varepsilon(3)$:

$$\varepsilon(2)\cdot\varepsilon(3) = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) \frac{1}{\langle 12\rangle [43]} \cdot 2 \cdot \langle 13\rangle [42] = -\frac{\langle 13\rangle [42]}{\langle 12\rangle [43]}$$

then

$$iM = \frac{1}{2}\cdot(ig^2)\frac{1}{\langle 23\rangle[32]}\langle 1(2-3)4]\left(-\frac{\langle 13\rangle[42]}{\langle 12\rangle[43]}\right)$$

Now $\langle 1(\not 2-\not 3)4] = -2\langle 1\not 3 4] = -2\langle 13\rangle[34]$

so

$$iM = ig^2\frac{\langle 13\rangle[34]\langle 13\rangle[42]}{\langle 23\rangle[32]\langle 12\rangle[43]}\cdot\frac{\langle 13\rangle}{\langle 13\rangle}$$

Now. $\langle 13\rangle[32] = -\langle 14\rangle[42] = \langle 41\rangle[42]$

$$= -ig^2\frac{\langle 13\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 41\rangle} = ig^2\frac{\langle 13\rangle^3\langle 43\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

For $g_2^- g_3^+$ choose

$$\varepsilon^\mu(2) = -\frac{1}{\sqrt{2}}\frac{[3\gamma^\mu 2\rangle}{[32]} \qquad \varepsilon^\mu(3) = \frac{1}{\sqrt{2}}\frac{\langle 2\gamma^\mu 3]}{\langle 23\rangle}$$

For this choice of reference vectors

$$\varepsilon(2)\cdot\varepsilon(3) = 0 \qquad 2\cdot\varepsilon(3) = 3\cdot\varepsilon(2) = 0$$

so only the first line of the expression for M contributes

$$iM = \frac{-ig^2}{2}\cdot\left(-\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right)\frac{\cdot 2\cdot 2\cdot}{[32]\langle 23\rangle}\frac{\langle 12\rangle[3(1+2)2\rangle[34]}{\langle 12\rangle[21]}$$

$$= ig^2\frac{\langle 12\rangle[31]\langle 12\rangle[34]}{[32]\langle 23\rangle\langle 12\rangle[21]}\cdot\frac{\langle 12\rangle}{\langle 12\rangle}\cdot\frac{\langle 42\rangle}{\langle 42\rangle}$$

using $[32]\langle 12\rangle = -\langle 12\rangle[23] = \langle 14\rangle[43] = \langle 41\rangle[34]$

$[21]\langle 42\rangle = \langle 42\rangle[21] = -\langle 43\rangle[31] = \langle 34\rangle[31]$

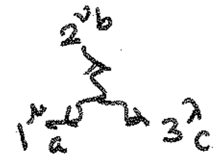
this simplifies to

$$iM = ig^2 \frac{\langle 12\rangle^2 \langle 42\rangle}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} = ig^2 \frac{\langle 12\rangle^3 \langle 42\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

so both nonzero color-ordered amplitudes take very simple forms with angle brackets only.

Let's next carry out the comparable analysis for $gg \to gg$. To do this, we should first formalize the rules for computation of color-ordered amplitudes.

The three-gauge boson vertex is

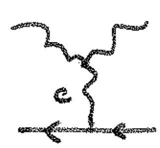


$$= -g f^{abc} \left[g^{\mu\nu}(1-2)^\lambda + g^{\nu\lambda}(2-3)^\mu + g^{\lambda\mu}(3-1)^\nu \right]$$

$$= (ig)\, i f^{abc} \left[g^{\mu\nu}(1-2)^\lambda + g^{\nu\lambda}(2-3)^\mu + g^{\lambda\mu}(3-1)^\nu \right]$$

$$= \frac{ig}{\sqrt{2}} \, \mathrm{tr}\left[T^a T^b T^c - T^b T^a T^c\right] \left[g^{\mu\nu}(1-2)^\lambda + g^{\nu\lambda}(2-3)^\mu + g^{\lambda\mu}(3-1)^\nu \right]$$

In a diagram computation, we will have

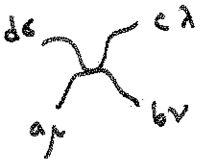


and the use of the identities on p. 2: $T^c_{ij} T^c_{kl} = \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}$

will inject $\left(T^a T^b - T^b T^a\right)$

into the line at the bottom. The first term contributes to the $T^a T^b$ color structure.

A similar analysis applies to the 4-gluon amplitude



$$= -ig^2 \left[f^{abe} f^{cde} (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\sigma} g^{\nu\lambda}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma}) \right]$$

Now

$$f^{abe} f^{cde} = -\frac{1}{2} \operatorname{tr} [T^a, T^b][T^c, T^d]$$

$$= -\frac{1}{2} \operatorname{tr} \left[T^a T^b T^c T^d - T^b T^a T^c T^d - T^a T^b T^d T^c + T^b T^a T^d T^c \right]$$

Among the three different structures in the expression above, there are 6 possible orderings for the T's in the traces. The coefficient of one of these orderings is:

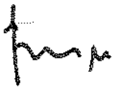
$$\frac{ig^2}{2} \operatorname{tr}(T^a T^b T^c T^d) \left[g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda} + 0 + (-1)(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma}) \right]$$

$$= \frac{ig^2}{2} \operatorname{tr}(T^a T^b T^c T^d) \left[2 g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\sigma} g^{\nu\lambda} \right]$$

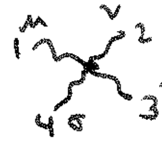
The coefficients of the other 5 terms can be found by Bose symmetry. However, those terms will, in general, contribute to different color structures. We can thus compute the amplitude associated with a fixed color structure

$$T^a T^b \cdots T^f \qquad \text{or} \qquad \operatorname{tr}[T^a T^b \cdots T^f]$$

using the "color-ordered Feynman rules":

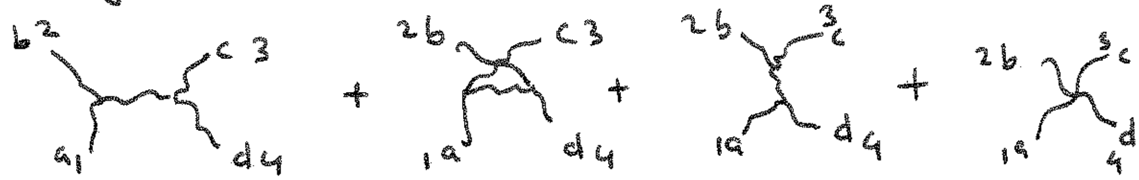
 $= \frac{ig}{\sqrt{2}}\gamma^\mu$

$= \frac{ig}{\sqrt{2}}\left[g^{\mu\nu}(1-2)^\lambda + g^{\nu\lambda}(2-3)^\mu + g^{\lambda\mu}(3-1)^\nu \right]$

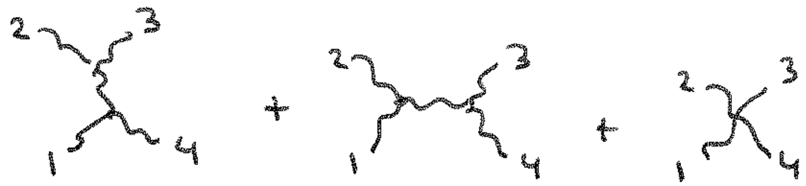


$= \frac{ig^2}{2}\left[2g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\nu\lambda} \right]$

Now we are ready to analyze $gg \to gg$. At tree level, there are four diagrams



Of these, only three contribute to each color structure. The coefficient of $\mathrm{tr}[T^a T^b T^c T^d]$ is



with the color-ordered rules:

$$i\mathcal{M} = \left(\frac{ig}{\sqrt{2}}\right)^2 \Big[\left(\frac{-i}{s_{14}}\right)\left[\varepsilon(1)\cdot\varepsilon(4)\,(4-1)^\lambda + 2\,\varepsilon^\lambda(1)\; 1\cdot\varepsilon(4) - 2\,\varepsilon^\lambda(4)\; 4\cdot\varepsilon(1)\right]$$
$$\cdot\left[\varepsilon(2)\cdot\varepsilon(3)\,(2-3)_\lambda + 2\,\varepsilon_\lambda(3)\; 3\cdot\varepsilon(2) - 2\,\varepsilon_\lambda(2)\; 2\cdot\varepsilon(3)\right]$$

$$+ \left(\frac{-i}{s_{34}}\right)\left[\varepsilon(1)\cdot\varepsilon(2)\,(1-2)^\lambda + 2\cdot\varepsilon^\lambda(2)\; 2\cdot\varepsilon(1) - 2\,\varepsilon^\lambda(1)\right.$$ [redacted] $$\left.\cdot\,\varepsilon(2)\right]$$
$$\cdot\left[\varepsilon(3)\cdot\varepsilon(4)\,(3-4)_\lambda + 2\,\varepsilon_\lambda(4)\; 4\cdot\varepsilon(3) - 2\,\varepsilon_\lambda(3)\; 3\cdot\varepsilon(4)\right]$$

$$+ (-i)\left[2\,\varepsilon(1)\cdot\varepsilon(3)\;\varepsilon(2)\cdot\varepsilon(4) - \varepsilon(1)\cdot\varepsilon(2)\;\varepsilon(3)\varepsilon(4) - \varepsilon(4)\cdot\varepsilon(1)\;\varepsilon(2)\cdot\varepsilon(3)\right] \Big]$$

We need to evaluate this for all possible values of the gluon helicities. Begin with all +: $g_1^+ g_2^+ g_3^+ g_4^+$. Choose all four polarization vectors to be of the form

$$\varepsilon_i^\mu = \frac{1}{\sqrt{2}} \frac{\langle r \gamma^\mu i]}{\langle r i \rangle} \quad \text{for the } \underline{\text{same}} \text{ vector } r$$

Then $\varepsilon_i \cdot \varepsilon_j = 0$ for all i, j. Every term in $M\mathrm{I}$ has at least one factor of $\varepsilon_i \cdot \varepsilon_j$. Then $M\mathrm{I} = 0$. The same argument applies to $M\mathrm{I}(g_1^- g_2^- g_3^- g_4^-)$.

Next, consider $M\mathrm{I}(g_1^- g_2^+ g_3^+ g_4^+)$. ($M\mathrm{I}$ is cyclically invariant, so the same analysis will apply for one − in any position.) Choose

$$\varepsilon_2^\mu = \frac{1}{\sqrt{2}} \frac{\langle 1 \gamma^\mu 2]}{\langle 12 \rangle} \qquad \varepsilon_3^\mu = \frac{1}{\sqrt{2}} \frac{\langle 1 \gamma^\mu 3]}{\langle 13 \rangle} \qquad \varepsilon_4^\mu = \frac{1}{\sqrt{2}} \frac{\langle 1 \gamma^\mu 4]}{\langle 14 \rangle}$$

$$\varepsilon_1 = -\frac{1}{\sqrt{2}} \frac{[r \gamma^\mu 1\rangle}{[r1]}$$

It follows that

$$\varepsilon_2 \cdot \varepsilon_3 = \varepsilon_2 \cdot \varepsilon_4 = \varepsilon_3 \cdot \varepsilon_4 = 0 \qquad \varepsilon_1 \cdot \varepsilon_2 = \varepsilon_1 \cdot \varepsilon_3 = \varepsilon_1 \varepsilon_4 = 0$$

and again $M\mathrm{I} = 0$. The same argument shows that $M\mathrm{I}(g_1^+ g_2^- g_3^- g_4^-)$ & cyclic cases w. one + vanish.

In fact, these arguments show that, for <u>any number</u> of external gluons (≥ 4), $M\mathrm{I} = 0$ when the gluons are <u>all +</u>, <u>all −</u>, <u>all + except for 1 −</u>, <u>all − except for one +</u>.

Then, for 4 gluons, there are only two cases in which

$\mathcal{M}$ can be nonzero: $\mathcal{M}(g_1^- g_2^- g_3^+ g_4^+)$, $\mathcal{M}(g_1^- g_2^+ g_3^- g_4^+)$, and cycles of these

To evaluate $\mathcal{M}(g_1^- g_2^- g_3^+ g_4^+)$ choose:

$$\varepsilon_1^\mu = -\frac{1}{\sqrt{2}} \frac{[4\gamma^\mu 1\rangle}{[41]} \qquad \varepsilon_2^\mu = -\frac{1}{\sqrt{2}} \frac{[4\gamma^\mu 2\rangle}{[42]}$$

$$\varepsilon_3^\mu = \frac{1}{\sqrt{2}} \frac{\langle 1\gamma^\mu 3]}{\langle 13\rangle} \qquad \varepsilon_4^\mu = \frac{1}{\sqrt{2}} \frac{\langle 1\gamma^\mu 4]}{\langle 14\rangle}$$

then

$$\varepsilon_1\cdot\varepsilon_2 = 0 \qquad \varepsilon_3\cdot\varepsilon_4 = 0 \qquad \varepsilon_1\cdot\varepsilon_4 = \varepsilon_2\cdot\varepsilon_4 = \varepsilon_1\cdot\varepsilon_3 = 0$$

and also $\quad 1\cdot\varepsilon_4 = 4\cdot\varepsilon_1 = 0$

The one nonzero product of polarizations is

$$\varepsilon_2\cdot\varepsilon_3 = -\frac{1}{2}\frac{1}{[42]\langle 13\rangle}\cdot 2\cdot [43]\langle 12\rangle = -\frac{\langle 12\rangle[43]}{\langle 13\rangle[42]}$$

In the formula for $\mathcal{M}$ on p.7, the first line is zero and the last line is zero. Thus only the second term survives, and this contains only one nonzero piece:

$$i\mathcal{M} = \left(\frac{ig}{\sqrt{2}}\right)^2 \frac{-i}{s_{34}} (-4)\ \varepsilon_2\cdot\varepsilon_3\ \ 2\cdot\varepsilon_1\ \ 3\cdot\varepsilon_4$$

$$= -ig^2\cdot 2\ \frac{1}{\langle 34\rangle[43]}\left(-\frac{1}{2}\right)\frac{[42]\langle 21\rangle}{[41]}\ \frac{\langle 13\rangle[34]}{\langle 14\rangle}\left(-\frac{\langle 12\rangle[43]}{\langle 13\rangle[42]}\right)$$

$$= -ig^2\ \frac{\langle 12\rangle^2[34]}{\langle 34\rangle\langle 41\rangle[41]}\cdot\frac{\langle 12\rangle}{\langle 12\rangle}$$

Now $\quad [41]\langle 12\rangle = -[43]\langle 32\rangle = -[34]\langle 23\rangle$

so

$$i\mathcal{M} = ig^2 \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} = ig^2 \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

Similarly, to evaluate $i\mathcal{M}(g_1^- g_2^+ g_3^- g_4^+)$ choose

$$\varepsilon_1 = -\frac{1}{\sqrt{2}} \frac{[4\gamma^\mu 1\rangle}{[41]} \qquad \varepsilon_2 = \frac{1}{\sqrt{2}} \frac{\langle 1\gamma^\mu 2]}{\langle 12\rangle}$$

$$\varepsilon_3 = -\frac{1}{\sqrt{2}} \frac{[4\gamma^\mu 3\rangle}{[43]} \qquad \varepsilon_4 = \frac{1}{\sqrt{2}} \frac{\langle 1\gamma^\mu 4]}{\langle 14\rangle}$$

then

$$\varepsilon_1\cdot\varepsilon_3 = 0 \qquad \varepsilon_2\cdot\varepsilon_4 = 0 \qquad \varepsilon_1\cdot\varepsilon_4 = \varepsilon_1\cdot\varepsilon_2 = \varepsilon_3\cdot\varepsilon_4 = 0$$

and $\quad 1\cdot\varepsilon_4 = 4\cdot\varepsilon_1 = 0$

The nonzero product of polarizations is

$$\varepsilon_2\cdot\varepsilon_3 = -\frac{1}{2}\frac{1}{\langle 12\rangle[43]}\cdot 2\,\langle 13\rangle[42] = -\frac{\langle 13\rangle[42]}{\langle 12\rangle[43]}$$

Once again, the first and third terms in $i\mathcal{M}$ vanish, and the second reduces to

$$i\mathcal{M} = \left(\frac{ig}{\sqrt{2}}\right)^2\left(\frac{-i}{s_{34}}\right)(-4)\quad \varepsilon_2\varepsilon_3 \quad 2\cdot\varepsilon_1 \quad 3\cdot\varepsilon_4$$

$$= -ig^2 \frac{2}{\langle 34\rangle[43]}\left(-\frac{1}{2}\right)\frac{[42]\langle 21\rangle}{[41]}\frac{\langle 13\rangle[34]}{\langle 14\rangle}\left(-\frac{\langle 13\rangle[42]}{\langle 12\rangle[43]}\right)$$

$$= +ig^2 \frac{\langle 13\rangle^2[42]^2}{\langle 34\rangle\langle 41\rangle\;[41][43]}\cdot\frac{\langle 13\rangle}{\langle 13\rangle}\frac{\langle 13\rangle}{\langle 13\rangle}$$

Using $[41]\langle 13\rangle = -[42]\langle 23\rangle$

$[43]\langle 13\rangle = -[43]\langle 31\rangle = [42]\langle 21\rangle = -[42]\langle 12\rangle$

this becomes

$$i\mathcal{M} = ig^2 \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

We have now found an interesting regularity, which turns out to apply not only to 4-point amplitudes but also to amplitudes with any number of gluons. For the color-ordered amplitudes with N gluons:

$$i\mathcal{M}(g_1^+ \cdots g_N^+) = 0$$

$$i\mathcal{M}(g_1^+ \cdots g_i^- \cdots g_N^+) = 0 \qquad 1-$$

$$i\mathcal{M}(g_1^+ \cdots g_i^- \cdots g_j^- \cdots g_N^+) = ig^{N-2} \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \cdots \langle N1\rangle}$$

These last amplitudes are called the Parke-Taylor maximal helicity violating (MHV) amplitudes. While I have proved the vanishing of the amplitudes with 0 and 1 − to all orders, I have only proved this formula for N = 4. I will give a complete proof in the next lecture.

For amplitudes with $q\bar{q}$ and gluons there is a similar set of regularities

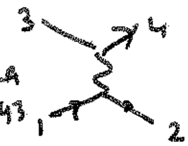
$$i\mathcal{M}(q_1^- g_2^+ \cdots g_{N-1}^+ \bar{q}_N^+) = 0$$

$$i\mathcal{M}(q_1^- g_2^+ \cdots g_j^- \cdots g_{N-1}^+ \bar{q}_N^+) = i g^{N-2} \frac{\langle 1j\rangle^3 \langle Nj\rangle}{\langle 12\rangle\langle 23\rangle \cdots \langle N1\rangle}$$

This is another MHV amplitude. I have proved it for $N=4$, but, again, as we will see, it is true to all orders.

To conclude this lecture, I will give an illustration of how the color-ordered amplitudes for $N=4$ can be assembled into complete cross sections. To do this, I will use these amplitudes to derive the standard cross sections for 2-parton elastic scattering.

To warm up, consider quark-quark scattering. For $u + d \to u + d$. For $u_L + d_L \to u_L + d_L$



$$i\mathcal{M} = \left(\frac{ig}{\sqrt{2}}\right)^2 \frac{-i}{s_{12}} \langle 4\gamma^\mu 3] \langle 2\gamma_\mu 1] T^a_{21} T^a_{43}$$

$$= ig^2 \frac{\langle 42\rangle [31]}{\langle 12\rangle [21]} \cdot \frac{\langle 42\rangle}{\langle 42\rangle} T^a T^a$$

$$= ig^2 \frac{\langle 42\rangle^2 [31]}{\langle 12\rangle(-\langle 43\rangle [31])} T^a T^a$$

$$= ig^2 \frac{\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle} T^a T^a$$

To compute $|\mathcal{M}|^2$ we need the color factor

$$\frac{1}{3}\cdot\frac{1}{3}\, \mathrm{tr}[T^a T^b]\, \mathrm{tr}[T^a T^b] = \frac{1}{9}\delta^{ab}\delta^{ab} = \frac{8}{9}$$

so $\quad \frac{1}{3}\cdot\frac{1}{3} \sum_{\text{colors}} |\mathcal{M}|^2 = \frac{8}{9} g^2 \frac{s^2}{t^2}$

and $u_R + d_R \to u_R + d_R$ gives the same result.

For $u_L + d_R \to u_L + d_R$ (or $u_R + d_L \to u_R + d_L$)

$$\frac{1}{3}\cdot\frac{1}{3} \sum_{\text{colors}} |\mathcal{M}|^2 = \frac{8}{9} g^4 \left| \frac{\langle 23\rangle^2}{\langle 12\rangle\langle 34\rangle} \right|^2$$

$$= \frac{8}{9} g^2 \frac{u^2}{t^2}$$

The whole polarization sum is then

$$\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{3} \sum_{\text{spin, color}} \quad \frac{8}{9} g^4 \cdot \frac{2}{4} \cdot \frac{s^2+u^2}{t^2}$$

so that

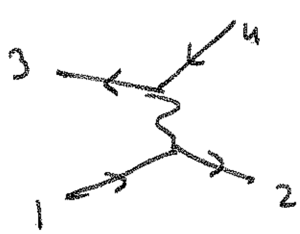
$$\frac{d\sigma}{d\cos\theta_*} = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{8}{9} g^4\,\frac{1}{2}\,\frac{s^2+u^2}{t^2}$$

or

$$\frac{d\sigma}{d\cos\theta_*}(ud \to ud) = \frac{2}{9}\frac{\pi\alpha_s^2}{s}\left(\frac{s^2+u^2}{t^2}\right)$$

For $u + u \to u + u$ we need to worry about identical particles.

For $u_L + u_R \to u_L + u_R$ there is only one diagram:

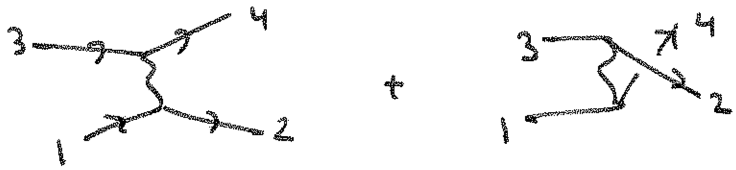


$$\frac{1}{3}\frac{1}{3}\sum_{\text{color}} |\mathcal{M}|^2 = \frac{8}{9} g^4 \frac{u^2}{t^2}$$

to be integrated over all of the final phase space, or we can symmetrize under $\cos\Theta \leftrightarrow -\cos\Theta$ and integrate over half of phase space

$$\frac{1}{3}\frac{1}{3}\sum_{\text{color}} |\mathcal{M}|^2 \longrightarrow \frac{8}{9} g^4 \left(\frac{u^2}{t^2} + \frac{t^2}{u^2}\right) \qquad \cos\Theta > 0 \text{ only}$$

For $u_L + u_L \to u_L + u_L$, there are two diagrams that interfere:



$$i\mathcal{M} = ig^2 \frac{\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle} T^a_{21} T^a_{43} - ig^2 \frac{\langle 42\rangle^2}{\langle 14\rangle\langle 32\rangle} T^a_{41} T^a_{23}$$

↑ fermions! ↑ $2 \leftrightarrow 4$

to square this, we need the cross term. The color factor is

$$\frac{1}{3}\frac{1}{3} \operatorname{tr} T^a T^b T^a T^b$$

To evaluate this, use the identity $T^b_{ij} T^b_{k\ell} = \delta_{i\ell}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{k\ell}$

$$= \frac{1}{3}\frac{1}{3}\left[(\operatorname{tr} T^a)(\operatorname{tr} T^a) - \frac{1}{3} \operatorname{tr} T^a T^a\right]$$

$$= \frac{1}{9} \cdot \left[0 - \frac{1}{3}\delta^{aa}\right] = -\frac{8}{27}$$

The spinor product part of the cross term is

$$\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle}\cdot\frac{([24])^2}{[23][41]} = \frac{s^2}{\langle 12\rangle[23]\langle 34\rangle[41]}$$

$$= -\frac{s^2}{\langle 12\rangle[21]\langle 14\rangle[41]} = -\frac{s^2}{tu}$$

so for $u_L u_L \to u_L u_R$ or $u_R u_R \to u_R u_R$

$$\sum_{\text{colors}} |\mathcal{M}|^2 = \frac{8}{9}\cdot g^4\left[\frac{s^2}{t^2}+\frac{s^2}{u^2}-2\left(-\frac{1}{3}\right)\left(-\frac{s^2}{tu}\right)\right]$$

Adding all of the pieces

$$\frac{d\sigma}{d\cos\theta_*}(uu\to uu) = \frac{2}{9}\frac{\pi\alpha_s^2}{s}\left[\frac{s^2+u^2}{t^2}+\frac{s^2+t^2}{u^2}-\frac{2}{3}\frac{s^2}{tu}\right]$$

to be integrated over $\cos\theta>0$ only.

Next, consider $u\bar{u}\to gg$. It suffices to analyze

$$\underset{1}{\bar{u}_R}\,\underset{2}{u_L} \to \underset{3}{g_L}\,\underset{4}{g_R}$$

$$i\mathcal{M} = i\mathbb{M}(\bar{q}_1^- g_3^- g_4^+ \bar{q}_2^+)\, T^a T^b + i\mathbb{M}(\bar{q}_1^- g_4^+ g_3^- \bar{q}_2^+)\, T^b T^a$$

$$= ig^2\left[\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle}T^aT^b + \frac{\langle 13\rangle^3\langle 23\rangle}{\langle 14\rangle\langle 43\rangle\langle 32\rangle\langle 21\rangle}T^bT^a\right]$$

The square of the first term involves the color factor

$$\frac{1}{3}\frac{1}{3}\,\mathrm{tr}\, T^aT^bT^bT^a = \frac{1}{3}\cdot\frac{1}{3}\,\mathrm{tr}\left(\frac{8}{3}\right)^2 = \frac{64}{27}$$

and the normalization factor

$$\left| \frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle} \right|^2$$

$$= \frac{t^3 u}{t s t s} = \frac{tu}{s^2}$$

The square of the second term involves the same color factor with

$$\left| \frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 14\rangle\langle 43\rangle\langle 32\rangle\langle 21\rangle} \right|^2 , \quad \frac{t^3 u}{us\,us} = \frac{t^3}{us^2}$$

and write

$$\frac{tu}{s^2} + \frac{t^3}{us^2} = \frac{t}{us^2}(t^2+u^2) = \frac{t}{us^2}\left[(t+u)^2 - 2tu\right]$$

$$= \frac{t}{u} - 2\frac{t^2}{s^2}$$

The cross term involves the color factor

$$\frac{1}{3}\frac{1}{3}\,\mathrm{tr}\, T^a T^b T^a T^b = -\frac{8}{27}$$

and the numerator factor

$$\left| \frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 21\rangle} \right|^2 \frac{1}{\langle 13\rangle\langle 34\rangle\langle 42\rangle} \cdot \frac{1}{[23][34][41]}$$

$$= \frac{t^3 u}{s} \frac{1}{\langle 34\rangle[43]} \cdot \frac{1}{\langle 13\rangle[32]\langle 24\rangle[41]}$$

$$= \frac{t^3 u}{s} \frac{1}{(-s)} \frac{1}{-\langle 14\rangle[42]\langle 24\rangle[41]}$$

$$= \frac{t^3 u}{s} \frac{1}{s} \frac{1}{ut} = \frac{t^2}{s^2}$$

In all
$$\sum_{\text{colors}} |\mathcal{M}|^2 = \frac{64}{27} g^4 \left[\frac{t}{u} - 2\frac{t^2}{s^2} - \frac{2}{8}\frac{t^2}{s^2} \right]$$

$$= \frac{64}{27} g^4 \left[\frac{t}{u} - \frac{9}{4}\frac{t^2}{s^2} \right]$$

Add the contribution from $\bar{u}_L u_R \to gg$ and symmetrize in $\cos\theta \to -\cos\theta$ so that we can integrate over $\cos\theta > 0$ only.

$$\frac{d\sigma}{d\cos\theta_*}(u\bar{u} \to gg) = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{64}{27} g^4 \left[\frac{t}{u} + \frac{u}{t} - \frac{9}{4}\frac{t^2+u^2}{s^2} \right] \cdot \frac{2}{4}$$

$$\frac{d\sigma}{d\cos\theta_*}(u\bar{u} \to gg) = \frac{16}{27}\,\frac{\pi\alpha_s^2}{s} \left[\frac{t}{u} + \frac{u}{t} - \frac{9}{4}\frac{t^2+u^2}{s^2} \right]$$

Finally, study $gg \to gg$. For this process

$$i\mathcal{M} = \mathrm{tr}[T_1 T_2 T_3 T_4] \quad i\mathbb{M}(g_1 g_2 g_3 g_4)$$

$$+ \quad (5 \text{ more terms})$$

This seems to be quite complicated to square, but a few tricks make it simple. First, note that, instead of using $SU(3)$ for the gauge group, we can use $U(3)$. The extra $U(1)$ gauge boson does not couple in the 3- or 4-gluon vertices, which are proportional to f^{abc}. This means that we can evaluate traces using the simpler T^a identity

$$T^a_{ij} T^a_{kl} = \delta_{il}\delta_{kj}$$

Now $(T^a)^2 = \frac{9}{3} = 3$. When we color average gluons, we must still divide by 8. When we square the 4-gluon matrix element, three different objects appear. In the square of any term, we find

$$\mathrm{tr}\, T^a T^b T^c T^d \cdot \mathrm{tr}\, T^d T^c T^b T^a$$

$$= \mathrm{tr}\, T^a T^b T^c T^c T^b T^a = \mathrm{tr}\, (T^2)^3 = 3 \cdot 3^3 = 81$$

In cross terms we find contractions with one permutation:

$$\mathrm{tr}\, T^a T^b T^c T^d \quad \mathrm{tr}\, T^d T^c T^a T^b$$

$$= \mathrm{tr}\, T^a T^b T^c T^c T^a T^b$$

$$= 3\, \mathrm{tr}\, T^a T^b T^a T^b = 3\, \mathrm{tr}\, T^a\, \mathrm{tr}\, T^a$$

$$= 3 \left(\frac{3}{\sqrt{3}} \delta^{a0}\right)^2 = 9$$

and with two permutations

$$\mathrm{tr}\, T^a T^b T^c T^d \quad \mathrm{tr}\, T^c T^d T^a T^b$$

$$= \mathrm{tr}\, T^a T^b T^c T^a T^b T^c$$

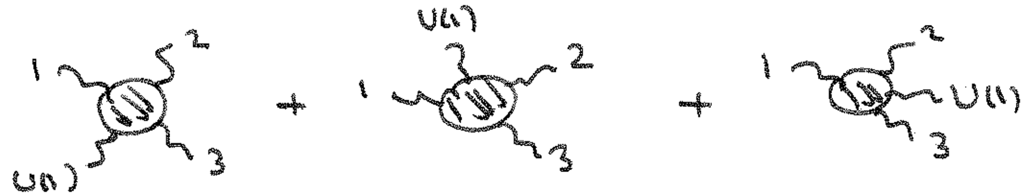
$$= \mathrm{tr}\, T^b T^c\, \mathrm{tr}\, T^b T^c = \delta^{bc} \delta^{bc} = 9$$

so

$$\sum_{\text{colors}} |M|^2 = 81 \sum_{\substack{\text{color orders} \\ \text{amps. } I}} |M_I|^2 + 9 \sum_{I \neq J} M_I^* M_J$$

$$= (81 - 9) \sum_I |M_I|^2 + 9 \sum_{I,J} M_I^* M_J$$

But, there is a further implication of the fact that the U(1) boson must decouple. The contribution of this boson is the sum of color-ordered amplitudes



so we must have

$$0 = i\mathcal{M}(g_0 g_1 g_2 g_3) + i\mathcal{M}(g_1 g_0 g_2 g_3) + i\mathcal{M}(g_1 g_2 g_0 g_3)$$

This is called the U(1) Ward identity. We can check special cases. For example:

$$i\mathcal{M}(g_0^- g_1^- g_2^+ g_3^+) + i\mathcal{M}(g_1^- g_0^- g_2^+ g_3^+) + i\mathcal{M}(g_1^- g_2^+ g_0^- g_3^+)$$

$$= i g^2 \langle 01\rangle^4 \left[\frac{1}{\langle 01\rangle\langle 12\rangle\langle 23\rangle\langle 30\rangle} + \frac{1}{\langle 10\rangle\langle 02\rangle\langle 23\rangle\langle 31\rangle} + \frac{1}{\langle 12\rangle\langle 20\rangle\langle 03\rangle\langle 31\rangle} \right]$$

$$= i g^2 \langle 01\rangle^4 \left[\frac{\langle 02\rangle\langle 13\rangle + \langle 12\rangle\langle 30\rangle - \langle 01\rangle\langle 23\rangle}{\langle 01\rangle\langle 12\rangle\langle 23\rangle\langle 30\rangle\langle 02\rangle\langle 13\rangle} \right]$$

$$= i g^2 \langle 01\rangle^4 \frac{\left(\langle 02\rangle\langle 13\rangle + \langle 03\rangle\langle 21\rangle + \langle 01\rangle\langle 32\rangle\right)}{\langle 01\rangle\langle 12\rangle\langle 23\rangle\langle 30\rangle\langle 02\rangle\langle 13\rangle}$$

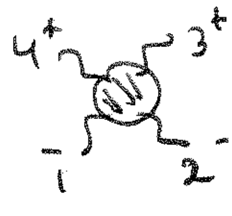
$$= 0$$

The quantity in parentheses vanishes by the Schouten identity.

The U(1) Ward identity implies $\sum_I \mathcal{M}_I = 0$, so the square of $\mathcal{M}$ reduces to

$$\frac{1}{8}\frac{1}{8}\sum_{\text{colors}}|\mathcal{M}|^2 = \frac{81-9}{64}\sum_I |\mathbb{M}|^2 = \frac{9}{8}\sum_I |\mathbb{M}|^2$$

We need to evaluate this for all possible helicity assignments for which $\mathbb{M}$ is non zero. Begin with



The 6 $\mathbb{M}$'s are:

$$|\mathbb{M}(g_1^- g_2^- g_3^+ g_4^+)|^2 = g^4\left|\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right|^2 = g^4\frac{s^4}{stst} = g^4\frac{s^2}{t^2}$$

$$= |\mathbb{M}(g_1^- g_4^+ g_3^+ g_2^-)|^2$$

$$|\mathbb{M}(g_1^- g_3^+ g_2^- g_4^+)|^2 = g^4\left|\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}\right|^2 = g^4\frac{s^4}{t^2u^2}$$

$$= |\mathbb{M}(g_1^- g_4^+ g_2^- g_3^+)|^2$$

$$|\mathbb{M}(g_1^- g_2^- g_4^+ g_3^+)|^2 = g^4\left|\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}\right|^2 = g^4\frac{s^2}{u^2}$$

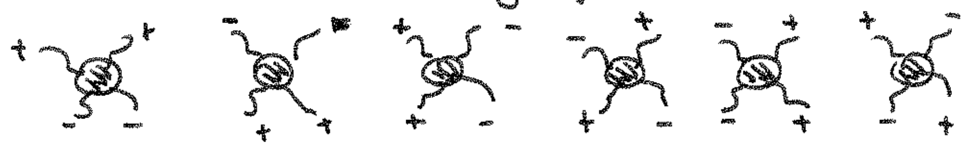
$$= |\mathbb{M}(g_1^- g_3^+ g_4^+ g_2^-)|^2$$

The sum of the six terms is

$$2\cdot g^4\left(\frac{s^2}{t^2}+\frac{s^2}{u^2}+\frac{s^4}{t^2u^2}\right) \qquad s^4 = s^2(u^2+t^2+2tu)$$

$$= 4g^4\left(\frac{s^2}{t^2}+\frac{s^2}{u^2}+\frac{s^2}{tu}\right)$$

Now we need to sum over all helicities of the gluons. There are six choices that give nonvanishing amplitudes



Then

$$\frac{1}{4}\,\frac{1}{8}\,\frac{1}{8} \sum_{\substack{\text{color}\\ \text{spin}}} |\mathcal{M}|^2 = \frac{1}{4}\,\frac{9}{8} \sum_{\text{spin}}{}' \left(\sum_I |\mathcal{M}_I|^2 \right)$$

$$= \frac{1}{4}\cdot\frac{9}{8}\cdot 2 \cdot 4g^4 \left(\frac{s^2}{t^2} + \frac{s^2}{u^2} + \frac{s^2}{tu} + \frac{t^2}{s^2} + \frac{t^2}{u^2} + \frac{t^2}{su} + \frac{u^2}{t^2} + \frac{u^2}{s^2} + \frac{u^2}{ts} \right)$$

now

$$\frac{s^2+u^2}{t^2} = \frac{(s+u)^2}{t^2} - 2\,\frac{su}{t^2} = 1 - 2\,\frac{su}{t^2}$$

$$\frac{s^2}{tu} = \frac{(u+t)^2}{ut} = \frac{u}{t} + \frac{t}{u} + 2$$

$$\frac{s^2}{tu} + \frac{t^2}{su} + \frac{u^2}{ts} = \frac{u+s}{t} + \frac{t+s}{u} + \frac{t+u}{s} + 3\cdot 2 = 3\cdot 2 + 3\cdot(-1) = 3$$

So!

$$= \frac{9}{4}\, g^4 \left(3 - 2\,\frac{su}{t^2} - 2\,\frac{st}{u^2} - 2\,\frac{tu}{s^2} + 3 \right)$$

$$= \frac{9}{2}\, g^4 \left(3 - \frac{su}{t^2} - \frac{st}{u^2} - \frac{tu}{s^2} \right)$$

$$\frac{d\sigma}{d\cos\theta_*}(gg \to gg) = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{9}{2}\, g^4 \left(3 - \cdots \right)$$

or, finally,

$$\frac{d\sigma}{d\cos\theta_*}(gg \to gg) = \frac{9}{4}\,\frac{\pi\alpha^2}{s} \left[3 - \frac{su}{t^2} - \frac{st}{u^2} - \frac{tu}{s^2} \right]$$

to be integrated over $\cos\theta_* > 0$ only.