Computation of Multiparticle Amplitudes in QCD: spinor products

There is one more aspect of QCD that I would like to discuss. At the LHC, we often need to understand processes with many QCD emissions. These arise in understanding the shapes of QCD events and in computing backgrounds to new physics. We have seen that we can go a certain distance in constructing these events using parton showers. However, the approximations we use there break down when we have many jets at large angles with respect to one another. For these configurations, we need a more exact analysis. In this and the next several lectures, I will introduce a set of methods for exactly computing QCD tree amplitudes with many partons in the final state.

The first method I will discuss is that of *spinor products*. I will take advantage of the fact that, in QCD at colliders, we are mainly dealing with particles that are massless to a very good approximation — quarks and gluons. We have already seen that it is effective to think of these as particles of definite helicity. Now we will use a further simplification: These particles have lightlike 4-vectors. Lightlike vectors can be decomposed into spinors. We will see that the QCD scattering amplitudes that relatively simple forms in terms of these spinors.

First, let's develop a formalism for massless fermions. Let the fermion momentum be p, $p^2 = 0$. The spinors for the fermions satisfy the Dirac equation

$$\not{p}\, u(p) = 0$$

There are two solutions to this equation

$$u_R(p) \qquad \gamma^5 u_R = +u_R$$

$$u_L(p) \qquad \gamma^5 u_L = -u_L$$

For example, for $\vec{p} \parallel \hat{3}$

$$\not{p} = \begin{pmatrix} & p\cdot\sigma \\ p\cdot\bar{\sigma} & \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$p\cdot\sigma = p^0 - \vec{p}\cdot\vec{\sigma} \qquad u_R = \sqrt{2p}\begin{pmatrix} 0 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix}$$

$$p\cdot\bar{\sigma} = p^0 + \vec{p}\cdot\vec{\sigma} \qquad u_L = \sqrt{2p}\begin{pmatrix} 0 \\ 1 \\ \hline 0 \\ 0 \end{pmatrix}$$

In this basis, the Dirac spinors become 2-component spinors

$$u_R = \begin{pmatrix} 0 \\ 0 \\ \hline u_R \end{pmatrix} \qquad u_L = \begin{pmatrix} u_L \\ \hline 0 \\ 0 \end{pmatrix}$$

The 2-component u_R is related to u_L by

$$u_R = -c\, u_L^* \qquad c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma^2$$

since, as you can check, if $(p^0 + \vec{p}\cdot\vec{\sigma})\, u_L = 0$, then

$(p^0 - \vec{p}\cdot\vec{\sigma})\, u_R = 0$. For antiparticles, v_L, v_R obey the same equation

$$\not{p}\, v(p) = 0$$

so we can use the same u_L, u_R for the antiparticles.

In this discussion, I will treat all external states as <u>outgoing</u>. Then we use

$\bar{u}_R(p) \to$ outgoing R-fermion

$\bar{u}_L(p) \to$ outgoing L-fermion

$u_L(p) \to$ outgoing R-antifermion

$u_R(p) \to$ outgoing L-antifermion

Now represent

$$u_R(p) = p\rangle \qquad\qquad u_L(p) = p]$$

$$\bar{u}_R(p) = [p \qquad\qquad \bar{u}_L(p) = \langle p$$

So that the Lorentz invariant scalar products are "spinor products"

$$\bar{u}_L(p)\, u_R(q) = \langle pq\rangle \qquad\qquad \bar{u}_R(p)\, u_L(q) = [pq]$$

We can decompose lightlike 4-vectors into these spinor objects

$$p\rangle[p = u_R(p)\,\bar{u}_L(p) = \left(\tfrac{1+\gamma^5}{2}\right)\not{p} = \not{p}\left(\tfrac{1-\gamma^5}{2}\right)$$

$$p]\langle p = \left(\tfrac{1-\gamma^5}{2}\right)\not{p}$$

Here are some properties of the spinor products:

$$\langle pq\rangle = \bar{u}_L(p)\, u_R(q) = [qp]^*$$

$$\langle pq\rangle [qp] = \bar{u}_L(p)\, u_R(q)\, \bar{u}_R(q)\, u_L(p) = |\langle pq\rangle|^2 = |[qp]|^2$$

$$= \mathrm{tr}\left(\tfrac{1+\gamma^5}{2}\right) \not{p} \not{q} = 2p\cdot q$$

again

$$\langle pq\rangle = [qp]^* \qquad |\langle pq\rangle|^2 = |[qp]|^2 = 2p\cdot q$$

that is, the spinor products are the square roots of the Lorentz products.

Next, note that, in terms of 2-component spinors

$$\langle pq\rangle = u_L^\dagger(p)\, u_R(q) = u_L^\dagger(p)(-c)\, u_L^*(q)$$

which is antisymmetric in (p,q) : $\langle pq\rangle = -\langle qp\rangle$, and similarly $[pq] = -[qp]$.

$$\bar{u}_L(p)\gamma^\mu u_L(q) = u_L^\dagger(p)\,\bar{\sigma}^\mu\, u_L(q)$$

$$= u_L^\dagger(p)\,\bar{\sigma}^\mu(-c^2)\, u_L(q)$$

$$= u_L^\dagger(p)(-c)^T(\sigma^\mu)^T(-c)\, u_L(q)$$

$$= u_R^T(p)(\sigma^\mu)^T u_R^*(q)$$

$$= u_R^\dagger(q)\,\sigma^\mu\, u_R(p)$$

so that

$$\langle p\gamma^\mu q] = [q\gamma^\mu p\rangle$$

Finally, there is a Fierz identity for combining γ-matrices

$$(\bar\sigma^\mu)_{\alpha\beta}\,(\bar\sigma_\mu)_{\gamma\delta} = 2\, C_{\alpha\gamma}\, C_{\beta\delta}$$

so that

$$u_L^\dagger(p)\,\bar\sigma^\mu u_L(q)\;\; u_L^\dagger(k)\,\bar\sigma_\mu u_L(\ell)$$

$$= 2\, u_L^\dagger(p)(-C)u_L^\dagger(k)\;\; u_L^T(q)(-C)u_L(\ell)$$

$$= 2\,\langle pk\rangle\,(-[q\ell]) = 2\,\langle pk\rangle\,[\ell q]$$

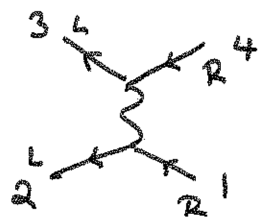
Thus,

$$\langle p\gamma^\mu q]\,\langle k\gamma_\mu \ell] = 2\,\langle pk\rangle\,[\ell q]$$

$$\langle p\gamma^\mu q]\,[k\gamma_\mu \ell\rangle = 2\,\langle p\ell\rangle\,[kq]$$

At this point, we are already able to perform some interesting calculations. Consider, for example, $e^-_L e^+_R \to \mu^-_L \mu^+_R$. Set this up with all momenta outgoing

$$e^-_L e^+_R \to \mu^-_L \mu^+_R$$
$$1 \quad 2 \qquad 3 \quad 4$$



$$i\mathcal{M} = (-ie)^2\,\frac{-i}{q^2}\;\bar u_L(3)\gamma^\mu u_L(4)\;\bar u_L(2)\gamma_\mu u_L(1)$$

$$= ie^2\,\frac{1}{q^2}\,\langle 3\gamma^\mu 4]\,\langle 2\gamma_\mu 1]$$

$$= 2ie^2\,\frac{1}{q^2}\,\langle 32\rangle\,[14]$$

so, up to phases, this is

$$= 2ie^2\,\frac{u}{s} = -ie^2(1+\cos\theta)$$

which is the right answer. This is a very simple derivation.

Actually, we can simplify the result further.

$$q^2 = 2p_1 \cdot p_2 = \langle 12\rangle [21]$$

so

$$i\mathcal{M} = 2ie^2 \frac{\langle 32\rangle [14]}{\langle 12\rangle [21]} \cdot \frac{\langle 32\rangle}{\langle 32\rangle}$$

Now $\langle 32\rangle [21] = \langle 3\ \gamma \cdot 2\ 1]$

but $2 = -1-3-4$ and $\langle 3\ \not{3} = 0$, $\not{1}\ 1] = 0$, so

$$\langle 32\rangle [21] = -\langle 34\rangle [41] = \langle 34\rangle [14]$$

then

$$i\mathcal{M} = 2ie^2 \frac{\langle 23\rangle^2}{\langle 12\rangle \langle 34\rangle}$$

The entire amplitude is given in terms of angle brackets, with no square brackets. Actually, if at the top of this page we had multiplied instead by

$$\frac{[14]}{[14]}$$

we could have derived instead

$$i\mathcal{M} = 2ie^2 \frac{[14]^2}{[12][34]}$$

Before going on, I would like to write one more identity obeyed by spinor products, the Schouten identity

$$\langle ij\rangle \langle kl\rangle + \langle ik\rangle \langle lj\rangle + \langle il\rangle \langle jk\rangle = 0$$

$$[ij][kl] + [ik][lj] + [il][jk] = 0$$

This identity follows from the fact that

$$u_{L\alpha}(p)\, u_{L\beta}(k)\, u_{L\gamma}(l)$$

antisymmetrized on $[\alpha\beta\gamma]$, gives zero, because $\alpha, \beta, \gamma = 1, 2$ only.

Now we need a way to treat external photons and gluons. I claim that the polarization vectors for outgoing massless vector particles can be represented by the following formulae:

$$\varepsilon_R^*(k) = \frac{1}{\sqrt{2}} \frac{\langle r\gamma^\mu k]}{\langle rk\rangle} \qquad \varepsilon_L^*(k) = -\frac{1}{\sqrt{2}} \frac{[r\gamma^\mu k\rangle}{[rk]}$$

where k is the momentum of the particle and r is another fixed lightlike vector. First, note the basic properties

$$[\varepsilon_R^*(k)]^* = \varepsilon_L^*(k)$$

$$k^\mu \varepsilon_{R,L}^*(k) = 0 \qquad \text{for example} \quad \langle r\, \not{k}\, k] = \langle rk\rangle [kk] = 0$$

$$|\varepsilon_R^\mu(k)|^2 = \frac{1}{2} \frac{\langle r\gamma^\mu k]\langle k\gamma_\mu r]}{\langle rk\rangle [kr]} = \frac{2}{2} \frac{\langle rk\rangle [rk]}{\langle rk\rangle [kr]} = -1$$

similarly $|\varepsilon_L^{*\mu}(k)|^2 = -1$

These are the basic properties of transverse vector polarizations. Next evaluate these formulae for a particular choice of r

$$k = (k, 0, 0, k) \qquad r = (r, 0, 0, -r) \qquad u_R(r) = \sqrt{2r} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then $[rk] = u_R^\dagger(r) u_L(k) = \sqrt{2r}\sqrt{2k}\, (0\;1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{4rk}$

$$[r\gamma^\mu k\rangle = \sqrt{2r}\sqrt{2k}\,(0\,1)\;\sigma^\mu \begin{pmatrix}1\\0\end{pmatrix}$$

$$= \sqrt{4rk}\;(0\,1)\;(1,\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \sqrt{4rk}\;\;(0\;\;1\;\;i\;\;0)$$

then

$$\epsilon_L^*(k) = -\frac{1}{\sqrt{2}}\frac{[r\gamma^\mu k\rangle}{[rk]} = -\frac{1}{\sqrt{2}}\,[(0\,,1,-i,\,0)]^*$$

which is correct. Similarly $\varepsilon_R^*(k)$ take the correct form for this choice of reference vector.

Finally what if we make a different choice for the lightlike reference vector r?

$$\varepsilon_R^{*\mu}(k;r) - \varepsilon_R^{*\mu}(k;s) = \frac{1}{\sqrt{2}}\left(\frac{\langle r\gamma^\mu k]}{\langle rk\rangle} - \frac{\langle s\gamma^\mu k]}{\langle sk\rangle}\right)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\langle rk\rangle\langle sk\rangle}\left(-\langle r\gamma^\mu k]\langle ks\rangle + \langle s\gamma^\mu k]\langle kr\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\langle rk\rangle\langle sk\rangle}\left(-\langle r\gamma^\mu \not{k}\, s\rangle + \langle s\gamma^\mu \not{k}\, r\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\langle rk\rangle\langle sk\rangle}\left(\langle s\,(\not{k}\gamma^\mu + \gamma^\mu \not{k})\,r\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\langle rk\rangle\langle sk\rangle}\;\;2k^\mu\,\langle sr\rangle$$

$$= \sqrt{2}\,\frac{\langle sr\rangle}{\langle rk\rangle\langle sk\rangle}\;k^\mu$$

so, replacing r by s gives an extra term proportional to k^μ, dotted into a photon or gluon on-shell amplitude

k^μ 

This will give zero by the Ward identity. Thus, we can make any choice of r that we wish, provided we make the same choice for all diagrams of a gauge-invariant set. Notice that we can use a different choice of r to compute different helicity amplitudes, as long as we use the same r for all diagrams contributing to the same amplitude.

From here on, I will make two final changes in notation. Since all particles are considered outgoing,

- drop the $*$ on $\varepsilon_R^* \varepsilon_L^*$; this is understood.
- replace R, L by $+, -$ the physical helicity of the outgoing particle.

Let's now compute an example with photons: $e^-_L e^+_R \to \gamma\gamma$. With all particles outgoing

$$\begin{array}{cccc} e^-_L & e^+_R & \to \gamma & \gamma \\ 1 & 2 & 3 & 4 \end{array}$$

$$i\mathcal{M} = (-ie)^2 \langle 2 \left\{ \gamma^\mu \frac{i(2+4)}{s_{24}} \gamma^\nu + \gamma^\nu \frac{i(2+3)}{s_{23}} \gamma^\mu \right\} 1] \; \varepsilon_\mu(4)\varepsilon_\nu(3)$$

where $s_{23} = (2+3)^2 = 2\, p_2 \cdot p_3 \qquad s_{24} = 2 p_2 \cdot p_4$.

There are four possible choices of the final photon helicities. Consider first $\gamma_+(3)\gamma_+(4)$. Choose: $r = 2$ for both particles.

$$\varepsilon^\mu(3) = \frac{1}{\sqrt{2}} \frac{\langle 2\gamma^\mu 3]}{\langle 23\rangle} \qquad \varepsilon^\mu(4) = \frac{1}{\sqrt{2}} \frac{\langle 2\gamma^\mu 4]}{\langle 24\rangle}$$

Then in $i\mathcal{M}$,

$$\langle 2\ \gamma\cdot\varepsilon(4) \cdots = \frac{1}{\sqrt{2}}\frac{1}{\langle 24\rangle} 2 \quad \langle 22\rangle [4 \cdots = 0$$

$$\langle 2\gamma\cdot\varepsilon(3) \cdots = \frac{1}{\sqrt{2}}\frac{1}{\langle 23\rangle}\ 2\cdot\ \langle 22\rangle [3 \cdots = 0$$

so $i\mathcal{M} = 0$. Similarly, for $\gamma_-(3)\gamma_-(4)$ take $r=1$ for both particles, and the amplitude is proportional to $[11] = 0$.

The remaining two cases are connected by crossing. Consider

$$\gamma_-(3)\ \gamma_+(4)$$

Choose
$$\varepsilon^\nu(3) = -\frac{1}{\sqrt{2}} \frac{[1\gamma^\mu 3\rangle}{[13]} \qquad \varepsilon^\mu(4) = \frac{1}{\sqrt{2}} \frac{\langle 2\gamma^\mu 4]}{\langle 24\rangle}$$

Then the first diagram is manifestly zero and the second gives

$$i\mathcal{M} = (-ie^2)\left(-\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right)\frac{2\cdot 2}{[13]\langle 24\rangle} \quad \langle 23\rangle \frac{[1\ \ (2+3)\ \ 2\rangle}{s_{23}} [41]$$

$$= 2ie^2 \ \frac{1}{[13]\langle 24\rangle}\ \frac{1}{\langle 23\rangle [32]} \quad \langle 23\rangle [13] \langle 32\rangle [41]$$

$$= 2ie^2 \ \frac{\langle 23\rangle [41]}{[23]\langle 24\rangle}$$

Up to phases, this is
$$2ie^2 \sqrt{\frac{u}{t}} = 2ie^2 \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

Again, we can simplify this further,

$$i\mathcal{M} = 2ie^2 \frac{\langle 23\rangle [41]}{[23]\langle 24\rangle} \cdot \frac{\langle 13\rangle}{\langle 13\rangle}$$

$$[41]\langle 13\rangle = -[42]\langle 23\rangle = [24]\langle 23\rangle$$

$$[23]\langle 13\rangle = -[23]\langle 31\rangle = [24]\langle 41\rangle$$

so

$$i\mathcal{M} = 2ie^2 \frac{\langle 23\rangle^2}{\langle 24\rangle\langle 41\rangle}$$

again, with all angle brackets and no square brackets.