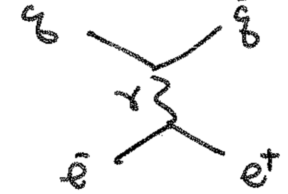
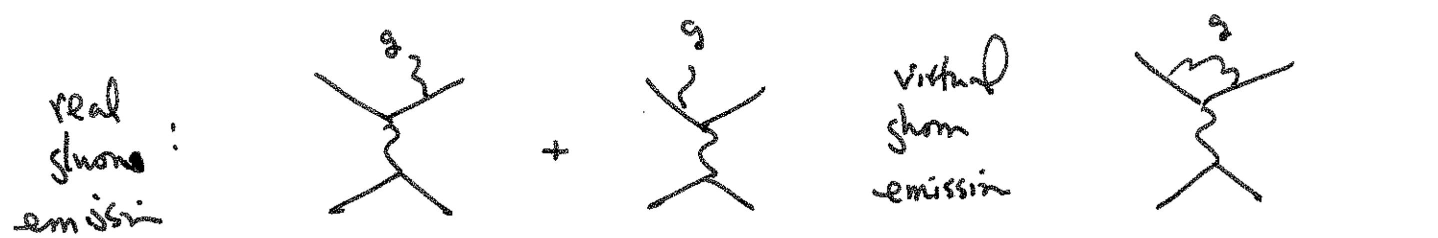
The Altarelli-Parisi equations

In the previous two lectures, we have begun our analysis of hard-scattering processes in QCD. So far, we have only discussed the leading order in α_s. In this and the next lecture, I will discuss the leading terms of the QCD corrections, which will essentially modify the physical picture we have obtained up to this point.

To begin, go back to the process $e^+e^- \to q\bar{q}$ and consider the first QCD corrections. We have already computed

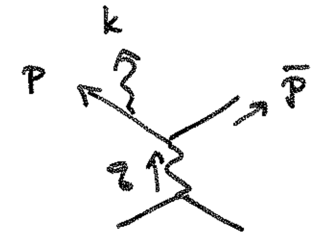


In the next order of perturbation theory, we have the following corrections:



Each individual diagram, it will turn out, is divergent. Later in the course, I will do the complete calculation and show that there is a finite correction to the total cross section when all of the effects are added. However, because the first two diagrams introduce real gluons, these diagrams will affect the form of the final states, and the infinities of these diagrams potentially signal a large correction. I would like to analyze the effect and see how we can account for it.

Let's study, first,



$$i\mathcal{M} = (-ie)(+ieQ_f)(ig_s)t^a\, \bar{u}(p)\, \gamma\cdot\varepsilon^*(k)\; i\frac{(\not p+\not k)}{(p+k)^2}\, \gamma^\mu\, v(\bar p)$$

$$\cdot \frac{-i}{q^2}\; \bar v \gamma_\mu u$$

The denominator is

$$(p+k)^2 = 2p\cdot k$$

This vanishes when p or $k \to 0$ ("soft" limit) and, more generally, when the lightlike vectors p and k become <u>parallel</u> ("collinear" limit). The collinear divergence is particularly important. The outgoing quark can give up an arbitrarily large amount of its momentum to the gluon while keeping the value of this diagram very large.

Let's evaluate this diagram approximately in the collinear region. For convenience, choose axes with $\bar{p}$ directed along the $\hat{3}$ axis:

$$\bar{p}^\mu = (\bar{P}, 0, 0, -\bar{P})$$

If p and k are exactly collinear

$$p^\mu = (1-z)\cdot(P, 0, 0, P)^\mu \qquad k^\mu = z\cdot(P, 0, 0, P^\mu)$$

$$0 < z < 1 \qquad \text{and} \qquad P = \bar{P} = \sqrt{s}/2$$

To ~~integrate~~ over the phase space of k, we might take p and k <u>almost</u> collinear:

$$p^\mu \cong ((1-z)P,\ p_T,\ 0,\ (1-z)P)$$

$$k^\mu = (zP,\ -p_T,\ 0,\ zP)$$

these vectors should be exactly light like $p^2 = k^2 = 0$, so, more accurately

$$p^\mu = \left((1-z)P,\ p_T,\ 0,\ (1-z)P - \frac{p_T^2}{2(1-z)P}\right)^\mu$$

$$k^\mu = \left(zP,\ -p_T,\ 0,\ zP - \frac{p_T^2}{2zP}\right)^\mu \qquad \text{to } O(p_T^4)$$

Then $$(k+p)^2 = 2k\cdot p = 2p_T^2\left[1 + \frac{z}{2(1-z)} + \frac{(1-z)}{2z}\right]$$

or $$(k+p)^2 = \frac{p_T^2}{z(1-z)}$$

and $$P \cong \bar{P} \cong \frac{\sqrt{s}}{2} \quad \text{up to } O(p_T^2)$$

We can now rewrite the diagram as

$$i\mathcal{M} \cong i g_s t^a \; \bar u(p)\, \gamma\cdot\epsilon^*(k)\, u(P) \; \frac{i}{\left(\frac{p_T^2}{z(1-z)}\right)}$$

$$\cdot \; (-ie)(+ieQ_f) \; \bar u(P)\gamma^\mu v(\bar p) \; \frac{-i}{q^2} \; (\bar v \gamma_\mu u)$$

I have set $P = p+k \approx (P, 0\, 0, P)$, $(\not p + \not k) = u(P)\,\bar u(P)$, which is correct up to $\mathcal{O}(p_T^2)$. (You will see ~~that~~ we will need the numerator to $\mathcal{O}(p_T^1)$ only.) The second line here is the zeroth-order diagram.

$$i\mathcal{M}(e^+e^- \to q g \bar q) = i g_s t^a \; \bar u(p)\gamma\cdot\epsilon^*(k)\, u(P) \; \frac{i\, z(1-z)}{p_T^2}$$

$$\cdot \; i\mathcal{M}_0(e^+e^- \to q\bar q)$$

Let's evaluate this matrix element. For definiteness, take the quark to be left-handed. The gluon may be either left- or right-handed; we must consider both cases. The spinors for left-handed massless fermions are:

$$u_L(P) = \sqrt{2P}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad u_L(p) = \sqrt{2(1-z)P}\begin{pmatrix}-p_T/2(1-z)P\\1\\0\\0\end{pmatrix}$$

(rotated by $\theta = +\frac{p_T}{(1-z)P}$)

The gluon polarization vectors (rotated from the $\hat{3}$ axis by $\theta = -\frac{p_T}{zP}$) are:

$$\epsilon_L^*(k) = \frac{1}{\sqrt{2}}\left(0, 1, i, \frac{p_T}{zP}\right) \qquad \epsilon_R^*(k) = \frac{1}{\sqrt{2}}\left(0, 1, -i, \frac{p_T}{zP}\right)$$

$$\bar{\sigma}^\mu \epsilon^*_{\mu L}(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} p_T/zP & 2 \\ 0 & -p_T/zP \end{pmatrix} \qquad \bar{\sigma}\cdot\epsilon^*_{\mu R}(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} p_T/zP & 0 \\ 2 & -p_T/zP \end{pmatrix}$$

then

$$\bar{u}(p)\,\gamma\cdot\epsilon_L^*\,u(P) = \sqrt{2(1-z)P}\,\sqrt{2P}\,\left(-\frac{p_T}{2(1-z)P}, 1\right)\frac{1}{\sqrt{2}}\begin{pmatrix} p_T/zP & 2 \\ 0 & -p_T/zP \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 2P\cdot\sqrt{(1-z)}\,\frac{1}{\sqrt{2}}\left[-\frac{p_T}{(1-z)P} - \frac{p_T}{zP}\right] = -\sqrt{2}\,\sqrt{(1-z)}\,\frac{p_T}{z(1-z)}$$

$$\bar{u}(p)\,\gamma\cdot\epsilon_R^*\,u(P) = \sqrt{2(1-z)P}\,\sqrt{2P}\,\left(-\frac{p_T}{2(1-z)P}, 1\right)\frac{1}{\sqrt{2}}\begin{pmatrix} p_T/zP & 0 \\ 2 & -p_T/zP \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 2P\cdot\sqrt{(1-z)}\,\frac{1}{\sqrt{2}}\left[-\frac{p_T}{zP}\right] = -\sqrt{2}\,\sqrt{(1-z)}\,\frac{(1-z)}{z(1-z)}\,p_T$$

Then

$$\left|\mathcal{M}(e^+e^- \to qg\bar{q})\right|^2 \cong g_s^2\,\underbrace{(t^a t^a)}_{\text{color factor} = 4/3}\,\frac{2(1-z)}{z^2(1-z)^2}\,p_T^2 \cdot \begin{pmatrix} 1 \\ (1-z)^2 \end{pmatrix}\begin{array}{l} g_L \\ g_R \end{array}$$

$$\cdot\;\frac{z^2(1-z)^2}{(p_T^2)^2}\cdot\left|i\mathcal{M}(e^+e^- \to q\bar{q})\right|^2$$

$$\sum_{\text{gluon pol.}} \left|\mathcal{M}(e^+e^- \to qg\bar{q})\right|^2 = \frac{4}{3}\,g_s^2\,\frac{2(1-z)}{p_T^2}\cdot\left[1 + (1-z)^2\right]\left|i\mathcal{M}(e^+e^- \to q\bar{q})\right|^2$$

To find the rate for gluon emission, ~~integrate~~ this over the gluon phase space:

$$\int \frac{d^3p\, d^3k\, d^3\bar{p}}{(2\pi)^3\, 2p\, 2k\, 2\bar{p}} \quad (2\pi)^4 \delta(\ )$$

$$k_{\parallel} = zP$$

$$\cong \int \frac{d^3P}{(2\pi)^3} \frac{1}{2P} \frac{d^3\bar{p}}{(2\pi)^3 2\bar{p}} \frac{(2\pi)^4\delta(\ )}{(1-z)} \cdot \frac{dk_{\parallel}\, d^2p_T}{(2\pi)^3\ 2zP}$$

$$= \int d\Pi_2 \cdot \int \frac{dz}{z(1-z)} \frac{\pi\, dp_T^2}{16\pi^3}$$

so

$$\sigma(e^+e^- \to qg\bar{q}) \cong \int \frac{dz\, dp_T^2}{z(1-z)} \frac{1}{16\pi^2} \cdot \frac{4}{3} g_s^2 \frac{2(1-z)}{p_T^2}\left[1+(1-z)^2\right] \cdot \sigma(e^+e^- \to q\bar{q})$$

finally

$$\sigma(e^+e^- \to qg\bar{q}) = \int_0^1 dz \int \frac{dp_T^2}{p_T^2} \frac{4}{3}\frac{\alpha_s}{2\pi} \frac{1+(1-z)^2}{z} \cdot \sigma(e^+e^- \to q\bar{q})$$

The collinear divergence is captured in the integral

$$\int \frac{dp_T^2}{p_T^2}$$

The largest value of p_T is of the order of $Q = \sqrt{s}$. When p_T is very small, the quark and gluon are very collinear and appear in the final state in the same jet or even in the same hadron. So, approximate the integral by

$$\int \frac{dp_T^2}{p_T^2} \cong \log \frac{Q^2}{(1\ \text{GeV})^2}$$

then we can write the above formula as:

(Probability for a quark to emit a collinear gluon)

$$= \int_0^1 dz \quad \frac{4}{3}\frac{\alpha_s}{2\pi} \log \frac{Q^2}{(1\,\text{GeV})^2} \cdot \frac{1+(1-z)^2}{z}$$

where z is the momentum fraction taken by the gluon.

Two aspects of this formula deserve additional comment. First, this is an order-α_s correction enhanced by an explicit factor of $\log Q^2$. Hence, it is an order-1 correction. Corrections of this type must be summed to all orders. I will explain how to do that in a ~~moment~~. Second, this formula has an additional infrared divergence in the factor

$$\int_0 \frac{dz}{z}$$

The limit $z \to 0$ corresponds to a soft gluon. This gluon does not cause a serious rearrangement of the momenta, and so it is not as important an effect as the collinear gluon. However, it indicates that the total cross section for real gluon emission has a double logarithmic infrared divergence which must be cancelled to obtain a finite total cross section. We will see how this works in a future lecture.

The expression above gives the probability of finding a collinear gluon at momentum fraction z. What is the probability of finding a collinear quarks at momentum fraction z. The first guess would be to change $z \leftrightarrow (1-z)$ in the above and write

$$\text{Probability} = \int dz \; \frac{4}{3}\frac{\alpha_s}{2\pi} \log\frac{Q^2}{(1\,\text{GeV})^2} \; \frac{1+z^2}{(1-z)}$$

But, this is not quite right. There is always one quark, before or after the gluon emission, so the total probability to find a quark should be 1. The formula above accounts only the probability to find the quark at a lower momentum fraction after the radiation. For a complete accounting, we should also decrement the probability to find the quark at the original momentum.

Thus:

$$\text{Probability} = \int dz \; \Big[\underbrace{\delta(z-1)}_{\text{original quark}} + \Big(\frac{4}{3}\frac{\alpha_s}{2\pi} \log\frac{Q^2}{(\text{GeV})^2} \Big[\frac{1+z^2}{(1-z)} - A\,\delta(z-1) \Big] \Big) \Big]$$

with A chosen so that

$$\int_0^1 dz \; \Big[\frac{1+z^2}{(1-z)} - A\,\delta(z-1) \Big] = 0$$

This δ-function term will be supplied by the virtual gluon diagrams.

There is some awkwardness in the notation here. The integral $\int_0^1 \frac{dz}{1-z}$ is divergent at $z=1$. Then we subtract a δ-function with support at $z=1$. It would be better to use a notation that automatically makes the infinite part of the cancellation and regularizes the divergence. This is conventionally done by defining the "+ -distribution", for any smooth function $f(z)$, by

$$\int_0^1 dz \; \frac{f(z)}{(1-z)_+} = \int_0^1 dz \; \frac{f(z) - f(1)}{(1-z)}$$

That is $\frac{1}{(1-z)_+} = \frac{1}{1-z} - b\,\delta(z-1)$ such that the integral is finite. With this definition

$$\int_0^1 dz \; \frac{1+z^2}{(1-z)_+} = \int_0^1 dz \; \frac{(1+z^2)-2}{1-z} = \int_0^1 dz \left[-(1+z)\right] = -\frac{3}{2}$$

and so we should fix A on the previous page by:

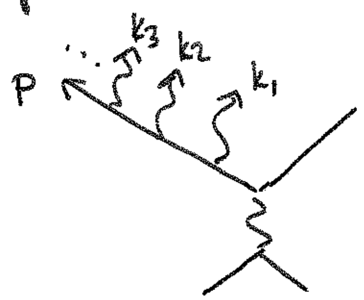
$$[\;\;] = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(z-1)$$

Then:

(Probability to find a quark after 1 emission)

$$= \int dz \left\{ \delta(z-1) + \frac{4}{3}\frac{\alpha_s}{2\pi} \log\frac{Q^2}{(GeV)^2} \cdot \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(z-1)\right]\right\}$$

Once the quark has emitted one gluon, it can emit further gluons. Each emission gives a factor $\alpha_s \log Q$, so we must sum up all collinear emissions. To do this, we should compute, for example



Consider the region where there are collinear emissions with transverse momentum p_{Ti} and longitudinal fraction z_i. If $p_{T3} \ll p_{T2} \ll p_{T1}$, the denominators of the three propagators shown will be

$$\frac{p_{T3}^2}{z_3(1-z_3)} \ll \frac{p_{T2}^2}{z_2(1-z_2)} \ll \frac{p_{T1}^2}{z_1(1-z_1)}$$

and ~~the integrals~~ over the shown phase space will give a triple log:

$$\int \frac{dp_{T3}^2}{p_{T3}^2} \int \frac{dp_{T2}^2}{p_{T2}^2} \int \frac{dp_{T1}^2}{p_{T1}^2}$$

However, if $p_{T3} \gg p_{T1}, p_{T2}$, all three denominators will be of the order of p_{T3}^2 and the first two integrals will not give logs. We can conclude that the contribution of the order of

$$\alpha_s^3 \log^3 \frac{Q^2}{(\mathrm{GeV})^2}$$

comes only from the region where $p_{T3} \ll p_{T2} \ll p_{T1}$. In general, diagrams with multiple emissions will give their dominant contributions only when the p_T's <u>decrease</u>

as we go outward from the hard scattering to the external states. This condition is called "strong ordering".

Let's now construct the probability density to find a gluon or a quark at a given value of the momentum fraction z. For the quarks, we need to add multiple emissions to the expression on p. 8:

(Probability to find a quark)

$$= \int dz\ \delta(z-1) + \int dz\ \frac{4}{3}\frac{\alpha_s}{2\pi}\int_{\text{GeV}}^{Q}\frac{dp_T^2}{p_T^2}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(z-1)\right]$$

$$+ \int dz_1\, dz_2\ \frac{4}{3}\frac{\alpha_s}{2\pi}\int_{\text{GeV}}^{Q}\frac{dp_{T1}^2}{p_{T1}^2}\left[\frac{1+z_1^2}{(1-z_1)_+} + \frac{3}{2}\delta(z_1-1)\right]\cdot\frac{4}{3}\frac{\alpha_s}{2\pi}\int_{p_{T1}}^{Q}\frac{dp_{T2}^2}{p_{T2}^2}\left[\frac{1+z_2^2}{(1-z_2)_+} + \frac{3}{2}\delta(z_2-1)\right]$$

$$+ \cdots$$

$$(\text{Probability to find a gluon}) = \int dz \int \frac{dp_T^2}{p_T^2}\ \frac{4}{3}\frac{\alpha_s}{2\pi}\ \frac{1+(1-z)^2}{z}$$

$$+ \int dz_1\, dz_2\ \frac{4\alpha_s}{3\cdot 2\pi}\int_{\text{GeV}}^{Q}\frac{dp_{T1}^2}{p_{T1}^2}\ \frac{1+(1-z_1)^2}{z_1}\ \frac{4\alpha_s}{3\,2\pi}\int\frac{dp_{T2}^2}{p_{T2}^2}\left[\frac{1+z_2^2}{(1-z_2)_+} + \frac{3}{2}\delta(z_2-1)\right]$$

$$+ \cdots$$

I would like to treat these formulae in the following way. Consider comparing these probabilities at two values of Q, one slightly larger than the other. Increasing Q increases the phase space for gluon emission. How do the probabilities change?

Let $f_q(z,Q)$ be the probability of finding a quark at momentum fraction z, including gluon emissions up to $p_T \sim Q$. Using

$$\int dz_1\, dz_2 = \int dz \int dz_1\, dz_2\, \delta(z - z_1 z_2)$$

$$= \int dz \int \frac{dz_1}{z_1} \Big|_{z_2 = \frac{z}{z_1}}$$

and ordering the p_T integrals from largest to smallest p_T:

$$\int_{GeV}^{Q} \frac{dp_{T1}^2}{p_{T1}^2} \int_{GeV}^{p_{T1}} \frac{dp_{T2}^2}{p_{T2}^2} \int_{GeV}^{p_{T2}} \frac{dp_{T3}^2}{p_{T3}^2} \cdots$$

we can write $f_q(z,Q)$ as

$$f_q(z,Q) = \delta(z-1) + \int^{Q} \frac{dp_{T1}^2}{p_T^2}\, \frac{4}{3}\frac{\alpha_s}{2\pi}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(z-1)\right] \int \frac{dz_1}{z_1}\, \delta\!\left(\frac{z}{z_1} - 1\right)$$

$$+ \int \frac{dz_1}{z_1} \int^{Q} \frac{dp_{T1}^2}{p_{T1}^2}\, \frac{4}{3}\frac{\alpha_s}{2\pi}\left[\frac{1+z_1^2}{(1-z_1)_+} + \frac{3}{2}\delta(z_1-1)\right] \int^{p_{T1}} \frac{dp_{T2}^2}{p_{T2}^2}\, \frac{4}{3}\frac{\alpha_s}{2\pi}\left[\frac{1+z_2^2}{(1-z_2)_+} + \frac{3}{2}\delta(z_2-1)\right]$$

$$+ \cdots \qquad z_2 = \frac{z}{z_1}$$

then

$$\frac{d}{d\log Q} f_q(z,Q) = \frac{4}{3}\frac{\alpha_s}{\pi} \int \frac{dz_1}{z_1} \left[\frac{1+z_1^2}{(1-z_1)_+} + \frac{3}{2}\delta(z_1-1)\right] f_q\!\left(\frac{z}{z_1}, Q\right)$$

Similarly, if $f_g(z,Q)$ is the probability to find a gluon at momentum fraction z, this function can be represented by

$$\frac{d}{d\log Q} f_g(z,Q) = \frac{4}{3}\frac{\alpha_s}{2\pi} \int \frac{dz_1}{z_1}\, \frac{1+(1-z_1)^2}{z}\, f_q\!\left(\frac{z}{z_1}, Q\right)$$

α_s should be evaluated at the p_T scale Q.

By integrating these differential equations, we sum the series of terms of order $(\alpha_s \log Q)^n$.

It will be useful to generalize these equations in several ways. First, because we are adding to the f's at the end of large p_T, the equations are valid whatever the final state is at small p_T. We can integrate down to 1–2 GeV and take the final particle to be a quark or gluon (or a jet), or we can include more soft and nonperturbative physics and take the final particle to be a hadron. For example, we could write the probability that a reaction $e^+e^- \to u\bar{u}$ that initially creates a u quark produces a π^+ that carries a fraction z of the original u quark momentum. This probability is called the fragmentation function

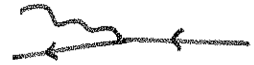
$$f_{\pi^+\leftarrow u}(z, Q)$$

The cross section for π^+ production at momentum p_π in e^+e^- annihilation at $\sqrt{s} = 2E$ is given by

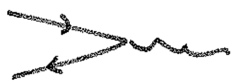
$$\frac{d\sigma}{dz}\left(e^+e^- \to \pi^+\left(z = \frac{p}{E}\right) + X\right) = \sum_f f_{\pi^+\leftarrow f}(z, \sqrt{s}) \cdot \frac{4\pi\alpha^2}{3s} \cdot 3Q_f^2$$

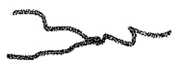
The sum is over all quark and antiquark flavors. If we can disentangle the contributions from the various species (for example, by seeing what particles are on the other side of the event) we can measure the fragmentation functions.

Second, we should understand that there are additional collinear emission processes in QCD. So far, we have accounted

$q \to qg$  also $\bar{q} \to \bar{q}g$

In addition, we have

$g \to q\bar{q}$ 

$g \to gg$ 

I will compute these collinear amplitudes in a moment. Let me anticipate, though, that they give emission probabilities of the same form that we have already found:

$$\frac{\alpha_s}{2\pi} \cdot \int dz \int \frac{dp_T^2}{p_T^2} \quad P(z)$$

$P(z)$ is called the splitting function. For emission of a gluon from a quark:

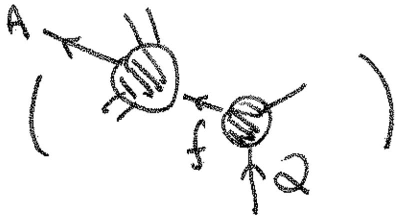
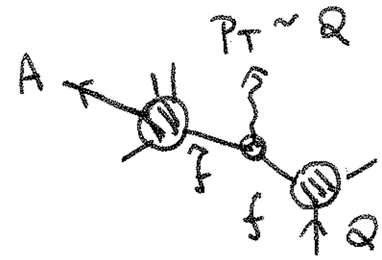
$$P_{g \leftarrow q}(z) = \frac{4}{3} \; \frac{1+(1-z)^2}{z}$$

Then the fragmentation function for a parton f coming from a hard process with momentum transfer Q to produce a final particle A (e.g. a hadron) at momentum fraction z is given as the solution of the differential equations:

$$\frac{d}{d\log Q} f_{A\leftarrow f}(z,Q) = \frac{\alpha_s(Q)}{\pi}\int_0^1 \frac{dw}{w} \sum_{\hat{f}} f_{A\leftarrow \hat{f}}(z/w)\, P_{\hat{f}\leftarrow f}(w)$$

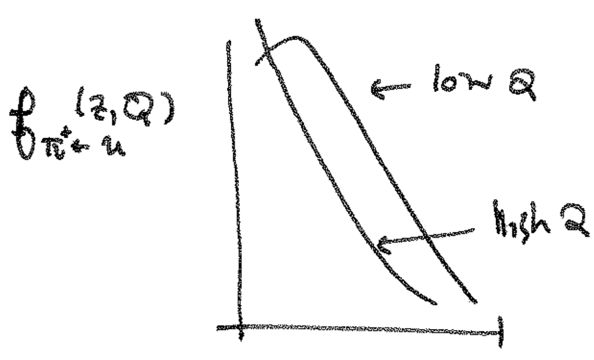
where $\hat{f}$ is summed over all species of quarks, antiquarks, and gluons.

Diagrammatically:

$\frac{d}{d\log Q}$ () $= \sum_{\hat{f}}$ 

These equations are called the Altarelli-Parisi equations (or, sometimes, the DGLAP - Dokhshitzer, Gribov, Lipatov, Altarelli, Parisi - equations). To use these equations, we measure the $f_{A\leftarrow f}(z,Q)$ at low Q and then integrate to predict the $f_{A\leftarrow f}(z,Q)$ at higher Q.

The basic qualitative effect of Altarelli-Parisi evolution should be clear. As we increase Q, we add radiation to the processes ~~that~~ produce the final state A. Thus, A appears at a lower momentum fraction z:



The figures show how this works in practice. Figs p. 2 shows fragmentation functions for $q \to$ (hadron), measured by the SLD experiment at $Q = m_Z = 91$ GeV. (The quark flavor is summed over the set of light quarks u, d, s, c produced in Z^0 decay.) The measurement takes advantage of the fact that, using polarized electrons to produce $e^-_L e^+_R \to Z^0 \to q\bar{q}$, the q dominantly goes forward and the $\bar{q}$ dominantly backward; thus, the kinematics seperates q- from $\bar{q}$- initiated fragmentation. The fragmentation functions show a pronounced leading particle effect: The highest z-hadrons are those ~~that~~ contain the flavor of quark that initiated the jet. For example

$$f_{p\leftarrow q}(z) \gg f_{\bar{p}\leftarrow q}(z) \qquad f_{\Lambda\leftarrow q}(z) \gg f_{\bar{\Lambda}\leftarrow q}(z)$$

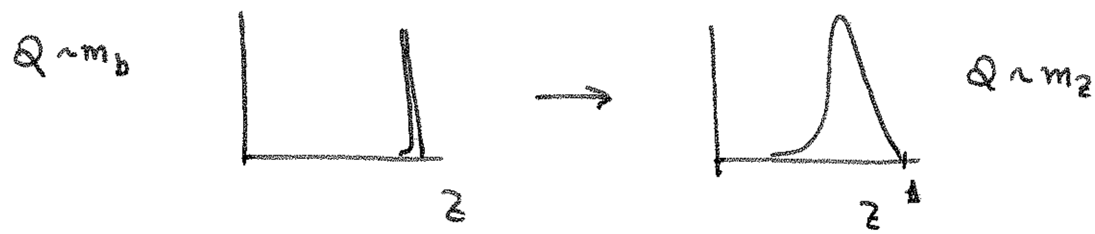
at high z, but not as $z \to 0$.

Figs. p. 3 shows the evolution of $f_{h^+\leftarrow q}(z, Q)$ with Q, using data from a variety of e^+e^- experiments from 14 to 202 GeV. The distributions get visibly steeper over this range. The solid lines show a fit to Altarelli-Parisi evolution.

Figs. p. 4 shows the measured B fragmentation function $f_{B\leftarrow b}(z)$ at $Q = m_Z$. Since the b is a heavy quark, we might expect that, at low Q, the b momentum would dominate the final B momentum. That is

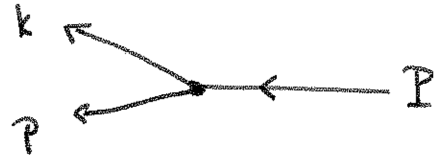
$$\text{at } Q \sim m_b \qquad f_{B\leftarrow b}(z) \sim \delta(z-1)$$

Altarelli Parisi evolution to $Q \sim m_Z$ should then produce



and this is indeed the shape measured.

We still need to complete our definition of the Altarelli-Parisi equations by computing the remaining splitting functions in QCD. To do this systematically, use the setup on p. 3



$$k^\mu = \left(zP,\ p_T,\ 0\ ,\ zP - \frac{p_T^2}{2zP}\right) \qquad k^2 = 0$$

$$p^\mu = \left((1-z)P,\ -p_T\ ,\ 0,\ (1-z)P - \frac{p_T^2}{2(1-z)P}\right) \qquad p^2 = 0$$

$$P = k+p = \left(P,\ 0\ \ 0,\ P - \frac{p_T^2}{2z(1-z)P}\right)$$

so that $$P^2 = (k+p)^2 = +\frac{p_T^2}{z(1-z)}$$

For $q \to q+g$ we computed the squared matrix element for the splitting and found.

$$|\mathcal{M}(q\to g+q)|^2 = \frac{4}{3} g_s^2 \frac{2(1-z)}{z^2(1-z)^2} p_T^2 \left(1+(1-z)^2\right)$$

This then combined with the factor $\frac{1}{(p+k)^2}$ and the integral over shower phase space as follows:

$$\left[\int \frac{dz}{z(1-z)} \frac{1}{16\pi^2} dp_T^2\right]\left[\frac{z^2(1-z)^2}{p_T^4}\right]\left[g_s^2 \frac{2(1-z)}{z^2(1-z)^2} \cdot p_T^2 \frac{4}{3}(1+(1-z)^2)\right]$$

$$= \int dz \; \frac{\alpha_s}{\pi} \int \frac{dp_T}{p_T} \; \frac{4}{3} \frac{(1+(1-z)^2)}{z}$$

We can then identify the splitting function as

$$\underset{b\leftarrow a}{P}(z) = \frac{z(1-z)}{2 g_s^2 p_T^2} \left|\mathcal{M}(a \to b+c)\right|^2$$

Apply this to the other two processes on p. 13: First, consider $g_L \to q_L \bar{q}_R$:

$$i\mathcal{M}(g_L \to q_L \bar{q}_R) = i g_s t^a \; \bar{u}_L(k) \gamma \cdot \varepsilon(P) \, v(p)$$

$$= i g_s t^a \; u_L^\dagger(k) \, \bar{\sigma} \cdot \varepsilon(P) \, v_R(p)$$

$$\varepsilon^\mu(P) = \frac{1}{\sqrt{2}} (0, 1, -i, 0)^\mu \quad \text{so} \quad \bar{\sigma} \cdot \varepsilon(P) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$u_L(k) = \sqrt{2zP} \begin{pmatrix} -p_T/2zP \\ 1 \end{pmatrix} \qquad v_R(p) = \sqrt{2(1-z)P} \begin{pmatrix} +p_T/2(1-z)P \\ 1 \end{pmatrix}$$

$$i\mathcal{M}(g_L \to q_L \bar{q}_R) = i g_s t^a \sqrt{4(z)(1-z)P^2} \, \frac{1}{\sqrt{2}} \left(-\frac{p_T}{2zP}, 1\right) \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} p_T/2(1-z)P \\ 1 \end{pmatrix}$$

$$= i g_s t^a \frac{2P}{\sqrt{2}} \sqrt{z(1-z)} \; \frac{p_T}{(1-z)P}$$

$$= i g_s t^a \sqrt{2} \sqrt{\frac{z}{1-z}} \; p_T$$

When squaring this, we need the color sum/average:

$$\frac{1}{8}\,\mathrm{tr}\,t^a t^a = \frac{1}{8}\cdot\frac{1}{2}\cdot 8 = \frac{1}{2}$$

then

$$|\mathcal{M}(g_L\to q_L\bar{q}_R)|^2 = 2g_s^2\,\frac{z}{1-z}\,p_T^2\cdot\frac{1}{2}$$

The formula on p. 17 then gives

$$P_{q_L\leftarrow g_L}(z) = \frac{1}{2}z^2$$

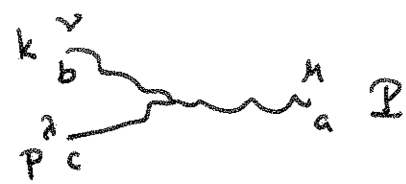
Changing $q_L\leftrightarrow\bar{q}_R$ or just $q_L\leftrightarrow q_R$ we find $z\leftrightarrow(1-z)$

$$P_{q_R\leftarrow g_L}(z) = \frac{1}{2}(1-z)^2$$

in all

$$P_{q\leftarrow g}(z) = \frac{1}{2}\left[z^2+(1-z)^2\right]$$

For $g\to gg$, we need to analyze



$$i\mathcal{M}(g(P)\to g(k)+g(p)) = gf^{abc}\left[\,\varepsilon(P)\cdot\varepsilon^*(k)\,(P+k)\cdot\varepsilon^*(p)\right.$$
$$\left.+\ \varepsilon^*(k)\cdot\varepsilon^*(p)\,(-k+p)\cdot\varepsilon(P) + \varepsilon^*(p)\cdot\varepsilon(P)\,(-p-P)\cdot\varepsilon^*(k)\right]$$

We need to consider all of the cases of polarization states for the three gluons. The color factor is the same in all

cases

$$\frac{1}{8}\sum_{\text{colors}}|f^{abc}|^2 = \frac{1}{8}f^{abc}f^{abc} = \frac{1}{8}\cdot\delta^{aa}\cdot 3 = 3$$

We can choose the initial g to be L:

$$\varepsilon(P) = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$$

Begin with $\underline{g_L \to g_R\, g_R}$ (We only need to keep terms to $\mathcal{O}(p_T^1)$.)

$$\varepsilon_R^*(k) = \frac{1}{\sqrt{2}}\left(0\ \ 1, -i, -\frac{p_T}{zP}\right) \qquad + \mathcal{O}(p_T^2)$$

$$\varepsilon_R^*(p) = \frac{1}{\sqrt{2}}\left(0\ \ 1, -i, +\frac{p_T}{(1-z)P}\right)$$

$$\varepsilon_L(P)\cdot\varepsilon_R^*(k) = 0 \qquad \varepsilon_L(P)\cdot\varepsilon_R^*(p) = 0 \qquad \varepsilon_R^*(k)\cdot\varepsilon_R^*(p) = \mathcal{O}(p_T^2)$$

so $i\mathcal{M}(g_L \to g_R g_R) = 0$ up to $\mathcal{O}(p_T)$

Next, $\underline{g_L \to g_L\, g_R}$

$$\varepsilon_L^*(k) = \frac{1}{\sqrt{2}}\left(0, 1, i, -\frac{p_T}{zP}\right) \qquad \varepsilon_R^*(p) = \frac{1}{\sqrt{2}}\left(0, 1, -i, \frac{p_T}{(1-z)P}\right)$$

$$i\mathcal{M} = g f^{abc}\left[(-1)(P+k)\cdot\varepsilon^*(p) + (-1)(-k+p)\cdot\varepsilon(P) + 0\right]$$

$$= g f^{abc}\left[-2P\cdot\varepsilon^*(p) + 2k\cdot\varepsilon(P)\right] \quad \text{using } p\cdot\varepsilon^*(p) = 0,\ P = k+p$$

$$= g f^{abc}\left[+\frac{2P}{\sqrt{2}}\cdot\frac{p_T}{(1-z)P} - \frac{2p_T}{\sqrt{2}}\right]$$

$$i\mathcal{M}(g_L \to g_L g_R) = \sqrt{2}\, g f^{abc}\, p_T \frac{z}{1-z}$$

Next, $\underline{g_L \to g_R\, g_L}$

$$\varepsilon_R^*(k) = \frac{1}{\sqrt{2}}\left(0\ \ 1\ -i, -\frac{p_T}{zP}\right) \qquad \varepsilon_L^*(p) = \frac{1}{\sqrt{2}}\left(0, 1, +i, \frac{p_T}{(1-z)P}\right)$$

$$i\mathcal{M} = g f^{abc}\left[\ 0 \ + (-1)(-2k\cdot\varepsilon(P)) + (-1)(-2P\cdot\varepsilon^*(k))\right]$$

$$= g f^{abc}\left[-\frac{2}{\sqrt{2}}p_T + 2P\frac{1}{\sqrt{2}}\frac{p_T}{zP}\right]$$

$$= g f^{abc}\sqrt{2}\,p_T\left(-1+\frac{1}{z}\right)$$

$$i\mathcal{M}(g_L\to g_R g_L) = \sqrt{2}\, g f^{abc}\, p_T\, \frac{1-z}{z}$$ ($z \leftrightarrow (1-z)$ from the above)

Finally, $g_L \to g_L g_L$

$$i\mathcal{M} = g f^{abc}\left[(-1)\ 2P\cdot\varepsilon^*(p) + (0) + (-1)(-2P\cdot\varepsilon^*(k))\right]$$

$$= g f^{abc}\left[+2P\frac{1}{\sqrt{2}}\frac{p_T}{(1-z)P} + 2P\frac{1}{\sqrt{2}}\frac{p_T}{zP}\right]$$

$$= g f^{abc}\sqrt{2}\,p_T\left(\frac{1}{(1-z)}+\frac{1}{z}\right)$$

$$i\mathcal{M}(g_L\to g_L g_L) = \sqrt{2}\, g_s f^{abc}\, p_T\, \frac{1}{z(1-z)}$$

so

$$\sum_{\substack{\text{fnd spin}\\ + \\ \text{color}}} \left|\mathcal{M}(g_L\to gg)\right|^2 = 2g_s^2\cdot 3\left[\frac{z^2}{(1-z)^2} + \frac{(1-z)^2}{z^2} + \frac{1}{z^2(1-z)^2}\right]$$

The prescription on p. 17 then gives

$$P_{g\leftarrow g}(z) = \frac{3}{z(1-z)}\left[z^4 + (1-z)^4 + 1\right]$$

$$= 6\left[\frac{(1-z)}{z} + \frac{z}{(1-z)} + z(1-z)\right]$$

As we did with $P_{q\leftarrow q}(z)$, we must subtract a term from $P_{g\leftarrow g}(z)$ to account for the fact that a gluon that undergoes splitting must be removed from the distribution. Of course, the process $g \to gg$ does not conserve the number of gluons. However, it conserves longitudinal momentum. Thus write

$$P_{g\leftarrow g}(z) = 6\left[\frac{(1-z)}{z} + \frac{z}{(1-z)} + z(1-z) - A\,\delta(z-1)\right]$$

and determine A by the condition $\int_0^1 dz\, z\, P_{g\leftarrow g}(z) = 0$. To do this explicitly, change $\frac{1}{(1-z)}$ to a + distribution. Then

$$\int_0^1 dz\; z\left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\right]$$

$$= \int dz\left[(1-z) + \frac{z^2-1}{(1-z)} + z^2(1-z)\right] = \frac{1}{2} - \frac{3}{2} + \frac{1}{3} - \frac{1}{4}$$

$$= -\frac{11}{12}$$

We also must remember that the splitting $g \to q + \bar{q}$ removes gluons and transfers longitudinal momentum:

$$\int dz \quad \frac{1}{2}\left[z^2 + (1-z)^2\right]\cdot\left[\underset{\substack{\uparrow\\ \text{mom of } q}}{z} + \underset{\substack{\uparrow\\ \text{mom of } \bar{q}}}{(1-z)}\right] = \frac{1}{2}\cdot\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{1}{3}$$

for each quark flavor. The final form of the gluon splitting function is then:

$$P_{g\leftarrow g}(z) = 6\left[\frac{(1-z)}{z} + \frac{z}{(1-z)_+} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18}\right)\delta(z-1)\right]$$

It would be good to concisely summarize this discussion. We have shown that final-state collinear quark and gluon radiation is described by the statement that fragmentation functions $f_{A\leftarrow f}(z,Q)$, where $f = (u\,d\,s\,c\,b,\ \bar{u}\,\bar{d}\,\bar{s},\bar{c},\bar{b},\ g)$, satisfy the Altarelli Parisi equation:

$$\frac{d}{d\log Q}\, f_{A\leftarrow f}(z,Q) = \frac{\alpha_s(Q)}{\pi}\int_0^1 \frac{dw}{w}\sum_{\hat{f}} P_{\hat{f}\leftarrow f}(w)\, f_{A\leftarrow \hat{f}}(z/w, Q)$$

This is a differential equation in $\log Q$, an integral equation in z. The kernels are the splitting functions:

$$P_{g\leftarrow q}(z) = P_{g\leftarrow \bar{q}}(z) = \frac{4}{3}\,\frac{1+(1-z)^2}{z}$$

$$P_{q\leftarrow q}(z) = P_{\bar{q}\leftarrow \bar{q}}(z) = \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(z-1)\right]$$

$$P_{q\leftarrow g}(z) = P_{\bar{q}\leftarrow g}(z) = \frac{1}{2}\, z^2 + (1-z)^2$$

$$P_{g\leftarrow g}(z) = 6\left[\frac{z}{(1-z)_+} + \frac{(1-z)}{z} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18}\right)\delta(z-1)\right]$$

For many purposes, it is useful to have the polarized splitting functions. Since I have worked these out, let me record them:

$$P_{g_L \leftarrow q_L} = \frac{4}{3}\,\frac{1}{z}$$

$$P_{g_R \leftarrow q_L} = \frac{4}{3}\,\frac{1}{z}\,(1-z)^2$$

$$P_{q_L \leftarrow q_L} = \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(z-1)\right]$$

$$P_{q_L \leftarrow g_L} = \frac{1}{2}\,z^2$$

$$P_{q_R \leftarrow g_L} = \frac{1}{2}\,(1-z)^2$$

$$P_{g_L \leftarrow g_L} = 3\left[\frac{1+z^4}{z(1-z)_+} + \left(\frac{11}{6} - \frac{n_f}{9}\right)\delta(z-1)\right]$$

$$P_{g_R \leftarrow g_L} = 3\,\frac{(1-z)^3}{z}$$

with other nonzero functions obtained by $L \leftrightarrow R$ or $q \leftrightarrow \bar{q}$ reflection (P, C). Certain P's are zero:

$$P_{q_R \leftarrow q_L} = 0 \qquad P_{\bar{q} \leftarrow q} = 0$$

In the gluon splitting we sum over the spectator gluon spin; without this sum:

$$P_{g_L(+g_L) \leftarrow g_L} = 3\,\frac{1}{z(1-z)} \qquad P_{g_L(+g_R) \leftarrow g_L} = 3\,\frac{z^4}{z(1-z)}$$

$$P_{g_R(+g_L) \leftarrow g_L} = 3\,\frac{(1-z)^4}{z(1-z)} \qquad P_{g_R(+g_R) \leftarrow g_L} = 0$$