QCD corrections to the Drell-Yan process

We have now developed a method for treating all of the various infrared divergences that appear in $\mathcal{O}(\alpha_s)$ QCD corrections to parton-parton scattering processes. In this lecture, I will test that method by applying it to the simplest process with two colliding hadrons in the initial state.

Consider, then, the Drell-Yan process $pp \to W^+ + X$. We have already derived the leading order formula:

$$\sigma(pp \to W^+ + X) = \int dx_1 \int dx_2 \left[f_u(x_1, m_W)\, f_{\bar d}(x_2, m_W) + f_{\bar d}(x_1, m_W)\, f_u(x_2, m_W) \right] \cdot \sigma(u\bar d \to W^+)$$

$$+ \text{(other } q\bar q \text{ combinations)}$$

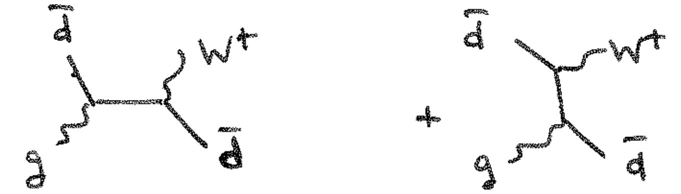
with $\sigma(u\bar d \to W^+) = \dfrac{\pi^2 \alpha_w}{3}\, \delta(\hat s - m_W^2)$. In the following, I will concentrate on the correction to this $u\bar d$ cross section. I will abbreviate $m_W \to m$.

To compute the $\mathcal{O}(\alpha_s)$ corrections, we must compute the following diagrams:

virtual corrections:

real gluon emission from $u\bar d$ annihilation:

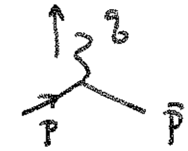
real gluon emission in $g\bar{d}$ or ug scattering ("Compton process")



To begin, we should redo the leading-order result in $d = 4-2\epsilon$ dimensions. Here and in the rest of the calculation, I will sum over the W^+ polarizations

$$\sum_{\varepsilon} \varepsilon^\mu(q)\, \varepsilon^{\nu *}(q) = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}\right)$$

The q^μ terms will give zero when contracted with the quark lines, as long as we ignore the quark masses. Then



$$i\mathcal{M} = \frac{ig}{\sqrt{2}}\, \bar{v}(\bar{p})\, \gamma^\mu \left(\frac{1-\gamma^5}{2}\right) u(p)\, \varepsilon_\mu(q)$$

$$= \frac{ig}{\sqrt{2}}\, M^\mu \varepsilon_\mu(q)$$

$$\frac{1}{4}\sum_{\text{pols.}} |M^\mu \varepsilon_\mu|^2 = \frac{1}{4}\, \mathrm{tr}\left(\bar{p}\!\!\!/\, \gamma^\mu p\!\!\!/\, \gamma^\nu \tfrac{1}{2}\right)(-g_{\mu\nu}) = 2(1-\epsilon)\cdot \frac{1}{2}\, p\cdot\bar{p}$$

$$= \frac{1}{2}(1-\epsilon)\, m^2$$

The phase space is

$$\int \frac{d^4 q}{(2\pi)^4}\, 2\pi\, \delta(q^2 - m^2)\, (2\pi)^4 \delta(p+\bar{p}-q) = 2\pi\, \delta(\hat{s}-m^2)$$

so finally

$$\sigma = \frac{1}{2\hat{s}}\, 2\pi\, \delta(\hat{s}-m^2)\cdot \frac{g^2}{2}\cdot \underline{\frac{1}{3}} \cdot \frac{1}{2}(1-\epsilon)\, m^2$$

color factor $\frac{1}{3}\cdot\frac{1}{3}\sum_{\text{colors}} 1 = \frac{1}{3}$

or

$$\sigma(u\bar{d}\to W^+) = \frac{\pi^2 \alpha_w}{3}\, \delta(\hat{s}-m_W^2)\, (1-\epsilon)$$

The effect of the virtual gluon diagrams is exactly as in our previous examples. The self-energy diagrams vanish, and the vertex diagram contributes a form factor

$$F_1(q^2) = 1 - \frac{4}{3}\frac{\alpha_s}{4\pi}\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\epsilon)[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8+\cdots\right)$$

q^2 is evaluated in the timelike region, at $q^2 = m^2$, so as on p. 18 of the lecture on $e^+e^- \to q\bar{q}(g)$, we must analytically continue $(-q^2)^{-\epsilon}$ to this region

$$(-q^2)^{-\epsilon} = (m^2)^{-\epsilon}\left(1 + i\pi\epsilon - \frac{\pi^2}{2}\epsilon^2 + \cdots\right)$$

Then

$$|F_1(m^2)|^2 = 1 - \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\epsilon)[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8\cdots\right)\left(1-\frac{\pi^2}{2}\epsilon^2+\cdots\right)$$

$$= 1 - \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\epsilon)[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8-\pi^2+\cdots\right)$$

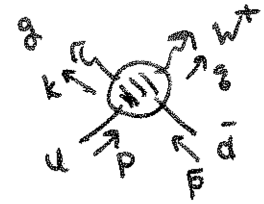
Now turn to the real corrections. Consider first the annihilation process $u\bar{d} \to gW^+$. As in the analysis of $\gamma^* q$ scattering it is useful to have a dimensionless variable whose kinematic range is $[0,1]$. Here, this is provided by

$$\tau = \frac{m^2}{s}$$

From here on, I will drop hats: τ, s, t, u refer to the parton subprocess. (But be careful; in other treatments of the Drell-Yan process, you will see $\tau = m^2/s(pp)$.)

Let's first work out the kinematics. $s = m^2/\tau$

In the CM frame



$$p = \bar{p} = \frac{1}{2}\sqrt{m^2/\tau}$$

$$k = \frac{s - m^2}{2\sqrt{s}} = \frac{m}{2}\,\frac{1-\tau}{\sqrt{\tau}} \qquad q^0 = \frac{s + m^2}{2\sqrt{s}} = \frac{m}{2}\,\frac{1+\tau}{\sqrt{\tau}}$$

Then the various invariants are

$$2p\cdot\bar{p} = m^2/\tau \qquad 2kq = m^2/\tau\,(1-\tau)$$

$$2p\cdot k = \frac{m^2}{\tau}(1-\tau)\left(\frac{1-\cos\theta}{2}\right) = \frac{m^2}{\tau}(1-\tau)\,\omega$$

$$2\bar{p}\cdot k = \frac{m^2}{\tau}(1-\tau)\left(\frac{1+\cos\theta}{2}\right) = \frac{m^2}{\tau}(1-\tau)\,(1-\omega)$$

where, as in the previous lecture, I have set $\frac{1-\cos\theta}{2} = \omega \quad \frac{1+\cos\theta}{2} = 1-\omega$

We also need the formula for 2-body phase space with one massive particle in $4-2\epsilon$ dimensions. Looking back at p. 6 of the previous lecture, this is

$$\int d\Pi_2 = \frac{1}{(2\pi)^{d-2}}\cdot\frac{1}{4}\int \frac{dk\; k^{d-2}}{k\, q^0}\,\delta(q^0 + k - \sqrt{s})\Big|_{q^0 = (k^2+m^2)^{1/2}} \int d\Omega_{d-1}$$

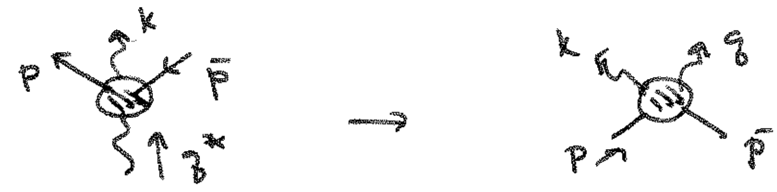
$$= \frac{1}{(2\pi)^{d-2}}\cdot\frac{1}{4}\, k^{d-3}\,\frac{1}{\left(1+\frac{k}{q^0}\right)}\int d\Omega_{d-2}\int d\cos\theta\,(\sin\theta)^{d-4}$$

$$= \frac{1}{16\pi}\left(\frac{4k^2}{4\pi}\right)^{-\epsilon}\left(\frac{2k}{E_{cm}}\right)\frac{1}{\Gamma(1-\epsilon)}\int d\cos\theta\,\left(\frac{1-\cos^2\theta}{4}\right)^{-\epsilon}$$

$$= \frac{1}{8\pi}\left[\frac{m^2(1-\tau)^2}{4\pi\tau}\right]^{-\epsilon}\frac{1-\tau}{\Gamma(1-\epsilon)}\int_0^1 d\omega\,[\omega(1-\omega)]^{-\epsilon}$$

The factors $(1-\tau)^{-\epsilon}$, $\omega^{-\epsilon}$, $(1-\omega)^{-\epsilon}$ will all provide needed convergence factors.

The matrix element is obtained, up to some obvious coupling constant factor, by making another crossing of the amplitude for $\gamma^* \to q\bar{q}g$



$$\frac{1}{4} \sum_{\text{pols.}} |\mathcal{M}|^2 = \frac{1}{4} \frac{g^2}{2} g_s^2 \cdot \frac{4}{3} (\mu^2)^\epsilon \, \mathbb{M} \frac{1}{2}, \quad \text{where } \mathbb{M} \text{ is again the}$$

quantity on p. 12 of the lecture on $e^+e^- \to q\bar{q}(g)$, crossed over to $k \to -k$. The last $\frac{1}{2}$ accounts the $(\frac{1-\gamma^5}{2})$ in the W coupling

$$\mathbb{M} = 8 \Big[(1-\epsilon)^2 \Big(\frac{2\bar{p}\cdot k}{2p\cdot k} + \frac{2p\cdot k}{2\bar{p}\cdot k} + \frac{2q^2 \, 2p\cdot\bar{p}}{2p\cdot k \, 2\bar{p}\cdot k} \Big] + 2\epsilon(1-\epsilon) \Big[\frac{q^2 \, 2p\bar{p}}{2p\cdot k \, 2\bar{p}\cdot k} - 1 \Big] \Big]$$

Using the expression on p. 4, this evaluates to:

$$\mathbb{M} = 8(1-\epsilon) \Big[(1-\epsilon) \Big[\frac{1-w}{w} + \frac{w}{1-w} + 2 \frac{\tau}{(1-\tau)^2 w (1-w)} \Big] + \epsilon \Big[\frac{2\tau}{(1-\tau)^2 w(1-w)} - 1 \Big] \Big]$$

Let's integrate this over w:

$$\int_0^1 dw \, [w(1-w)]^{-\epsilon} \frac{1-w}{w} = \frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} = -\frac{1}{\epsilon} \frac{(1-\epsilon)}{(1-2\epsilon)} \frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}$$

$$\int dw \, [w(1-w)]^{-\epsilon} \frac{1}{w(1-w)} = \frac{\Gamma(-\epsilon)\,\Gamma(-\epsilon)}{\Gamma(-2\epsilon)} = \frac{-2\epsilon}{(-\epsilon)^2} \frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}$$

so the ~~integral~~ of $\mathbb{M}$ is:

$\int dw\, [w(1-w)]^{-\epsilon}\, |M|$

$$= 8(1-\epsilon)\left[-\frac{1}{\epsilon}\right]\left[\frac{(1-\epsilon)^2}{1-2\epsilon}\cdot 2 + 2\cdot\frac{2\tau}{(1-\tau)^2}\right]\frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)} + \mathcal{O}(\epsilon)$$

$$= 8(1-\epsilon)\left(-\frac{2}{\epsilon}\right)\left(\frac{1+\tau^2}{(1-\tau)^2}\right)\frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)} + \mathcal{O}(\epsilon)$$

where — it is important to see — the terms of $\mathcal{O}(\epsilon)$ are not singular as $\tau \to 1$. Then the real gluon emission cross section is:

$$\sigma(u\bar{d}\to gW^+) = \frac{1}{2s}\,\frac{1}{8\pi}\left(\frac{m^2(1-\tau)}{4\pi\tau}\right)^{-\epsilon}\frac{1-\tau}{\Gamma(1-\epsilon)}\cdot\frac{1}{8}\, g_s^2 g_W^2 (\mu^2)^{\epsilon}\, \overset{\text{color}}{\frac{4}{3}\cdot\frac{1}{3}}\cdot\frac{1}{2}$$

$$\frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}\left[8(1-\epsilon)\left(-\frac{2}{\epsilon}\right)\frac{1+\tau^2}{(1-\tau)}+\cdots\right]$$

or

$$\sigma(u\bar{d}\to gW^+) = \frac{\pi^2\alpha_W}{3}\cdot\frac{4}{3}\,\frac{\alpha_s}{2\pi}\cdot\frac{\tau}{m^2}\,(1-\epsilon)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{m^2(1-\tau)^2}{4\pi\mu^2\tau}\right]^{-\epsilon}$$

$$\cdot\left[-\frac{2}{\epsilon}\,\frac{1+\tau^2}{1-\tau}+\cdots\right]$$

This cross section is singular as $\tau \to 1$. The limit $\tau\to 1$ gives the kinematics of the leading order process $u\bar{d}\to W^+$. So it is useful in this case, as in the previous lecture, to replace the singular factor with a $+$ distribution, plus a delta function term to be added to the virtual terms. To do this, we need:

$$A = \int_0^1 d\tau\left(\frac{(1-\tau)^2}{\tau}\right)^{-\epsilon}\left(-\frac{2}{\epsilon}\right)\frac{1+\tau^2}{1-\tau} = \frac{\Gamma(1-2\epsilon)\,\Gamma(1+\epsilon)}{\Gamma(1+\epsilon)}\left\{\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{9}{2}+\cdots\right\}$$

so we can write $\left[-\frac{2}{\epsilon}\,\frac{1+\tau^2}{1-\tau}\left(\frac{1-\tau}{\tau}\right)^{-\epsilon}\right] = \left[-\frac{2}{\epsilon}\,\frac{1+\tau}{1-\tau}\left(\frac{1-\tau}{\tau}\right)^{-\epsilon}\right]_+ + A\,\delta(\tau-1)$

Now combine this cross section with that for $u\bar{d} \to W^+$. We need to integrate over parton distributions

$$\int dx_1 dx_2 \; f_u(x_1) f_{\bar{d}}(x_2)$$

Since $\hat{s} = m^2/\tau = x_1 x_2 s_{pp}$, we can trade one of $x_1 x_2$ for τ.

Let $x_1^0 = \dfrac{m^2}{x_2 s_{pp}}$, the correct value of x_1 for $u\bar{d} \to W^+$.

Then $x_1 = \dfrac{x_1^0}{\tau}$ $\quad dx_1 = \dfrac{d\tau}{\tau^2} x_1^0 = \dfrac{d\tau}{\tau}\left(\dfrac{\hat{s}}{x_2 s_{pp}}\right)$

We can use this factor to convert $\frac{1}{\hat{s}} = \frac{\tau}{m^2}$ on the previous page to $\frac{1}{s_{p\bar{p}}}$. The zeroth order cross section becomes

$$\frac{\pi^2 \alpha_w}{3} \delta(\hat{s} - m^2)(1-\epsilon) = (1-\epsilon)\frac{\pi^2 \alpha_w}{3} \frac{1}{m^2}\delta(\tau - 1) = (1-\epsilon)\frac{\pi^2 \alpha_w}{3}\frac{1}{\hat{s}}\delta(\tau-1)$$

Then we can put the pieces together:

$$\sigma(pp \to W^+ + X) = \frac{\pi^2 \alpha_w}{3 s_{pp}}(1-\epsilon)\int \frac{d\tau}{\tau}\int \frac{dx_2}{x_2}\left[f_u\left(\frac{x_1^0}{\tau}\right) f_{\bar{d}}(x_2) + f_{\bar{d}}\left(\frac{x_1^0}{\tau}\right) f_u(x_2)\right]$$

$$\cdot \Big\{ \delta(\tau-1)\left[1 - \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1+\epsilon)[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 - \pi^2 + \cdots\right)\right]$$

$$+ \frac{4}{3}\frac{\alpha_s}{2\pi}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\left[\left(\frac{(1-\tau)^2}{\tau}\right)^{-\epsilon}\left[-\frac{2}{\epsilon}\frac{1+\tau^2}{1-\tau}\right] + \cdots\right] +$$

$$+ \frac{4}{3}\frac{\alpha_s}{2\pi}\Gamma(1+\epsilon)\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{9}{2} + \cdots\right)\delta(\tau-1)\Big\}$$

$$+ \quad (\text{Compton})$$

Now, $\dfrac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)} = 1 - \epsilon^2 \dfrac{\pi^2}{6} + \cdots$ so the complete $\delta(\tau-1)$

term is: $\left[1 + \frac{4}{3}\frac{\alpha_s}{2\pi}\Gamma(1+\epsilon)\left[\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{9}{2}\right) + \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \frac{\pi^2}{3}\right) + \cdots\right] \cdot (m^2/4\pi\mu^2)^{-\epsilon}\right]\delta(\tau-1)$

This has a smooth $\epsilon \to 0$ limit

$$\left(1 + \frac{4}{3}\frac{\alpha_s}{2\pi}\left[-\frac{7}{2} + \frac{4\pi^2}{3}\right]\right)$$

The term that remains contains the $\frac{1}{\epsilon}$ singularity

$$\int \frac{d\tau}{\tau}\left(f_u\left(\frac{x_1^0}{\tau}\right) f_{\bar d}(x_2) + \cdots\right) \frac{4}{3}\frac{\alpha_s}{2\pi}\cdot\left[-\frac{1}{\epsilon}\left(\frac{1+\tau^2}{1-\tau}\right)_+\right]\cdot 2$$

If the final 2 were a 1, this factor would be absorbed by converting $f_u(x_1^0)$ to $f_u^{(\overline{MS})}(x_1^0)$. The full factor is absorbed by converting both $f_u(x_1)$ and $f_{\bar d}(x_2)$. Then, in all,

$$\sigma(pp \to W^+ + X) = \frac{\pi^2 \alpha_w}{3 s_{pp}} \int \frac{d\tau}{\tau} \int \frac{dx_2}{x_2} \left[f_u^{(\overline{MS})}\left(\frac{x_1^0}{\tau}, \mu_f^2\right) f_{\bar d}^{(\overline{MS})}(x_2, \mu_f^2) + f_{\bar d}^{(\overline{MS})}\left(\frac{x_1^0}{\tau}, \mu_f^2\right) f_u^{\overline{MS}}(x_2, \mu_f^2)\right]$$

$$\Big\{ \delta(\tau - 1)\left(1 + \frac{4}{3}\frac{\alpha_s}{2\pi}\left[\frac{4}{3}\pi^2 - 7/2\right]\right)$$

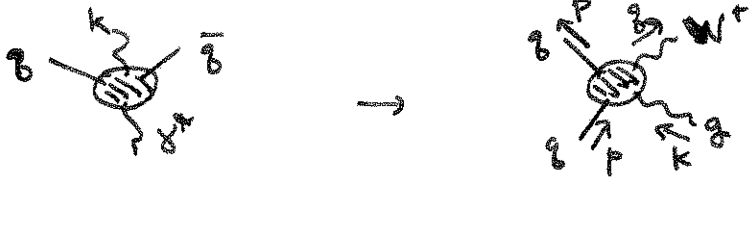
$$+ \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{1+\tau^2}{1-\tau}\right)_+ \log\frac{m^2}{\mu_f^2}\cdot 2$$

$$+ \frac{4}{3}\frac{\alpha_s}{2\pi}\cdot 2 \cdot \left[\left(\frac{1+\tau^2}{1-\tau}\right)\log\frac{(1-\tau)^2}{\tau}\right]_+ \Big\}$$

$$+ \text{(Compton terms)}$$

As before, the term with $\log m^2/\mu_f^2$ can be absorbed by evaluating the pdf's at $Q = m$.

To finish this calculation, we need to include the Compton terms. This process is another cross of the $\gamma^* \to q\bar q g$ amplitude:



From our ~~first~~ expression for $\mathbb{M}$, we need to replace

$$p \to p' \qquad \bar{p} \to -p \qquad k \to -k \qquad q \to -q \qquad q^2 = +m^2$$

(and multiply by (-1) because we cross 1 fermion line).

$$\frac{1}{4}\sum |\mathcal{M}|^2 = \frac{1}{4}\,\frac{g_w^2}{2}\,g_s^2 \cdot \frac{1}{6} \cdot \mathbb{M} \cdot \frac{1}{2}$$

with

$$\mathbb{M} = 8(1-\epsilon)\Big[(1-\epsilon)\Big[\frac{2p\cdot k}{2p'\cdot k} + \frac{2p'\cdot k}{2p\cdot k} - 2\,\frac{m^2\; 2p\cdot p'}{2p\cdot k\; 2p'\cdot k}\Big] - 2\epsilon\Big[\frac{m^2\; 2p\cdot p'}{2p\cdot k\; 2p'\cdot k} - 1\Big]\Big]$$

The factor $\frac{1}{6}$ is the color factor: $\frac{1}{3}\cdot\frac{1}{8}\,\mathrm{tr}\, t^a t^a = \frac{1}{8}\cdot\frac{4}{3} = \frac{1}{6}$

We can read the kinematics from p. 4:

$$2p\cdot p' = \frac{m^2}{\tau}(1-\tau)\left(\frac{1-\cos\theta}{2}\right) \qquad 2k\cdot p' = \frac{m^2}{\tau}(1-\tau)\left(\frac{1+\cos\theta}{2}\right)$$

$$2p\cdot k = m^2/\tau$$

$$\mathbb{M} = 8(1-\epsilon)\Big[(1-\epsilon)\Big[\frac{1}{(1-\tau)(1-\omega)} + (1-\tau)(1-\omega) - \frac{2\tau\omega}{(1-\omega)}\Big] - 2\epsilon\Big[\frac{\tau\omega}{1-\omega} - 1\Big]\Big]$$

The ~~integral~~ of this expression $\int_0^1 d\omega\, [\omega(1-\omega)]^{-\epsilon}$ is

$$8(1-\epsilon)\frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}\,\frac{1}{(1-\tau)}\left\{-\frac{1}{\epsilon}\left[\tau^2 + (1-\tau)^2\right] + \left(\frac{3}{2} + \tau - \frac{3}{2}\tau^2\right) + \dots\right\}$$

In all

$$\sigma(ug \to W^+ d) = \frac{1}{2\hat{s}}\,\frac{1}{8\pi}\left(\frac{m^2(1-\tau)^2}{4\pi\mu^2\tau}\right)^{-\epsilon}\frac{(1-\tau)}{\Gamma(1-\epsilon)}\,\frac{g^2 g_s^2}{8\cdot 6\cdot 2}\cdot 8(1-\epsilon)$$

$$\cdot\,\frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}\,\frac{1}{(1-\tau)}\left\{-\frac{1}{\epsilon}\left[\tau^2 + (1-\tau)^2\right] + (\;) + \dots\right\}$$

a

$$\sigma(ug \to W^+ d) = \frac{\pi^2 \alpha_w}{3\hat{s}} \cdot \left(\frac{m^2(1-\tau)^2}{4\pi\mu^2\tau}\right)^{-\epsilon} \frac{\alpha_s}{2\pi}$$

$$\cdot \left[\left(-\frac{1}{\epsilon}\right) \frac{1}{2}\left[\tau^2 + (1-\tau)^2\right] + \left(\frac{3}{4} + \frac{\tau}{2} - \frac{3}{4}\tau^2\right) + \ldots \right\}$$

we can recognize the coefficient of $-\frac{1}{\epsilon}$ as $P_{\bar{q} \leftarrow g}(\tau)$. So this term in $\sigma(pp)$:

$$\sigma(pp \to W^+ + X) = \frac{\pi^2 \alpha_w}{3 S_{pp}} \int \frac{d\tau}{\tau} \int \frac{dx_2}{x_2} \left[f_g\left(\frac{x_1^0}{\tau}\right) f_u(x_2) + \ldots \right]$$

$$\cdot \left[\frac{\alpha_s}{2\pi} \left(-\frac{1}{\epsilon}\right) P_{\bar{d} \leftarrow g}(\tau) \right]$$

gives the $g \to \bar{d}$ splitting term that we need to absorb into $f_{\bar{d}}^{(\overline{MS})}(x_1^0)$

The ug cross section is not singular as $\tau \to 1$, so there are no new contributions to the $u\bar{d} \to W^+$ term. Now we have absorbed all of the IR divergences and we can write the final answer:

$$\sigma(pp \to W^+ + X) = \frac{\pi^2 \alpha_w}{3 S_{pp}} \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \Big[\left[f_u^{(\overline{MS})}(x_1, m) f_{\bar{d}}^{(\overline{MS})}(x_2, m) + f_{\bar{d}}^{(\overline{MS})}(x_1, m) f_u^{(\overline{MS})}(x_2, m) \right]$$

$$\left\{ \delta(\tau - 1) \left(1 + \left(\frac{8\pi^2}{9} - \frac{7}{3}\right) \frac{\alpha_s(m)}{\pi} \right) + \frac{4}{3}\frac{\alpha_s}{2\pi} \cdot 2 \cdot \left[\frac{1+\tau^2}{1-\tau} \log \frac{(1-\tau)^2}{\tau} \right]_+ \right\}$$

$$+ \left[f_u^{(\overline{MS})}(x_1, m) f_g^{\overline{MS}}(x_2 m) + f_{\bar{d}}^{(\overline{MS})}(x_1 m) f_g(x_2 m) + f_g^{(\overline{MS})}(x_1 m) f_u(x_2 m) + f_g^{(\overline{MS})}(x_1 m) f_{\bar{d}}^{(\overline{MS})}(x_2 m) \right]$$

$$\cdot \frac{\alpha_s}{4\pi} \left\{ \left[\tau^2 + (1-\tau)^2 \right] \log \frac{(1-\tau)^2}{\tau} + \left(\frac{3}{2} + \tau - \frac{3}{2}\tau^2\right) \right\} \Big]$$

Unlike the previous cases, the correction to the δ-function coefficient is not small

$$\frac{8\pi^2}{9} - \frac{7}{3} = 6.44$$

so that

$$\left(1 - \left(\frac{8\pi^2}{9} - \frac{7}{3}\right)\frac{\alpha_s(Q)}{\pi}\right) = \begin{cases} 1.25 & Q = m_W \\ 1.6 & Q = 2\ \text{GeV} \end{cases}$$

(relevant to $pp \to \gamma^* \to \mu^+\mu^-$)

From this factor and the other corrections, we find a large positive correction to the normalization of the cross section in the Drell-Yan process. Such large corrections are typical in QCD corrections to parton cross sections at hadron colliders. These corrections are particularly large at the LHC, where the terms involving gluon pdf's give large effects.

Very recently, Hamburg, van Neerven, and Matsuma, Harlander and Kilgore, and Anastasiou, Dixon, Melnikov, and Petriello computed the order α_s^2 corrections to the Drell-Yan cross section, so we have some idea of how the series converges. Figs. p.3, comparing the blue, green, and red bands, shows the successive estimates of the Drell-Yan cross section at the Tevatron at leading order, order α_s (NLO), and order α_s^2 (NNLO). The three curves are computed with MRST parton distributions. The pink curve is computed from an alternative set of NNLO parton distributions (and so gives an idea of the error due to the uncertainty in the proton distributions). We see a correction by about a factor 1.5 from LO to NLO and a further, but converging, correction to NNLO.

The ratio of the NLO (or NNLO) cross section to that at leading order is called the <u>K-factor</u>. The idea of this definition is that the NLO corrections mainly change the normalization of the LO cross section while preserving their shape. Usually it is true ~~that~~ the change in normalization is the strongest effect. Figs <u>p.3</u> shows the modification of the cross section as a function of the $\mu\mu$ pair rapidity y. Note the suppressed zero. The shift in normalization is about 1.5–1.6, with about a $\pm 10\%$ dependence on y, at this relatively low energy point: $\sqrt{s} = 39$ GeV, $Q = 8$ GeV. At the LHC, in $pp \to W^\pm$ or Z, Anastasiou et al estimate an overall cross section enhancement of about 1.2. Figs. <u>p.4 and 5</u> show their predictions for the rapidity distribution of Z, W^-, and W^+ at the LHC. Since the LHC gives pp collisions, the rapidity distribution for all three bosons must be symmetric under $y \leftrightarrow -y$. However, the W^- and W^+ distributions are expected to have different shapes, reflecting the fact that the u quarks in the proton are harder that the d quarks.

At lowest order, the total cross section for boson production at the LHC are:

$\sigma_0(W^+) \sim 83\,\text{nb} \qquad \sigma_0(W^-) \sim 63\,\text{nb} \qquad \sigma_0(Z^0) \sim 45\,\text{nb}.$

To convert this to observable cross sections, we need to multiply by the branching ratios to leptons:

$$BR(W^+ \to \mu^+\nu) = 11\% \qquad BR(Z^0 \to \mu^+\mu^-) = 3.4\%$$

apply the K-factor, and take account of experimental cuts.

For example,

$$\sigma_0(W^-) \times \underbrace{1.2}_{K} \times BR(W^- \to \mu^-\nu) = 8.3\,\text{nb}$$

For a Breit-Wigner shape $\frac{d\sigma}{dm^2} = \frac{1}{\pi m\Gamma}\sigma$ or

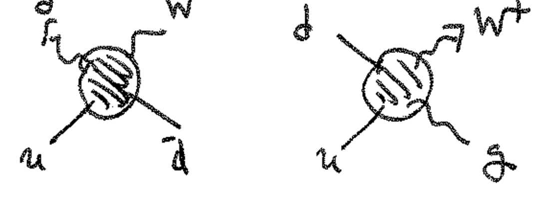
$\frac{d\sigma}{dm} = \frac{2}{\pi\Gamma}\sigma$ at the peak of the cross section in $m(\ell\ell)$.

For W^-, this is $\left.\frac{d\sigma}{dm}\right|_{peak} = 2.6\,\text{nb}/\text{GeV}$, or about

$$\frac{d\sigma}{dm\,dy} \sim 600\,\text{pb}/\text{GeV}$$

This brings us into the range of the scale in Fig. p. 5, with the remaining factor given by taking account of the experimental acceptance for the μ^-.

At leading order, the Drell-Yan lepton pair is produced with zero transverse momentum. However, the order α_s corrections also generate a transverse momentum distribution for the massive bosons. In particular, in the real emission diagrams, the boson recoils against a quark or gluon:



I gave the formula for p_T distributions in an earlier lecture:

$$\frac{d\sigma}{dp_T} = \sum_{1,2} \int dx_1 dx_2 \, f_1(x_1) f_2(x_2) \frac{4p_T}{\sqrt{\hat{s}(\hat{s}-4p_T^2)}} \frac{d\sigma}{d\cos\theta_*}(1+2 \to 3+4)$$

and we have computed the elementary cross sections $\frac{d\sigma}{d\cos\theta_*}$ earlier in this lecture. For the annihilation process (p.5)

$$\frac{d\sigma}{d\cos\theta_*}(u\bar{d} \to gW^+) = \underbrace{\frac{1}{2s}\frac{1}{16\pi}\left(1-\frac{m^2}{s}\right)}_{\text{phase space}} \cdot \frac{1}{4}\,\frac{g^2}{2}\,g_s^2 \cdot \frac{4}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$$

$$\cdot\, 8 \cdot \left[\frac{u}{t} + \frac{t}{u} + \frac{2m^2 s}{tu}\right]$$

or

$$\frac{d\sigma}{d\cos\theta_*}(u\bar{d} \to gW^+) = \frac{1}{9}\frac{\pi\alpha_w\alpha_s}{s}\left(1-\frac{m^2}{s}\right)\left[\frac{u}{t}+\frac{t}{u}+\frac{2m^2 s}{tu}\right]$$

Similarly, using the expression on p.9, we find for the Compton process:

$$\frac{d\sigma}{d\cos\theta_*}(ug \to dW^+) = \frac{1}{24}\frac{\pi\alpha_w\alpha_s}{s}\left(1-m^2/s\right)\left[-\frac{s}{u}-\frac{u}{s}-\frac{2m^2 t}{su}\right]$$

This expression can also be obtained from the one above by crossing and a change in the color average: $\frac{1}{3}\cdot\frac{1}{3} \to \frac{1}{3}\cdot\frac{1}{8}$.

In pp collisions, there are more gluons than $\bar{d}$'s, so the Compton process dominates at large p_T. Figs. p.6 shows a comparison of the two separate contributions to the p_T distribution of $\mu^+\mu^-$ pairs in $pp \to \mu^+\mu^- + X$ at the ISR. Fig. p.7

shows a comparison of the DØ measurement of the W boson p_T distribution at the Tevatron to a QCD calculation, including corrections at the next order in α_s computed by Arnold and Kauffman.

Note that the QCD expressions for $d\sigma/dp_T$ diverge as $p_T \to 0$

$$\frac{d\sigma}{dp_T} \sim \frac{1}{p_T} \qquad p_T \to 0$$

a nonintegrable singularity. The hard work in this lecture was devoted to integrating this cross section to obtain the correct total cross section by combining this ~~integral~~ with virtual corrections. Still, there remains the problem of obtaining the correct shape of $d\sigma/dp_T$ at low p_T. Two effects must be accounted. First, multiple gluon emission can smear the p_T distribution around $p_T = 0$. This can in principle be accounted for by resumming QCD emissions to all orders. Second, nonperturbative effects generate a finite intrinsic p_T for the initial partons. At the Tevatron, both effects are important. Figs. p. 8 shows three attempts to combine these perturbative and nonperturbative effects and their comparison to DØ data.