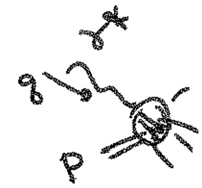
QCD Corrections to Hadronic Reactions

In the previous lecture, we computed the order α_s corrections to the total cross section for $e^+e^- \to$ hadrons. For a process like this, with no hadrons in the initial state, the Kinoshita-Lee-Nauenberg theorem tells us that the IR divergences must cancel — and we found that they did. However, this theorem does not apply to processes with hadrons in the initial state. Here we find a cross section that depends on parton distribution functions, and the order α_s corrections must reflect the evolution of the pdf's. In this lecture, I will do a relatively simple calculation to show how these ideas fit together.

It is not so difficult to compute the $O(\alpha_s)$ corrections to deep inelastic scattering. However, I would like to simplify the calculation even further. Instead of considering electron scattering

I would like to consider the scattering from a proton of a virtual photon summed over polarizations such that



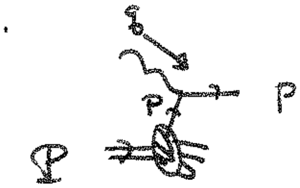
$$Q^2 = -q^2$$

$$\sum_\epsilon \epsilon^\mu(q)\,\epsilon^{\nu *}(q) = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)$$

The result will have a somewhat different structure in y from

$e^- p$ inelastic scattering, but it will address the question that I have posed above.

Let's first compute the zeroth-order cross section for this virtual Compton scattering.



$p = \xi P \qquad 0 < \xi < 1$

The parton level matrix element is

$$\mathcal{M} = +ieQ_f \quad \bar{u}(p')\gamma\cdot\varepsilon(q)\, u(p)$$

so

$$\frac{1}{2}\sum_{\text{pols.}} |\mathcal{M}|^2 = e^2 Q_f^2 \cdot \frac{1}{2} \cdot \mathrm{tr}[\not{p}'\gamma^\mu \not{p}\gamma^\nu]\,(-g_{\mu\nu})$$

$$= e^2 Q_f^2 \cdot \frac{1}{2}\,\mathrm{tr}[\not{p}'\gamma^\mu\not{p}\gamma_\mu]\,(-1)$$

$$= e^2 Q_f^2 \cdot \frac{1}{2}\cdot 2 \cdot 4\, p'\cdot p$$

$$= e^2 Q_f^2 \cdot 2\,(2p'\cdot p) = 2e^2 Q_f^2\,(-q^2)$$

We can now construct the cross section. The flux factor is

$$q = (q^0, 0, 0, q) \qquad p = (p, 0\,0\,-p)$$

$$\frac{1}{2E_1 2E_2 |v_1 - v_2|} = \frac{1}{2q^0 2p\left(\frac{q}{q^0}+1\right)} = \frac{1}{2\cdot 2p\cdot q}$$

The phase space is :

$$\int \frac{d^4p'}{(2\pi)^4}\, 2\pi\, \delta((p')^2)\, (2\pi)^4 \delta(p+q-p')$$

$$= \quad 2\pi\, \delta((p+q)^2) \quad = \quad 2\pi\, \delta(2p\cdot q - Q^2)$$

$p = \xi P \qquad 2pq = \xi\, 2P\cdot q$ so this

$$= \quad \frac{2\pi}{2P\cdot q}\, \delta(\xi - x) \qquad \text{where } x = \frac{Q^2}{2P\cdot q} \text{ as before}$$

Finally

$$\sigma_0(\gamma^* q) = \frac{1}{2\; 2pq} \cdot 2e^2 Q_f^2 (-q^2)\; \frac{2\pi\, \delta(\xi - x)}{2P\cdot q}$$

$$\sigma_0(\gamma^* q) = 8\pi^2\alpha\, Q_f^2 \cdot \frac{x}{Q^2}\; \delta(\xi - x)$$

then

$$\sigma_0(\gamma^* p) = \int d\xi \sum_f f_f(\xi)\, \sigma_0(\gamma^* q_f)$$

$$= \sum_f f_f(x)\, Q_f^2\, \frac{x}{Q^2}\; 8\pi^2\alpha$$

or

$$\sigma_0(\gamma^* p) = F_2(x) \cdot \left(\frac{8\pi^2\alpha}{Q^2}\right)$$

with $F_2(x) = \sum_f x\, Q_f^2\, f_f(x)$ as before. Note that the factor $[1+(1-y)^2]$, which reflects the virtual photon polarizat sum in deep inelastic scatterg, is no longer here.

When we compute the order α_s corrections, we will need to regulate the IR divergences using dimensional regularization. So we will need to work in d dimensions, and thus it will be convenient to have this zeroth-order result in d dimensions. The only part of the calculation that is affected is the computation of $\Sigma |\mathcal{M}|^2$, which now becomes:

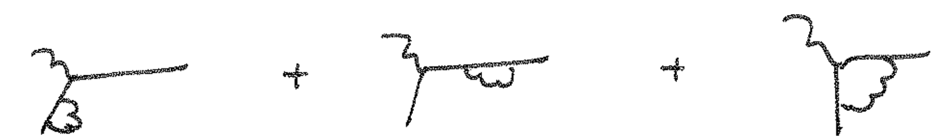
$$\frac{1}{2}\sum |\mathcal{M}|^2 = + e^2 Q_f^2 \left(-\tfrac{1}{2}\right) \mathrm{tr}[\not{p}' \gamma^\mu \not{p} \gamma_\mu]$$

$$= e^2 Q_f^2 \left(-\tfrac{1}{2}\right)(-2)(1-\epsilon)\ \mathrm{tr}\, \not{p}' \not{p}$$

$$= 2 e^2 Q_f^2 (-q^2) \cdot (1-\epsilon)$$

so $$\sigma_0(\gamma^* q \to q) = 8\pi^2 \alpha\, Q_f^2\, \frac{x}{Q^2}\, (1-\epsilon)\ \delta(\xi - x)$$

Now add the virtual gluon diagrams:



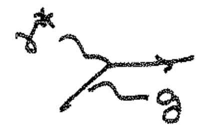
In the previous lecture, I argued that the self-energy diagrams are zero for on shell massless quarks. Then the expression for σ_0 above is modified only by the inclusion of the form factor generated by the vertex diagram

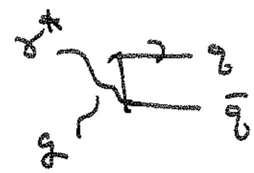
$$\sigma(\gamma^* q \to q) = (\text{above}) \times \left| F_1(q^2 = -Q^2) \right|^2$$

where $F_1(q^2)$ was computed in the previous lecture:

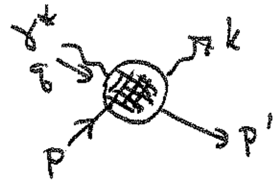
$$|F_1(q^2)|^2 = 1 - \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\epsilon)[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8+\cdots\right)$$

Now we only need to compute the diagrams with real gluon emission. There are two class of these diagrams.

$q\gamma^* \to qg$  + 

$g\gamma^* \to q\bar{q}$  + 

Look first at the $q\gamma^*$ diagrams. The kinematics is



For the $\gamma^* q \to q$ process, we had $Q^2 = 2p\cdot q$. Now

$$(p+q)^2 = (p'+k)^2 = 2p'\cdot k > 0 \qquad \text{so} \qquad 2p\cdot q - Q^2 > 0$$

Let
$$z = \frac{Q^2}{2p\cdot q} \qquad\qquad 0 < z < 1$$

The cross section for $\gamma^* q \to qg$ takes the form

$$\sigma(\gamma^* q \to gq) = \frac{1}{2(2p\cdot q)}\int d\Pi_2 \quad \frac{1}{2}\sum_{\text{pols.}}{}' |\mathcal{M}|^2$$

In the previous lecture, we worked out a formula for 2-body massless phase space in d dimensions:

$$\int d\Pi_2 = \frac{1}{8\pi}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\left(\frac{Q^2}{4\pi}\right)^{-\epsilon}$$

However, now we need this formula differential in the CM scattering angle θ_*!

$$\int d\Pi_2 = \frac{1}{(2\pi)^{d-2}}\,\frac{1}{4}\int \frac{dp\, p^{d-2}}{p\cdot k}\,\delta(p+k-Q)\Big|_{k=p}\int d\Omega_{d-1}$$

$$= \frac{1}{(2\pi)^{d-2}}\,\frac{1}{4\cdot 2}\left(\frac{\sqrt{s}}{2}\right)^{d-4}\int d\Omega_{d-2}\int d\cos\theta_*\,[\sin\theta_*]^{d-4}$$

$$= \frac{1}{2\cdot 16\pi^2}\left(\frac{s}{4\pi}\right)^{-\epsilon}\frac{1}{(4\pi)^{-\epsilon}}\,\frac{2\pi^{(d-2)/2}}{\Gamma(\frac{d-2}{2})}\int d\cos\theta_*\,[1-\cos^2\theta_*]^{-\epsilon}$$

$$= \frac{1}{16\pi}\left(\frac{s}{4\pi}\right)^{-\epsilon}\frac{1}{\Gamma(1-\epsilon)}\int_{-1}^{1} d\cos\theta_*\left[\frac{1-\cos^2\theta_*}{4}\right]^{-\epsilon}$$

Doing the integral using the Euler beta function gives the previous result.

The matrix element has the form

$$i\mathcal{M} = (+ieQ_f)(ig\,t^a)\;\bar{u}(p')\,[\;\cdots\;]\,u(p)$$

so

$$\frac{1}{2}\sum|\mathcal{M}|^2 = \frac{1}{2}e^2Q_f^2\,g^2\,\frac{4}{3}\cdot \mathbb{M}$$

where $\mathbb{M}$ is the squared matrix element from the previous lecture

crossed over to the current kinematics

$$p \to p' \qquad \bar{p} \to -p$$

(and an extra (-1) for crossing one fermion line.) This gives

$$|M| = 8\left[(1-\epsilon)^2 \left(\frac{2p\cdot k}{2p'\cdot k} + \frac{2p'\cdot k}{2p\cdot k} - 2\,\frac{q^2\, 2p\cdot p'}{2p\cdot k\; 2p'\cdot k} \right) - 2\epsilon(1-\epsilon)\left(\frac{q^2\, 2pp' - 2p\cdot k\, 2p'\cdot k}{2p\cdot k\; 2p'\cdot k} \right) \right]$$

For this kinematics $q^2 = -Q^2$. In the CM frame

$$q = (q^0, 0, 0, q) \qquad p = (p, 0, 0, -p)$$

$$q^0 = \frac{s - Q^2}{2\sqrt{s}} \qquad p = q = \frac{s+Q^2}{2\sqrt{s}} \qquad \text{so} \quad (q^0)^2 - q^2 = -Q^2$$

$$2p\cdot q = s + Q^2$$

then $$z = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{s+Q^2} \qquad \text{so} \qquad s = Q^2\,\frac{(1-z)}{z}$$

$$p = q = \frac{Q^2}{2z}\left[\frac{z}{Q^2(1-z)}\right]^{1/2} = \frac{Q}{2}\,\frac{1}{\sqrt{z(1-z)}}$$

$$p' = k' = \frac{\sqrt{s}}{2} = \frac{Q}{2}\sqrt{\frac{(1-z)}{z}}$$

then $$2p\cdot q = \frac{Q^2}{z} \qquad 2p'\cdot k = \frac{Q^2}{z}(1-z)$$

$$2p\cdot p' = \frac{Q^2}{z}\left(\frac{1-\cos\theta_*}{2}\right) \qquad 2p\cdot k = \frac{Q^2}{z}\left(\frac{1+\cos\theta_*}{2}\right)$$

let $$w = \frac{1-\cos\theta_*}{2} \qquad 1-w = \frac{1+\cos\theta_*}{2} \qquad dw = \frac{d\cos\theta_*}{2}$$

then,

$$\mathcal{M} = 8(1-\epsilon)\Big\{ (1-\epsilon)\left(\frac{1-w}{1-z} + \frac{1-z}{1-w} + 2\,\frac{zw}{(1-z)(1-w)}\right)$$

$$+ 2\epsilon\left(\frac{zw + (1-z)(1-w)}{(1-z)(1-w)}\right)\Big\}$$

then

$$\sigma(\gamma^* q \to g q) = \frac{1}{2}\,\frac{z}{Q^2}\,\frac{1}{2}\,e^2 Q_f^2\, g_s^2 \cdot \frac{4}{3}(\mu^2)^{\epsilon}\,\frac{1}{16\pi}\cdot 8(1-\epsilon)$$

$$\cdot\left(\frac{s}{4\pi}\right)^{-\epsilon}\frac{1}{\Gamma(1-\epsilon)}\cdot 2\int_0^1 dw\,[w(1-w)]^{-\epsilon}$$

$$\left[(1-\epsilon)\left(\frac{1-w}{1-z} + \frac{1-z}{1-w} + 2\,\frac{zw}{(1-z)(1-w)}\right)(1+\epsilon) + 2\epsilon\right]$$

$$= \frac{8\pi^2\alpha Q_f^2}{Q^2}\cdot z\cdot\frac{4}{3}\,\frac{\alpha_s}{2\pi}\,(1-\epsilon)\,\frac{1}{\Gamma(1-\epsilon)}\left[\frac{Q^2(1-z)}{4\pi\mu^2 z}\right]^{-\epsilon}$$

$$\int_0^1 dw\,[w(1-w)]^{-\epsilon}\Big\{(1-\epsilon)\left[\frac{1+z^2}{(1-w)(1-z)} - 2\,\frac{z}{1-z} + \frac{(1-w)}{1-z}\right]$$

$$+ 2\epsilon\,\frac{zw}{(1-z)(1-z)} + 2\epsilon\Big\}$$

This cross section is infrared divergent, but the divergence is regularized by the factor $[w(1-w)]^{-\epsilon}$ for $\epsilon<0$. For example,

$$\int_0^1 dw\,[w(1-w)]^{-\epsilon}\,\frac{1}{1-w} = \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} = \frac{[\Gamma(1-\epsilon)]^2}{\Gamma(2-\epsilon)}\left(-\frac{1}{\epsilon}\right)$$

Doing the w integrals using the Euler beta function, we find

$$\sigma(\gamma^* q \to g q) = \frac{8\pi^2 \alpha\, Q_f^2}{Q^2} \cdot z \cdot \frac{4}{3}\frac{\alpha_s}{2\pi} \frac{(1-\epsilon)\,\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{Q^2(1-z)}{4\pi\mu^2 z}\right]^{-\epsilon}$$

$$\cdot \left\{ -\frac{1}{\epsilon}\frac{1+z^2}{1-z} + \frac{3/2 - 4z + z^2}{(1-z)} + \epsilon\, \frac{5/2 - 6z}{1-z} + \cdots \right\}$$

Let's assemble the pieces we have computed so far. To obtain the $\gamma^* p$ cross section, we need to combine this with $f_q(\xi)$ and integrate over ξ. Since

$$x = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{2p\cdot q}\cdot \xi = z\xi$$

we see ~~that~~

$$\int d\xi\; f_q(\xi) = \int \frac{dz}{z}\,\xi\, f_q\!\left(\frac{x}{z}\right) = \int \frac{dz}{z}\, f_q\!\left(\frac{x}{z}\right)\cdot \frac{x}{z}$$

so

$$\sigma(\gamma^* p) = 8\pi^2\alpha\, Q_f^2\, \frac{x}{Q^2}\,(1-\epsilon)$$

$$\cdot \Bigg\{ f_q(x)\left[1 - \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2}{4\pi\mu}\right)^{-\epsilon} \frac{\Gamma(1+\epsilon)[\Gamma(1-\epsilon)]^2}{\Gamma(1-2\epsilon)}\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 + \cdots\right)\right]$$

$$+ \int_0^1 \frac{dz}{z}\, f_q\!\left(\frac{x}{z}\right) \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2(1-z)}{4\pi\mu z}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\left[-\frac{1}{\epsilon}\frac{1+z^2}{1-z} + \frac{3/2 - 4z + z^2}{1-z} + \epsilon\,\frac{5/2 - 6z}{1-z} + \cdots\right]$$

$$+ \left[\text{another term involving } f_g\!\left(\frac{x}{z}\right)\right] \Bigg\}$$

There are $\frac{1}{\epsilon}$ IR divergences all over this formula, and it is not so obvious how to combine them. Notice that the second term also has $O(\frac{1}{\epsilon^2})$ divergences after we integrate ~~over~~ z up to $z=1$.

In our study of the Altarelli-Parisi equations, we regularized kernels of the form $\frac{1}{1-z}$ by turning them into + distributions. Let's now define a + distribution more generally as

$$[f(z)]_+ = f(z) - A\,\delta(z-1) \quad \text{s.t.} \quad \int_0^1 dz\, [f(z)]_+ = 0$$

Then, e.g. $\quad P_{q\leftarrow q}(z) = \frac{4}{3}\left[\frac{1+z^2}{1-z}\right]_+ .$

We can then replace the term in brackets on the previous page by a + distribution. With the dimensional regularization, the coefficient A of the delta function is

$$A = \int_0^1 dz \left[\frac{(1-z)}{z}\right]^{-\epsilon} \left[-\frac{1}{\epsilon}\frac{1+z^2}{1-z} + \frac{3/2 - 4z + z^2}{1-z} + \epsilon\,\frac{5/2 - 6z}{1-z} + \cdots\right]$$

$$= \frac{\Gamma(1-\epsilon)\,\Gamma(1+\epsilon)}{\Gamma(1)}\left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{13}{2} + \cdots\right]$$

The term $A\,\delta(z-1)$ can be subtracted in the second term and added to the first term on p. 9; this gives

$$\sigma(\gamma^* p) = 8\pi^2 \alpha\, Q_f^2\, \frac{x}{Q^2}(1-\epsilon)$$

$$\cdot \Big\{ f_q(x)\Big[1 + \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1+\epsilon)\,(\Gamma(1-\epsilon))^2}{\Gamma(1-2\epsilon)}$$

$$\cdot \Big\{\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \cdots\right) + \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{13}{2} + \cdots\right)\Big\}$$

$$+ \int_0^1 \frac{dz}{z}\, f_q\!\left(\frac{x}{z}\right) \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\cdot \left[\left(\frac{1-z}{z}\right)^{-\epsilon}\left(-\frac{1}{\epsilon}\frac{1+z^2}{1-z} + \frac{3/2 - 4z + z^2}{1-z} + \cdots\right)\right]_+$$

$$+ \text{ term involving } f_g\!\left(\frac{x}{z}\right) \Big\}$$

Now the first term gives a finite rescaling of the zeroth order term, by a factor

$$1 + \frac{4}{3}\frac{\alpha_s}{2\pi}\left(-8 + \frac{13}{2}\right) = 1 - \frac{\alpha_s}{\pi}$$

The second term is not finite. The divergent term is of the form

$$-\frac{1}{\epsilon}\int \frac{dz}{z}\, f_q\!\left(\frac{x}{z}\right)\ \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{1+z^2}{1-z}\right)_+$$

This is the change in $f_q(x)$ induced by Altarelli-Parisi evolution, since, as we recognize

$$P_{q\leftarrow q}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)_+$$

We can absorb this divergence into a redefinition of the parton distribution functions. This is like a renormalization scheme for parton distributions. In the Minimal Subtraction scheme of renormalization, we absorb just the $\frac{1}{\epsilon}$ divergence. In the $\overline{MS}$ scheme, we absorb also some conventional factors $(-\gamma + \log 4\pi)$ from the expansion of the Γ functions. Thus, define the $\overline{MS}$-subtracted parton distribution functions (to order α_s) by

$$f_i^{(\overline{MS})}(x,\mu_F) = f_i(x) - \frac{\alpha_s}{2\pi}\left(\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\frac{\mu_F^2}{\mu^2}\right)$$

where $i,j = u, d, s, \ldots,$
$\bar{u}\, \bar{d} \cdots,\ g$

$$\cdot \int \frac{dz}{z} \sum_j P_{i\leftarrow j}(z)\, f_j\!\left(\frac{x}{z}\right)$$

Having absorbed this divergence, we can take $\epsilon \to 0$ in the formula on p.10 and find (picking up all terms $\mathcal{O}(\frac{1}{\epsilon}\cdot\epsilon)$)

$$\sigma(\gamma^* p) = 8\pi^2 \alpha Q_f^2 \frac{x}{Q^2}$$

$$\cdot \int \frac{dz}{z} \Big\{ \Big[\delta(z-1)\left(1 - \frac{\alpha_s}{\pi}\right) + \frac{\alpha_s}{2\pi}\left(\log \frac{Q^2}{\mu_F^2}\right) P_{q\leftarrow q}(z)$$

$$+ \frac{4}{3}\frac{\alpha_s}{2\pi}\Big[\frac{1+z^2}{1-z} \log\frac{(1-z)}{z} + \frac{3/2 - 4z + z^2}{1-z}\Big]_+ \Big] f_q^{(\overline{MS})}\left(\frac{x}{z}, \mu_F\right)$$

$$+ \text{ term involving } f_g^{(\overline{MS})}\left(\frac{x}{z}, \mu_F\right) \Big\}$$

The second term in the bracket is the result of integrating the Altarelli - Parisi equations from μ_F to Q. μ_F is called the "factorization scale". It is effectively the cutoff such that $p_T < \mu_F$ is assigned to the pdf's and $p_T > \mu_F$ is assigned to the explicit perturbation theory calculation. To resum terms of order $\alpha_s \log Q^2$, we should take $\mu_F \sim Q$.

To obtain $O(\alpha_s)$ accuracy, we should also include the solution of the Altarelli Parisi equations computed with the 2-loop QCD β-function and the $O(\alpha_s)$ corrections to the splitting functions. In the tabulated "NLO" parton distribution functions, these effects are included. Then we can write

$$\sigma(\gamma^* p) = 8\pi^2 \alpha Q_f^2 \frac{x}{Q^2} \cdot \int \frac{dz}{z} \Big\{ \delta(z-1)\left(1 - \frac{\alpha_s}{\pi}\right)$$

$$+ \frac{4}{3}\frac{\alpha_s}{2\pi}\Big[\frac{1+z^2}{1-z} \log\frac{(1-z)}{z} + \frac{3/2 - 4z + z^2}{1-z}\Big]_+ \Big\} f_q^{\overline{MS}}\left(\frac{x}{z}, Q\right)$$

$$+ \left(f_g^{(\overline{MS})} \text{ term}\right)$$

We can test these ideas by computing the f_g term. This must contribute two pieces that need to be included to reach the full result we have written at the bottom of p. 12:

$$8\pi^2\alpha Q_f^2\,\frac{x}{Q^2}\cdot\int\frac{dz}{z}\cdot\Big[\,\frac{\alpha}{2\pi}\log\frac{Q^2}{\mu_F^2}\,P_{q\leftarrow g}(z)$$

$$-\Big(\frac{1}{\epsilon}-\gamma+\log 4\pi-\log\frac{\mu_F^2}{\mu^2}\Big)P_{q\leftarrow g}(z)\Big]\;f_g\!\left(\frac{x}{z}\right)$$

Let's see if they are there:

For $\gamma^* g\to q\bar q$, the cross section formula on p. 8 reads.

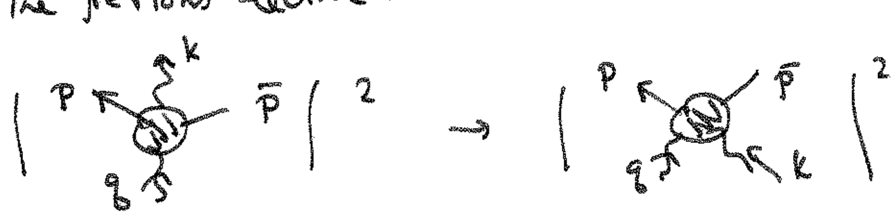
$$\sigma(\gamma^* g\to q\bar q) = \frac{1}{2}\,\frac{z}{Q^2}\cdot\frac{1}{2}\,e^2 Q_f^2\;g_s^2(\mu^2)^{\epsilon}\cdot\frac{1}{2}\cdot\frac{1}{16\pi}$$

$$w=\frac{1-\cos\theta_*}{2}\qquad\cdot\left(\frac{s}{4\pi}\right)^{-\epsilon}\frac{1}{\Gamma(1-\epsilon)}\cdot 2\int_0^1 dw\;[w(1-w)]^{-\epsilon}\cdot[\![M]\!]$$

where $[\![M]\!]$ is the squared matrix element summed over spins and stripped of coupling constants. The only change from p. 8 is that the color factor is now

$$\frac{1}{8}\,\mathrm{tr}\,t^a t^a = \frac{1}{2}\qquad\text{instead of}\qquad \frac{1}{3}\,\mathrm{tr}\,t^a t^a = \frac{4}{3}$$

$[\![M]\!]$ is given by a different cross of the squared matrix element from the previous lecture:



that is $\quad k\to -k$

thus,

$$\mathbb{M} = 8 \Big\{ (1-\epsilon)^2 \left(\frac{2\bar{p}\cdot k}{2p\cdot k} + \frac{2p\cdot k}{2\bar{p}\cdot k} + \frac{2q^2\, 2p\cdot\bar{p}}{2p\cdot k\; 2\bar{p}\cdot k} \right)$$

$$+ 2\epsilon(1-\epsilon) \left[\frac{q^2\; 2p\cdot\bar{p}}{2p\cdot k\; 2\bar{p}\cdot k} - 1 \right] \Big\}$$

We can borrow the kinematics from p. 7

$$q^2 = -Q^2 \qquad s = Q^2\,\frac{(1-z)}{z}$$

$$2k\cdot q = \frac{Q^2}{z} \qquad 2p\cdot\bar{p} = \frac{Q^2}{z}(1-z)$$

$$2p\cdot k = \frac{Q^2}{z}\,\omega \qquad 2\bar{p}\cdot k = \frac{Q^2}{z}(1-\omega)$$

$$\mathbb{M} = 8(1-\epsilon) \Big\{ (1-\epsilon) \left(\frac{1-\omega}{\omega} + \frac{\omega}{1-\omega} - 2\,\frac{z(1-z)}{\omega(1-\omega)} \right)$$

$$-2\epsilon \left(\frac{z(1-z)}{\omega(1-\omega)} + 1 \right) \Big\}$$

Now we can do the ~~integrals~~ over ω. For example

$$\int_0^1 d\omega\; [\omega(1-\omega)]^{-\epsilon} \left(\frac{\omega}{1-\omega}\right) = \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)} = \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{1}{\epsilon}\,\frac{(1-\epsilon)}{(1-2\epsilon)} \right]$$

$$\int_0^1 d\omega\; [\omega(1-\omega)]^{-\epsilon}\, \frac{1}{\omega(1-\omega)} = \frac{\Gamma(-\epsilon)\,\Gamma(-\epsilon)}{\Gamma(-2\epsilon)} = \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(-\frac{2}{\epsilon} \right)$$

so

$$\int_0^1 d\omega\; [\omega(1-\omega)]^{-\epsilon}\, \mathbb{M} = \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\; 8(1-\epsilon)^2 \left(-\frac{2}{\epsilon}\right)(1 - 2z + 2z^2) + \mathcal{O}(1)$$

$$= \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\; 8(1-\epsilon)^2 \left(-\frac{2}{\epsilon}\right) \left(z^2 + (1-z)^2 \right) + \mathcal{O}(1)$$

Now everything begins to fit together

$$\sigma(\gamma^* g) = 8\pi^2 \alpha Q_f^2 \frac{z}{Q^2}(1-\epsilon)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \cdot \left[\frac{Q^2(1-z)}{4\pi\mu^2 z}\right]^{-\epsilon}$$

$$\cdot \frac{1}{2} \cdot 2 \cdot \left[-\frac{1}{\epsilon}\left[z^2+(1-z)^2\right] + \mathcal{O}(\epsilon)\right]$$

The contribution to $\sigma(\gamma^* p)$ is

$$\sigma(\gamma^* p) = \cdots + \quad 8\pi^2 \alpha Q_f^2 \frac{x}{Q^2}(1-\epsilon)$$

$$\cdot \int_0^1 \frac{dz}{z} \; f_g\left(\frac{x}{z}\right) \; \frac{1}{2}\; \frac{\alpha_s}{2\pi}\left[\frac{Q^2(1-z)}{4\pi\mu^2 z}\right]^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\cdot\; 2 \cdot \left[-\frac{1}{\epsilon}\left[z^2+(1-z)^2\right] + \cdots\right]$$

There are two important features of this equation. First, even when integrated over z, it has no $\frac{1}{\epsilon^2}$ divergences. This is fortunate, because we already cancelled the $\mathcal{O}(\frac{1}{\epsilon^2})$ divergences. Second, the $\mathcal{O}(\frac{1}{\epsilon})$ divergence is of the form

$$8\pi^2 \alpha Q_f^2 \frac{x}{Q^2}(1-\epsilon) \;\cdot\; 2 \cdot$$

$$\cdot \int \frac{dz}{z}\left[\;\frac{\alpha_s}{2\pi}\; P_{q\leftarrow g}(z)\left[-\frac{1}{\epsilon}+\gamma-\log 4\pi+\log\frac{Q^2}{\mu^2}\right]\right] f_{\tilde g}\left(\frac{x}{z}\right)$$

since $P_{q\leftarrow g}(z) = \frac{1}{2}\left[z^2+(1-z)^2\right]$. The extra factor of 2 appears because g also generates the anti-quark distribution, which should also be summed over in our formulae.

Here is the final formula for the $\gamma^* p$ cross section:

$$\sigma(\gamma^* p) = \sum_{\substack{f = \text{quark} + \\ \text{antiquark flavors}}} \frac{8\pi^2 \alpha Q_f^2}{Q^2} x \cdot \int_0^1 \frac{dz}{z} \Bigg[\Big\{ \delta(z-1) \left(1 - \frac{\alpha_s(Q)}{\pi}\right)$$

$$+ \frac{4}{3}\frac{\alpha_s}{2\pi} \cdot \left[\frac{1+z^2}{1-z} \log\left(\frac{1-z}{z}\right) + \frac{3/2 - 4z + z^2}{1-z} \right]_+ \Big\} f_f^{(\overline{MS})}\left(\frac{x}{z}, Q\right)$$

$$+ \frac{\alpha_s}{2\pi} \left[z^2 + (1-z)^2 \right] \log\left(\frac{1-z}{z}\right) f_g^{\overline{MS}}\left(\frac{x}{z}, Q\right) \Bigg]$$

To determine the NLO $\overline{MS}$ parton distribution functions, we would take the corresponding formulae for

$$\frac{d\sigma}{dx\,dy}(\bar{e}p \to \bar{e}X)$$

and other observable deep inelastic cross sections and fit them to the data to determine

$$f_j^{(\overline{MS})}(x, \mu_F) \qquad j = f, g$$

These parton distributions depend on the regularization scheme. However, if we apply the same scheme in the computation of other cross sections, we obtain well-defined and observable $O(\alpha_s)$ corrections.