QCD corrections to $e^+e^- \to$ hadrons

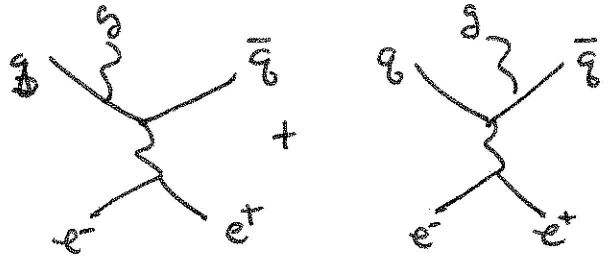
In the past two lectures, we studied the leading QCD corrections to quark-gluon processes. These terms came from quark and gluon emissions collinear to the quarks and gluons in the original process. These corrections were large, of the order of

$$\left[\frac{\alpha_s}{2\pi} \log \frac{Q^2}{(GeV)^2} \right]^n$$

and we discussed how to sum them to all orders.

Now I would like to go back and compute the order-α_s corrections more accurately. I would like to find the complete order α_s corrections to our basic processes of $e^+e^- \to q\bar{q}$, deep inelastic scattering, and Drell-Yan production. We will also need to see how these corrections fit together with the large corrections that we sum to all orders.

Let's begin with $e^+e^- \to$ hadrons. In the previous lectures, we analyzed the leading term coming from the diagrams



Now I would like to compute these diagrams completely, and combine them with the results from the virtual gluon diagrams.

In the process of this calculation, we will meet ultraviolet divergences, from the virtual diagrams, and infrared divergences, from both sets of diagrams. The Ward identity implies that the ultraviolet divergences cancel to order α_s. (At higher orders, we must include counterterms for the QCD vertices.) In our earlier analysis, we saw that the gluon emission diagrams are doubly infrared divergent

$$\sigma \sim \frac{\alpha_s}{2\pi} \int_0 \frac{dz}{z} \int_0 \frac{dp_T}{p_T}$$

However, a theorem of Kinoshita, Lee, and Nauenberg shows that, for processes with no quarks or gluons in the initial state, the total cross section is infrared finite when all real and virtual effects are summed. This is not true when there are colored particles in the initial state. In this and the next lectures, we will compute some example processes explicitly to see how the infrared behavior works.

To carry out these calculations, we will need to introduce ultraviolet and infrared regulators. In QED, we can regulate in the infrared by introducing a small photon mass. In QCD, however, a gluon mass violates gauge invariance. Various schemes have been proposed to avoid this problem. The simplest method, though, is to use dimensional regularization to control both UV and IR divergences. I will continue QCD in dimensionality according to

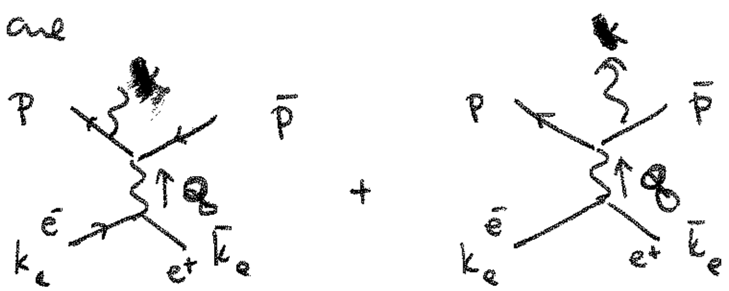
$$d = 4 - 2\epsilon$$

When d is slightly smaller than 4 ($\varepsilon > 0$), we control the UV divergences. If the UV divergences cancel, we can analytically continue to $d > 4$ ($\varepsilon < 0$). This gives a regularization of the IR divergences. This odd procedure turns out to be very effective, as we will see when we apply it to these problems.

[In Peskin + Schroeder, the final project to Part I, you will find a calculation of the total cross section for $e^+e^- \to$ hadrons to order α_s using what is effectively a gluon mass IR regulator. You can check that this approach gives the same answer that we will find here.]

To begin, compute the cross section for $e^+e^- \to q\bar{q}g$.

The diagrams are



To simplify, I would like to compute the energy distribution of the $q\bar{q}$ and g, integrated over the relative orientation of the $q\bar{q}g$ system and the initial e^+e^- beam axis. Let's first discuss the kinematics. Convenient variables are

$$x_1 = \frac{2p\cdot q}{q^2} \qquad x_2 = \frac{2\bar{p}\cdot q}{q^2} \qquad x_3 = \frac{2k\cdot q}{q^2}$$

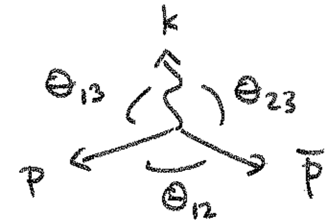
In the CM frame, $q = (Q,0,0,0)$ and the maximum energy for any parton is $Q/2$. So

$$x_1 = \frac{p^0}{Q/2} \qquad x_2 = \frac{\bar{p}^0}{Q/2} \qquad x_3 = \frac{k^0}{Q/2}$$

These variables obey

$$0 < x_i < 1 \qquad\qquad x_1 + x_2 + x_3 = 2$$

The three particles in the final state define a plane, the <u>event plane</u>



The x_i define the momenta in this plane completely. For example,

$$\begin{aligned} 2p\cdot\bar{p} &= 2p\bar{p}(1-\cos\theta_{12}) \\ &= (p+\bar{p})^2 = (q-k)^2 = q^2 - 2q\cdot k = Q^2(1-x_3) \end{aligned}$$

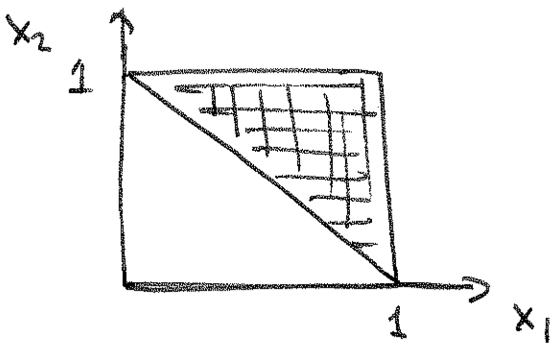
so

$$\frac{(p+\bar{p})^2}{Q^2} = (1-x_3) \qquad\qquad \cos\theta_{12} = 1 - 2\frac{(1-x_3)}{x_1 x_2}$$

The same logic applies if the final particles are massive, but the formulae are slightly more complicated.) The formula for 3-body phase space — integrated over all other angles — is

$$\int d\Pi_3 = \frac{Q^2}{128\pi^3}\int dx_1\, dx_2$$

I will derive this in a moment. For all massless final particles, $x_1\ x_2$ are integrated over the region



$$x_1, x_2 < 1$$

$$x_3 = 2 - x_1 - x_2 < 1$$

The limits $x_1\ x_2\ x_3 \to 1$ correspond to the configurations

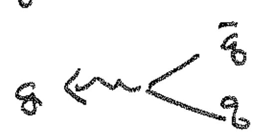
$x_1 \to 1$

collinear $\bar{q}g$

$x_2 \to 1$

collinear qg

$x_3 \to 1$



collinear $q\bar{q}$

The diagrams on p. 3 give

$$i\mathcal{M} = (-ie)(+ieQ_f)\,\bar{v}(\bar{k}_e)\gamma^\mu u(k_e)\,\frac{-i}{q^2}\cdot(ig_s t^a)$$

$$\bar{u}(p)\left[\gamma\cdot\epsilon^*(k)\,\frac{i(\not{p}+\not{k})}{(p+k)^2}\gamma_\mu + \gamma_\mu\,\frac{i(-\bar{\not{p}}-\not{k})}{(\bar{p}+k)^2}\gamma\cdot\epsilon^*(k)\right]v(\bar{p})$$

The square of the lepton factor $\bar{v}(\bar{k}_e)\gamma^\mu u(k_e)$ is

$$\frac{1}{4}\,\mathrm{tr}\,[\bar{k}_e \gamma^\mu k_e \gamma^\nu] = [\bar{k}_e^\mu k_e^\nu + \bar{k}_e^\nu k_e^\mu - g^{\mu\nu} k_e\cdot\bar{k}_e]$$

I would like to average this expression over orientations of the initial axis relative to the event plane. This is trivially done as follows: The tensor satisfies

$$q_\mu [\;\;] = [\;\;] q_\nu = 0$$

After average over the orientations, the final result can only depend on q. Thus it must be of the form

$$\langle [\bar{k}_e^\mu k_e^\nu + \bar{k}_e^\nu k_e^\mu - g^{\mu\nu} \bar{k}_e\cdot k_e] \rangle = A\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)$$

To evaluate A, take the trace:

$$2\bar{k}_e k_e - 4\bar{k}_e k_e = A\cdot 3 \quad \text{or} \quad A = -\frac{2}{3} k_e\cdot\bar{k}_e = -\frac{1}{3} q^2$$

so

$$\left\langle \frac{1}{4}\sum \bar{v}\gamma^\mu u\, \bar{u}\gamma^\nu v \right\rangle = -\frac{1}{3}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

By the Ward identity, the quark tensor also satisfies

$$q^\mu [\;\;] = 0$$

so we can drop the second term and keep only $\left(-\frac{1}{3} q^2 g^{\mu\nu}\right)$

We can also use the Ward identity to represent the sum over gluon polarizations:

$$\sum_\epsilon \epsilon^\alpha(k)\, \epsilon^{*\beta}(k) \to -g^{\alpha\beta}$$

then

$$\left\langle \frac{1}{4}\sum_{\text{spins}} |i\mathcal{M}(e^+e^- \to q\bar{q}g)|^2 \right\rangle$$

$$= \frac{e^4 Q_f^2}{(q^2)^2} \; g_s \; \underbrace{t^a t^a}_{4/3} \cdot \frac{q^2}{3} \cdot \mathbb{M}$$

$$= \frac{e^2 Q_f^2}{3} \; \frac{1}{q^2} \cdot \frac{4}{3} g_s \cdot \mathbb{M}$$

where

$$\mathbb{M} = \mathrm{tr}\Big\{ \not{p} \Big[\gamma^\alpha \frac{(\not{p}+\not{k})}{2p\cdot k} \gamma^\mu - \gamma^\mu \frac{(\not{\bar{p}}+\not{k})}{2\bar{p}\cdot k} \gamma_\alpha \Big]$$

$$\cdot \; \not{\bar{p}} \Big[\gamma_\mu \frac{(\not{p}+\not{k})}{2p\cdot k} \gamma_\alpha - \gamma_\alpha \frac{(\not{\bar{p}}+\not{k})}{2\bar{p}\cdot k} \gamma_\mu \Big] \Big\}$$

this takes the form

$$\mathbb{M} = \frac{\mathrm{I}}{(2p\cdot k)^2} + \frac{\mathrm{II} + \mathrm{III}}{2p\cdot k \; 2\bar{p}\cdot k} + \frac{\mathrm{IV}}{(2\bar{p}\cdot k)^2}$$

where I – IV are traces

$$\mathrm{I} = \mathrm{tr}\left[\not{p}\gamma^\alpha(\not{p}+\not{k})\gamma^\mu \not{\bar{p}} \gamma_\mu (\not{p}+\not{k})\gamma_\alpha\right] = 8\cdot 2p\cdot k \; 2\bar{p}\cdot k = \mathrm{IV}$$

$$\mathrm{II} = -\mathrm{tr}\left[\not{p}\gamma^\alpha(\not{p}+\not{k})\gamma^\mu \not{\bar{p}} \gamma_\alpha (\not{\bar{p}}+\not{k})\gamma_\mu\right] = 8\, q^2 \, 2p\cdot\bar{p} = \mathrm{III}$$

Now we can compute the cross section

$$2p\cdot\bar{p} = q^2(1-x_3) \qquad 2p\cdot k = q^2(1-x_2) \qquad 2\bar{p}\cdot k = q^2(1-x_1)$$

then,

$$\mathbb{M} = 8 \cdot \left[\frac{(1-x_1)(1-x_2)}{(1-x_2)^2} + 2\frac{(1-x_3)}{(1-x_1)(1-x_2)} + \frac{(1-x_1)(1-x_2)}{(1-x_1)^2} \right.$$

$$= 8 \frac{1}{(1-x_1)(1-x_2)} \left[(1-x_1)^2 + 2(1-x_3) + (1-x_2)^2 \right]$$

$$= 8 \frac{1}{(1-x_1)(1-x_2)} \left[1 + 1 + 2 - 2x_1 - 2x_3 - 2x_2 + x_1^2 + x_2^2 \right]$$

$$= 8 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

then

$$\sigma(e^+e^- \to q\bar{q}g) = \frac{1}{2s} \frac{Q^2}{128\pi^3} \int dx_1 dx_2 \; e^4 g_s^2 Q_f^2 \, \frac{4}{3} \cdot \frac{8}{3\cancel{Q^2}} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

$$\frac{d\sigma}{dx_1 dx_2}(e^+e^- \to q\bar{q}g) = \underbrace{\frac{4}{3}\frac{\pi\alpha^2}{s}Q_f^2}_{\text{0th-order cross sect. for } e^+e^- \to q\bar{q}g} \cdot \frac{4}{3}\frac{\alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

again,

$$\frac{d\sigma}{dx_1 dx_2}(e^+e^- \to q\bar{q}g) = \sigma_0 \cdot \frac{4}{3}\frac{\alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

The double IR divergence is clear in this expression. The $\bar{q}g$ collinear region is $x_2 \to 0$

$$(1-x_2) = \frac{2p \cdot q}{Q^2} \sim \frac{p_T^2}{Q^2} \quad \text{so} \quad \frac{dx_2}{(1-x_2)} \approx \frac{dp_T^2}{p_T^2}$$

In this kinematics $x_2 \approx 1$ so $x_1 = (1-z)$ $x_3 = z$

$$\sigma(e^+e^- \to q\bar{q}g) \cong \sigma_0 \cdot \int dz \frac{dp_T^2}{p_T^2} \frac{4}{3}\frac{\alpha_s}{2\pi} \frac{1 + (1-z)^2}{z}$$

which is just the splitting behavior we found earlier. The limit $x_1 \to 1$ gives the same behavior in the $\bar{q}g$ collinear region. The limit $x_1 \to 1$, $x_2 \to 1$ corresponds to $x_3 \to 0$, the soft gluon region. This region gives a double infrared log.

I would now like to regularize these logs using dimensional regularization, setting $d = 4 - 2\epsilon$ $(\epsilon < 0)$. To implement this, we need to redo both the phase space and the matrix element calculations in d dimensions. [This analysis is described nicely in Field's book.]

First, analyze the phase space. 2-body phase space in d dimension is

$$\int d\Pi_2 = \int \frac{d^{d-1}p \; d^{d-1}k}{(2\pi)^{d-1}2p(2\pi)^{d-1}2k} (2\pi)^d \delta(p+k-Q)$$

$$= \frac{1}{(2\pi)^{d-2}4} \cdot \int \frac{dp \; p^{d-2}}{pk} \delta(p+k-Q)\Big|_{|\vec{k}|=p} \int d\Omega_{d-1}$$

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad \text{so} \quad \int d\Omega_{d-1} = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \quad ; \quad p = k = \frac{Q}{2}$$

$$= \frac{1}{(2\pi)^{d-2}} \left(\frac{Q}{2}\right)^{d-4} \frac{1}{4 \cdot 2} \cdot \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}$$

$$= \frac{1}{4} \cdot \frac{1}{2^{2d-6}} \frac{1}{\pi^{(d-3)/2}} (Q^2)^{(d-4)/2} \frac{1}{\Gamma(\frac{d-1}{2})}$$

$$= \frac{1}{16} \frac{1}{\sqrt{\pi}} \frac{(Q^2)^{-\epsilon}}{4^{-\epsilon}(4\pi)^{-\epsilon}} \frac{1}{\Gamma(3/2-\epsilon)}$$

using $\Gamma(z)\Gamma(z+1/2) \cdot \frac{2^{2z-1/2}}{(2\pi)^{1/2}} = \Gamma(2z)$, $\frac{1}{\Gamma(3/2-\epsilon)} = \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \cdot \frac{4^{(1-\epsilon)}}{(4\pi)^{1/2}}$

so finally

$$\int d\Pi_2 = \frac{1}{8\pi}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\left(\frac{Q^2}{4\pi}\right)^{-\epsilon} \qquad \left(= \frac{1}{8\pi} \text{ for } \epsilon=0\right)$$

We can make a similar reduction of 3-body phase space

$$\int d\Pi_3 = \int \frac{d^{d-1}p \; d^{d-1}\bar{p} \; d^{d-1}k}{(2\pi)^{3(d-1)}\, 2p\, 2\bar{p}\, 2k} (2\pi)^d \delta(p+\bar{p}+k-q)$$

Work in the CM frame. Integrate out k. ~~Integrate~~ over all other angles except for the angle between p and $\bar{p}$, which we have seen is determined by $x_1\, x_2\, x_3$. The integral of $\hat{p}$ over all angles gives $\int d\Omega_{d-1}$. The integral of $\hat{\bar{p}}$ over all angles except $\cos\theta_{12}$ gives:

$$\int d\Omega_{d-2} \int (\sin\theta_{12})^{d-4}\, d\cos\theta_{12}$$

then

$$\int d\Pi_3 = \frac{1}{(2\pi)^{2d-3}}\cdot\frac{1}{8}\int d\Omega_{d-1}\int d\Omega_{d-2}$$

$$\cdot \int \frac{dp\, p^{d-2}\; d\bar{p}\; \bar{p}^{d-2}}{p\,\bar{p}\,k}\int (\sin\theta_{12})^{d-4}\, d\cos\theta_{12}\; \delta(p+\bar{p}+k-Q)$$

with $\vec{k} = -\vec{p}-\vec{\bar{p}}$, $|k| = [p^2+\bar{p}^2+2p\bar{p}\cos\theta_{12}]^{1/2}$. The integral over the δ-function then gives

$$\int d\cos\theta_{12}\; \delta(p+\bar{p}+k-Q) = \frac{1}{\left(\frac{p\bar{p}}{k}\right)}$$

From p.4 $\quad \cos\theta_{12} \equiv z = \left(1 - 2\frac{(1-x_3)}{x_1 x_2}\right)$

Assemble the pieces:

$$\int d\Pi_3 = \frac{1}{(2\pi)^{2d-3}} \frac{1}{8} \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \frac{2\pi^{(d-2)/2}}{\Gamma(\frac{d-2}{2})}$$

$$\cdot \int \frac{dp\, p^{d-2}}{p} \frac{d\bar{p}\, \bar{p}^{d-2}}{\bar{p}\; k} \frac{k}{p\bar{p}} (1-z^2)^{(d-4)/2}$$

$$= \frac{1}{2^{5-4\epsilon}} \frac{1}{2} \left(\frac{Q}{2}\right)^{2(1-2\epsilon)} \frac{1}{\pi^{d-3/2}} \frac{1}{\Gamma(1-\epsilon)\Gamma(3/2-\epsilon)}$$

$$\cdot \int dx_1\, x_1^{-2\epsilon} \int dx_2\, x_2^{-2\epsilon} (1-z^2)^{-\epsilon}$$

We saw above that: $\dfrac{1}{\Gamma(1-\epsilon)\Gamma(3/2-\epsilon)} = \dfrac{2\; 4^{-\epsilon}}{\sqrt{\pi}}$

so, finally,

$$\int d\Pi_3 = \frac{Q^2}{128\pi^3} \left(\frac{Q^2}{4\pi}\right)^{-2\epsilon} \frac{1}{\Gamma(2-2\epsilon)} \int dx_1 dx_2\; x_1^{-2\epsilon} x_2^{-2\epsilon} \left(\frac{1-z^2}{4}\right)^{-\epsilon}$$

Since $\epsilon < 0$, the last three factors act as convergence factors for the various sorts of infrared divergences.

To work consistently in d dimensions, we also must redo the Dirac algebra. Conventionally, in dimensional regularization, we keep $\mathrm{tr}[1] = 4$. Then

$$\mathrm{tr}[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu} \qquad \mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4[g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma}]$$

etc. But $\gamma^\mu \gamma_\mu = d = 4 - 2\epsilon$

Then

$$\gamma^\mu \not{a}\, \gamma_\mu = -2\not{a} + 2\epsilon \not{a}$$

$$\gamma^\mu \not{a}\not{b}\, \gamma_\mu = 4 a\cdot b - 2\epsilon \not{a}\not{b}$$

$$\gamma^\mu \not{a}\not{b}\not{c}\, \gamma_\mu = -2\not{c}\not{b}\not{a} + 2\epsilon \not{a}\not{b}\not{c}$$

In the calculation on p. 6, the last stages change to

$$\langle [\, \bar{k}_e^\mu k_e^\nu + \bar{k}_e^\nu k_e^\mu - g^{\mu\nu} \bar{k}_e k_e \,] \rangle = A \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right)$$

$$-(d-2)\, \bar{k}_e \cdot k_e = A \cdot (d-1)$$

$$-(1-\epsilon)\, q^2 = (3-2\epsilon)\, A$$

so $$A = -\frac{q^2(1-\epsilon)}{(3-2\epsilon)}$$

There are more changes in the calculation of $\mathbb{M}$

$$(\mathrm{I}) = (\mathrm{IV}) = 8(1-\epsilon)^2 \; 2p\cdot k \; 2\bar{p}\cdot k$$

$$(\mathrm{II}) = (\mathrm{III}) = 8(1-\epsilon)^2 \; 2p\bar{p}\; q^2 + 8\epsilon(1-\epsilon)\left(2p\cdot\bar{p}\, q^2 - 2p\cdot k\, 2\bar{p}\cdot k\right)$$

Then

$$\mathbb{M} = 8\left[(1-\epsilon)^2 \left(\frac{2\bar{p}\cdot k}{2p\cdot k} + \frac{2p\cdot k}{2\bar{p}\cdot k} + \frac{2q^2\; 2p\cdot\bar{p}}{2p\cdot k\; 2\bar{p}\cdot k} \right) + 2\epsilon(1-\epsilon)\frac{(q^2\, 2p\cdot\bar{p} - 2p\cdot k\, 2\bar{p}\cdot k)}{2p\cdot k\; 2\bar{p}\cdot k} \right]$$

evaluating in terms of $x_1\, x_2$

$$\mathbb{M} = 8\left[(1-\epsilon)^2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} - 2\epsilon(1-\epsilon)\frac{2 - 2x_1 - 2x_2 + x_1 x_2}{(1-x_1)(1-x_2)} \right]$$

There is one more modification that we need to make to go to d dimensions. In d dimensions, the QCD coupling constant g_s is dimensionful. We can keep it dimensionless

by replacing
$$g_s^2 \to g_s^2 (\mu^2)^{\epsilon}$$
This introduces the dimensional regularization scale μ.

Now we can construct the cross sections. Begin with the cross section for $e^+e^- \to q\bar{q}$. The quark tensor is

$$\mathfrak{m} = \mathrm{tr}(\not{p}\, \gamma^\mu \not{\bar{p}}\, \gamma^\nu)(-g_{\mu\nu}) = 2(1-\epsilon)\, \mathrm{tr}\, p\bar{p} = 4(1-\epsilon)\, 2p\cdot\bar{p}$$
$$= 4(1-\epsilon)\, q^2$$

A, p.12

$$\sigma(e^+e^- \to q\bar{q}) = \frac{1}{2s}\, \frac{1}{8\pi}\, \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{Q^2}{4\pi}\right)^{-\epsilon} \cdot \frac{e^4 Q_f^2}{(q^2)^2}\, (4(1-\epsilon)q^2) \cdot \frac{q^2(1-\epsilon)}{3(1-\frac{2}{3}\epsilon)}$$

$$= \frac{4}{3}\, \frac{\pi\alpha^2}{s}\, Q_f^2 \cdot \frac{\Gamma(1-\epsilon)\,(1-\epsilon)}{\Gamma(2-2\epsilon)}\, \frac{(1-\epsilon)}{(1-\frac{2}{3}\epsilon)} \left(\frac{Q^2}{4\pi}\right)^{-\epsilon}$$

or
$$\sigma(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3s} Q_f^2 \underbrace{\left(\frac{Q^2}{4\pi}\right)^{-\epsilon}} \frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{1-\epsilon}{1-\frac{2}{3}\epsilon}\right)$$

(this factor would be properly dimensionless if we had taken care to make e^2 dimensionless)

for $e^+e^- \to q\bar{q}g$:

$$\sigma(e^+e^- \to q\bar{q}g) = \frac{1}{2s}\, \frac{Q^2}{128\pi^3} \left(\frac{Q^2}{4\pi}\right)^{-2\epsilon} \frac{1}{\Gamma(2-2\epsilon)} \int dx_1\, dx_2\, x_1^{-2\epsilon} x_2^{-2\epsilon} \left(\frac{1-z^2}{4}\right)^{-\epsilon}$$

$$\cdot\; e^4 Q_f^2\, g_s^2\, (\mu^2)^{\epsilon} \cdot \frac{4}{3} \cdot \frac{1}{(q^2)^2}\, \frac{q^2(1-\epsilon)}{3(1-\frac{2}{3}\epsilon)} \cdot 8$$

$$\left[(1-\epsilon)^2 \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} - 2\epsilon(1-\epsilon) \frac{2-2x_1-2x_2+x_1x_2}{(1-x_1)(1-x_2)} \right]$$

Now we just have to do the integrals. There is a problem that the x_1 and x_2 integrals are coupled through z. This is nicely addressed with the change of variables

$$x_2 = (1 - x_1 v) \qquad dx_2 = -x_1\, dv$$

then
$$\int_0^1 dx_1 \int_{1-x_1}^1 dx_2 = \int_0^1 dx_1\, x_1 \int_0^1 dv$$

$$(1-x_2) = x_1 v$$

$$(1-x_3) = (x_1 + x_2 - 1) = x_1 (1-v)$$

$$z = 1 - 2\,\frac{(1-x_3)}{x_1 x_2} = 1 - 2\,\frac{(1-v)}{(1-v x_1)}$$

$$\frac{(1-z)}{2} = \frac{(1-v)}{(1-v x_1)} \qquad \frac{(1+z)}{2} = \frac{v(1-x_1)}{1-v x_1}$$

$$x_2^{-2\epsilon}\left(\frac{1-z^2}{4}\right)^{-\epsilon} = (1-x_1 v)^{-2\epsilon}\,\frac{[(1-v)\, v\, (1-x_1)]^{-\epsilon}}{(1-x_1 v)^{-2\epsilon}}$$

and the integral falls apart as

$$\int dx_1\, x_1^{-2\epsilon} \int dx_2\, x_2^{-2\epsilon} \left(\frac{1-z^2}{4}\right)^{-\epsilon} = \int_0^1 dx_1\, x_1^{1-2\epsilon}(1-x_1)^{-\epsilon} \int_0^1 dv\, [v(1-v)]^{-\epsilon}$$

Now we can integrate term by term using the Euler beta function

$$\int_0^1 dz\, z^{\alpha-1}(1-z)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

The most singular terms in the integrand have the form

$$\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \xrightarrow[x_1, x_2 \to 1]{} \frac{2}{(1-x_1)(1-x_2)} = \frac{2}{x_1 (1-x_1)\, v}$$

the integral of this expression is

$$\int_0^1 dx_1 \; x_1^{-2\epsilon}(1-x_1)^{-\epsilon-1} \int_0^1 dv \; v^{-\epsilon-1} v^{-\epsilon} \cdot 2$$

$$= 2\,\frac{\Gamma(1-2\epsilon)\,\Gamma(-\epsilon)}{\Gamma(1-3\epsilon)}\;\frac{\Gamma(1-\epsilon)\,\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} \sim \frac{2}{\epsilon^2}$$

This is the infrared double log, successfully regularized by dimensional regularization.

After carrying out all of the integrals, the final result becomes:

$$\sigma(e^+e^- \to q\bar{q}g) = \frac{4}{3}\frac{\pi\alpha^2}{s} Q_f^2 \left(\frac{Q^2}{4\pi}\right)^{-\epsilon} \frac{1}{\Gamma(2-2\epsilon)} \cdot \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \cdot \frac{(1-\epsilon)}{(1-\frac{2}{3}\epsilon)}$$

$$\cdot\,(1-\epsilon)\cdot\frac{[\Gamma(1-\epsilon)]^3}{\Gamma(1-3\epsilon)\,\epsilon^2}\left\{2+3\epsilon+\frac{19}{2}\epsilon^2+\cdots\right\}$$

The term of order ϵ^2 in the bracket is the finite term as $\epsilon\to 0$; we can drop all terms after that one. Then

$$\frac{\sigma(e^+e^-\to q\bar{q}g)}{\sigma_0(e^+e^-\to q\bar{q})} = \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-3\epsilon)}\left\{\frac{2+3\epsilon+\frac{19}{2}\epsilon^2+\cdots}{\epsilon^2}\right\}$$

We still need to compute the virtual gluon diagrams

These diagrams are both UV and IR divergent, so we need to use dimensional regularization to regulate both sets of divergences. In this case, though, the sum of the loop diagrams is UV finite, so there is less need for extreme care.

Let's now compute the loop diagrams. The self-energy diagram is

$$-i\Sigma_1 =$$ 

$$= (ig_s t^a)^2 (\mu^2)^\epsilon \int \frac{d^dk}{(2\pi)^d}\, \gamma_\mu \frac{i\not k}{k^2} \gamma^\mu \frac{-i}{(p-k)^2}$$

Introduce Feynman parameters:

$$= -g_s^2 \frac{4}{3} (\mu^2)^\epsilon [-2(1-\epsilon)] \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\not k}{[\mathbb{k}^2 - x(1-x)p^2]^2} \qquad k = \mathbb{k} + xp$$

$$= 2(1-\epsilon)\, g_s^2 \cdot \frac{4}{3} (\mu^2)^\epsilon \not p \frac{i}{(4\pi)^{d/2}} \frac{1}{[-p^2]^{2-d/2}} \int_0^1 dx \frac{x}{[x(1-x)]^{2-d/2}}$$

This behaves as $\Sigma_1 \sim \not p \cdot \left(\frac{p^2}{\mu^2}\right)^{-\epsilon}$

If we take $\epsilon < 0$, then $p \to 0$, the field strength renormalization $d\Sigma/d\not p$ is zero. It turns out to be consistent to set $\Sigma_1(p) = 0$ for an external on-shell massless quark.

Now look at the vertex diagram.

$$= (ig_s t^a)^2 (\mu^2)^\epsilon \int \frac{d^dk}{(2\pi)^d} \left(\frac{-i}{k^2}\right)$$

$$\cdot\ \bar u(p) \gamma_\lambda \frac{i(p+k)}{(p+k)^2} \gamma^\mu \frac{i(+k-\bar p)}{(\bar p - k)^2} \gamma^\lambda v(\bar p)$$

Introduce Feynman parameters

$$\mathbb{k} = k + xp - y\bar p \qquad \text{Denom} = [\mathbb{k}^2 - xy q^2]$$

$$k = \mathbb{k} - xp + y\bar p \qquad (k+p) = \mathbb{k} + (1-x)p + y\bar p \qquad k - \bar p = \mathbb{k} - xp - (1-y)\bar p$$

The numerator becomes:

$$\bar{u}(p)\gamma_\alpha (k+p)\gamma^\mu (k-\bar{p})\, v(\bar{p})$$

$$= (-2)\ \bar{u}(p)\Big\{ (k - xp - (1-y)\bar{p})\gamma^\mu (k + (1-x)p + y\bar{p})$$

$$- \epsilon\ (k + (1-x)p + y\bar{p})\gamma^\mu (k - xp - (1-y)\bar{p}) \Big\} v(\bar{p})$$

Use $\bar{u}(p) p = 0 = \bar{p}\, v(\bar{p})$ to simplify this. Then integrate over k:

$$= \left(-ig_s^2 \tfrac{4}{3}(\mu^2)^\epsilon\right)(-2)\int_0^1 dx \int_0^{1-x} dy\ \frac{-i}{(4\pi)^{d/2}}$$

$$\bar{u}(p)\Big\{ -\frac{1}{2}\frac{\Gamma(2-d/2)(1-\epsilon)}{(-q^2xy)^{2-d/2}}\gamma^\alpha\gamma^\mu\gamma_\alpha + \frac{\Gamma(3-d/2)}{[-q^2xy]^{3-d/2}}\left[-(1-x)(1-y) + \epsilon xy\right]\bar{p}\gamma^\mu p\Big\} v(\bar{p})$$

$$= +2ig_s^2\ \tfrac{4}{3}(\mu^2)^\epsilon\ \frac{-i}{(4\pi)^{2-\epsilon}}\ \Gamma(1+\epsilon)\int_0^1 dx\int_0^{1-x} dy$$

$$\bar{u}(p)\Big\{ (1-\epsilon)^2\gamma^\mu\cdot\frac{1}{\epsilon}\frac{1}{(-q^2xy)^\epsilon} + \frac{(-(1-x)(1-y)+\epsilon xy)}{(-q^2xy)^{1+\epsilon}}\ \bar{p}\gamma^\mu p\Big\} v(\bar{p})$$

the second term is simplified by

$$\bar{u}(p)(\bar{p}\gamma^\mu p)v(\bar{p}) = \bar{u}(p)(2\bar{p}^\mu p\!\!\!/ - 2p\cdot\bar{p}\,\gamma^\mu)v(\bar{p}) = (-q^2)\,\bar{u}\gamma^\mu v$$

so

$$= 2g_s^2\ \tfrac{4}{3}(\mu^2)^\epsilon\ \frac{1}{(4\pi)^{2-\epsilon}}\ \Gamma(1+\epsilon)\int_0^1 dx\int_0^{1-x} dy\ (-q^2)^{-\epsilon}\frac{1}{(xy)^{1+\epsilon}}$$

$$\bar{u}(p)\Big\{\gamma^\mu\left[\frac{(1-\epsilon)^2}{\epsilon}xy - (1-x)(1-y) + \epsilon xy\right]\Big\} v(\bar{p})$$

It is unclear whether we should interpret the $\frac{1}{\epsilon}$ as a UV or an IR divergence, but $\delta_2 = 0 \Rightarrow \delta_1 = 0$, so this must be an IR divergence. Let's proceed to analyze the integral for $\epsilon < 0$.

In that case, ϵ provides convergence factors for the x and y ~~integrals~~, we can again do the ~~integrals~~ using the Euler beta function, and we find

$$= \frac{4}{3}\frac{\alpha_s}{4\pi}\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\epsilon)\left[\Gamma(1-\epsilon)\right]^2}{\Gamma(1-2\epsilon)}\left\{-\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8\right)+\cdots\right\}$$

Now there is one more tricky point. The term $(-q^2)^{-\epsilon}$ must be analytically continued from negative q^2 to positive $q^2+i\varepsilon$

$$(-q^2)^{-\epsilon} = (q^2)^{-\epsilon}\left(e^{-i\pi}\right)^{-\epsilon}$$

$$= (q^2)^{-\epsilon}\left(1+i\pi\epsilon-\frac{\epsilon^2\pi^2}{2}+\cdots\right)$$

The imaginary part gives the cut of the diagram 

However, the order α_s correction to the rate is

$$\frac{\sigma(e^+e^-\to q\bar{q})}{\sigma_0(e^+e^-\to q\bar{q})} = \left|F_1(q^2)\right|^2$$

$$= 1+\frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\epsilon)\left[\Gamma(1-\epsilon)\right]^2}{\Gamma(1-2\epsilon)}\left\{-\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8\right)+\cdots\right\}$$

$$\cdot\left(1-\frac{\epsilon^2\pi^2}{2}\right)$$

Finally, we need to combine this result with the result for the correction from the real diagrams on p. 15. The Γ functions are not quite identical, so we need to expand

$$\Gamma(1+\epsilon) = e^{-\epsilon\gamma}\left(1+\frac{\epsilon^2\pi^2}{12}+\cdots\right)$$

then $$\frac{\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \cdot \Gamma(1-3\epsilon) = 1 + \frac{\epsilon^2\pi^2}{12}(1+9-4) + \cdots$$
$$= 1 + \frac{\epsilon^2\pi^2}{2} + \cdots$$

Now we have all of the pieces:

$$\frac{\sigma(e^+e^- \to q\bar{q}g)}{\sigma_0} = \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-3\epsilon)}\left\{\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \cdots\right\}$$

$$\frac{\sigma(e^+e^- \to q\bar{q})}{\sigma_0} = 1 - \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1-3\epsilon)}$$
$$\cdot \left\{\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8\right)\left(1 + \frac{\epsilon^2\pi^2}{12}\right)\left(1 - \frac{\epsilon^2\pi^2}{12}\right) + \cdots\right\}$$

All of the divergent terms cancel! The final result is

$$\frac{\sigma_{tot}(e^+e^- \to \text{hadrons})}{\sigma_0} = 1 + \frac{4}{3}\frac{\alpha_s}{2\pi}\cdot\left(\frac{19}{2} - 8\right)$$

or $$\sigma_{tot}(e^+e^- \to \text{hadrons}) = \frac{4}{3}\frac{\pi\alpha^2}{s}\left(1 + \frac{\alpha_s}{\pi} + \cdots\right)$$

This result — a +4% correction at $Q = m_Z$ — does agree well with the measured e^+e^- cross sections!