Physics 332 - Problem Set #8
Solutions

1.) a) The Adler-Bell-Jackiw anomaly equation

$$\partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2}\,\varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$$

can be rewritten as follows.

$$j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi = \bar{\psi}_R\gamma^\mu\psi_R - \bar{\psi}_L\gamma^\mu\psi_L$$

$$= j_R^\mu - j_L^\mu$$

$$\varepsilon^{\alpha\beta\gamma\delta} = 4\cdot\epsilon^{ijk} F_{0i} F_{jk}$$

$$= 8\, F_{0i}\cdot\left(\tfrac{1}{2}\epsilon^{ijk}F_{jk}\right)$$

$$= 8\, E^i(-B^i)$$

If $\quad N_R = \int d^3x\, j_R^0 \qquad N_L = \int d^3x\, j_L^0$

$\int d^4x$ of the ABJ equation is

$$\Delta N_R - \Delta N_L = \int d^4x\, \frac{e^2}{2\pi^2}\,\vec{E}\cdot\vec{B}$$

now consider

$$A^\mu = (0, 0, Bx^1, A(t))$$

$$B^3 = \partial_1 A^2 = B \qquad E^3 = -\partial_0 A^3 = -\partial_0 A$$

so if A goes adiabatically from 0 to $\frac{2\pi}{eL}$, ABJ implies

$$\Delta N_R - \Delta N_L = -\frac{e^2}{2\pi^2} \cdot \frac{2\pi}{eL} \cdot L^3 \cdot B$$

$$= -\frac{eB}{\pi} L^2$$

b.) $$\mathcal{H} = \int d^3x \; \psi^\dagger \gamma^0 (-i\vec{\gamma}\cdot\vec{D}) \psi$$

$$\gamma^0 \vec{\gamma} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \text{ on } \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$\mathcal{H} = \int d^3x \left(\psi_R^\dagger (-i\vec{\sigma}\cdot\vec{D}) \psi_R - \psi_L^\dagger (-i\vec{\sigma}\cdot\vec{D}) \psi_L \right)$$

$$(\vec{D})^i = D_i = (\partial_i - ieA^i)$$

c.) We now need to diagonalize $-i\vec{\sigma}\cdot\vec{D}$

Look for eigenvectors of the form

$$\psi_R = \begin{pmatrix} \phi_1(x^1) \\ \phi_2(x^1) \end{pmatrix} e^{i(k_2 x^2 + k_3 x^3)}$$

$$-i\vec{\sigma}\cdot\vec{D}\,\psi_R = \left[-i\sigma^1\partial_1 - i\sigma^2(ik_2 - ieBx^1) - i\sigma^3(ik_3 - ieA)\right]\psi_R$$

$$= \left[-i\sigma^1\partial_1 + \sigma^2(k_2 - eBx^1) + \sigma^3(k_3 - eA)\right]\psi_R$$

so that $\quad -i\vec{\sigma}\cdot\vec{D}\,\psi_R = E\,\psi_R$

reads

$$-i\partial_1\phi_2 - i(k_2 - eBx^1)\phi_2 + (k_3 - eA)\phi_1 = E\phi_1$$

$$-i\partial_1\phi_1 + i(k_2 - eBx^1)\phi_1 - (k_3 - eA)\phi_2 = E\phi_2$$

so

$$-i\left[\partial_1 - (k_2 - eBx^1)\right]\phi_1 = \left[E + (k_3 - eA)\right]\phi_2$$

$$-i\left[\partial_1 + (k_2 - eBx^1)\right]\phi_2 = \left[E - (k_3 - eA)\right]\phi_1$$

$$-\left(\partial_1 + (k_2 - eBx^1)\right)\left(\partial_1 - (k_2 - eBx^1)\right)\phi_1$$

$$= \left[E^2 - (k_3 - eA)^2\right]\phi_1$$

$$\left[\begin{array}{l} -\partial_1^2 + (eBx^1 - k_2)^2 + (k_3 - eA)^2 \\ \qquad - \partial_1(eBx^1) \end{array}\right]\phi_1 = E^2\phi_1$$

better:

$$\frac{1}{2}\left[-\partial_1^2 + (eB)^2\left(x^1 - \frac{k_2}{eB}\right)^2\right]\phi_1 = \left(eB + \left[E^2 - (k_3 - eA)^2\right]\right)\phi_1$$

for ϕ_2 the differential operators are in the other order, so

$$\left\{ \left[-\tfrac{1}{2}\partial_1^2 + \tfrac{1}{2}(eB)^2\left(x^1 - \frac{k_2}{eB}\right)^2 \right] - \tfrac{1}{2}eB \right\} \phi_1 = \tfrac{1}{2}\left[E^2 - (k_3 - eA)^2\right] \phi_1$$

$$\left\{ \left[-\tfrac{1}{2}\partial_1^2 + \tfrac{1}{2}(eB)^2\left(x^1 - \frac{k_2}{eB}\right)^2 \right] + \tfrac{1}{2}eB \right\} \phi_2 = \tfrac{1}{2}\left[E^2 - (k_3 - eA)^2\right] \phi_2$$

the operator

$$\mathcal{H} = \left[-\tfrac{1}{2}\partial_1^2 + \tfrac{1}{2}(eB)\left(x^1 - \frac{k_2}{eB}\right)^2 \right]$$

is a harmonic oscillator Hamiltonian with eigenvalues

$$\partial_n = (n + \tfrac{1}{2})(eB)$$

so ϕ_1 has $(\mathcal{H} - \tfrac{1}{2}eB)$ acting on it; this has a

<u>zero</u> eigenvalue

ϕ_2 has $(\mathcal{H} + \tfrac{1}{2}eB)$ acts on it; this has only

positive eigenvalues

For $(\mathcal{H} - \tfrac{1}{2}eB) = eBn > 0$

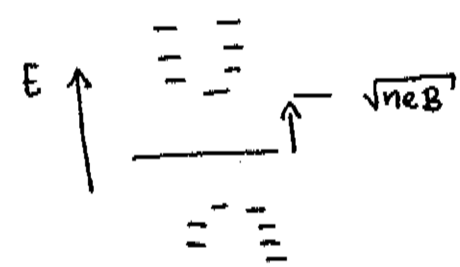
$$E = \pm\left[(k_3 - eA)^2 + n\,eB \right]^{1/2}$$

For $(\mathcal{H} - \tfrac{1}{2}eB) = 0$, $\phi_2 = 0$

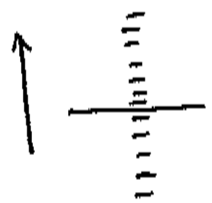
$$\Rightarrow \quad E = (k_3 - eA)$$

d.) if $k^3 = \frac{2\pi m}{L}$, then spectra look like

$n > 0$



$n = 0$



spacing $= \frac{2\pi}{L}$

each level is a Harmonic Oscillator centered at

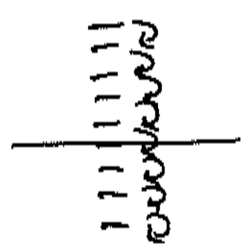
$$x' = \frac{k_2}{eB}$$

for $0 < x' < L$, $0 < k_2 < eBL$. If k_2 is quantized: $k_2 = \frac{2\pi m}{L}$

there are $\frac{eBL^2}{2\pi}$ states in each level.

e.) Now increase A adiabatically from 0 to $\frac{2\pi}{eL}$

this causes the $n=0$ states to move down one notch:



$$\Delta N_R = -\frac{eBL^2}{2\pi}$$

For the left-handed spectrum,

$$E\psi_L = +i\vec{\sigma}\cdot\vec{D}\psi_L$$

for $n=0$ $\quad E = -(k_3 - eA)$

so $$\Delta N_L = +\frac{eBL^2}{2\pi}$$

in all $$\Delta N_R - \Delta N_L = -\frac{eBL^2}{\pi}$$

as predicted on p. 2.

2.) a) $$\bar{u}_L\gamma^\mu d_L = \bar{Q}\gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\left(\frac{1-\gamma^5}{2}\right) Q$$

where $Q = \begin{pmatrix} u \\ d \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2}\sigma^1 + i\tfrac{1}{2}\sigma^2 = \tau^1 + i\tau^2$$

so $$\bar{u}_L\gamma^\mu d_L = \tfrac{1}{2}\left(j^{\mu 1} + i j^{\mu 2} - j^{\mu 5 1} - i j^{\mu 5 2}\right)$$

$|\pi^+\rangle$ is annihilated by $\bar{d}_L \gamma^\mu u_L = \frac{1}{2}(j^{\mu 1} - i j^{\mu 2} - j^{\mu 5 1} + i j^{\mu 5 2})$

so $|\pi^+\rangle = \frac{1}{\sqrt{2}}(|\pi^1\rangle + i|\pi^2\rangle)$

$$\langle 0|\bar{d}_L \gamma^\mu u_L|\pi^+\rangle = -ip^\mu f_\pi \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \cdot 2 \cdot (-1)$$

$$= +ip^\mu f_\pi \frac{1}{\sqrt{2}}$$

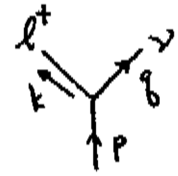
then the weak ~~interaction~~ effective Lagrangian

$$\Delta\mathcal{L} = \frac{4G_F}{\sqrt{2}} \left[\bar{\nu}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\ell\right]\left[\bar{d}\gamma_\mu \frac{1-\gamma^5}{2} u\right]$$

since

$$i\mathcal{M}(\pi^+ \to \ell^+\nu) = i\frac{4G_F}{\sqrt{2}} \bar{u}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)v \cdot \frac{ip_\mu}{\sqrt{2}} f_\pi$$

$$= -2G_F \bar{u} \not{p}\left(\frac{1-\gamma^5}{2}\right)v \cdot f_\pi$$



for the ν: $\bar{u}\not{q} = 0$

for the ℓ^+: $\not{p} v = -m_\ell v$

$$\not{p}\left(\frac{1-\gamma^5}{2}\right)v = -\left(\frac{1+\gamma^5}{2}\right)m_\ell v$$

b.) $i\mathcal{M}(\pi^+ \to \ell^+\nu) = 2G_F m_\ell \bar{u}(q)\left(\frac{1+\gamma^5}{2}\right)v(k) \cdot f_\pi$

$$\sum_{\text{spin}} |\mathcal{M}|^2 = 4 G_F^2 m_\ell^2 f_\pi^2 \ \mathrm{tr}\ \not q \left(\frac{1+\gamma^5}{2}\right)(\not k + m_\ell)\left(\frac{1-\gamma^5}{2}\right)$$

$$= 4 G_F^2 m_\ell^2 f_\pi^2 \cdot 2\, q\cdot k$$

$$2 q\cdot k = (q+k)^2 - k^2 = m_\pi^2 - m_\ell^2$$

$$= 4 G_F^2 f_\pi^2 m_\ell^2 (m_\pi^2 - m_\ell^2) \qquad \frac{2p}{\sqrt{s}}$$

$$\Gamma(\pi^+ \to \ell^+ \nu) = \frac{1}{2m_\pi} \cdot \frac{1}{8\pi} \cdot 4 G_F^2 f_\pi^2 m_\ell^2 (m_\pi^2 - m_\ell^2) \cdot \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)$$

$$= \frac{1}{4\pi} G_F^2 f_\pi^2 m_\pi^3 \frac{m_\ell^2}{m_\pi^2} \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2$$

note that

$$\frac{\Gamma(\pi^+ \to e^+ \nu_e)}{\Gamma(\pi^+ \to \mu^+ \nu_\mu)} = \frac{m_e^2}{m_\mu^2} \frac{(1 - m_e^2/m_\pi^2)^2}{(1 - m_\mu^2/m_\pi^2)^2} = 1.27 \times 10^{-4}$$

indeed! $\quad BR(\pi^+ \to e^+ \nu) = 1.23 \times 10^{-4}$!

now

$$\Gamma(\pi^+ \to \mu^+ \nu) = \frac{1}{4\pi} G_F^2 m_\pi^3 f_\pi^2 \cdot \underbrace{\frac{m_\nu^2}{m_\pi^2}}_{0.104} \cdot \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

$$\Gamma(\pi^+\to\mu^+\nu) = \frac{1}{4\pi}\cdot(1.166\times10^{-5}\ \mathrm{GeV}^{-2})^2\cdot(0.14\ \mathrm{GeV})^3$$

$$\cdot(0.1\ \mathrm{GeV})^2\cdot\left(\frac{f_\pi}{100\,\mathrm{MeV}}\right)^2\cdot\ 0.104$$

$$= 3.1\times10^{-17}\ \mathrm{GeV}\cdot\left(\frac{f_\pi}{100\,\mathrm{MeV}}\right)^2$$

$$\times\ (6.58\times10^{-25}\ \mathrm{GeV\,sec})^{-1}$$

$$= \left[2.1\times10^{-8}\ \mathrm{sec}\cdot\left(\frac{100\ \mathrm{MeV}}{f_\pi}\right)^2\right]^{-1}$$

from $\tau_{\pi^+} = 2.6\times10^{-8}\,\mathrm{sec}$ $\qquad f_\pi = 90\ \mathrm{MeV}$.

3.) The anomaly coefficient of a representation r of $SU(N)$ is defined by

$$\mathrm{tr}\,[t^a_r\,\{t^b_r, t^c_r\}] = \tfrac{1}{2}\,A(r)\,d^{abc}$$

so $A(N) = 1$

a) In $r_1\times r_2$ $\qquad t^a_{r_1\times r_2} = t^a_{r_1}\otimes 1 + 1\otimes t^a_{r_2}$

$$\mathrm{tr}\,[t^a_{r_1\times r_2}\,\{t^b_{r_1\times r_2},\ t^c_{r_1\times r_2}\}]$$

$$= \mathrm{tr}\,(t^a_{r_1}\otimes 1 + 1\otimes t^a_{r_2})\,\{(t^b_{r_1}\otimes 1 + 1\otimes t^b_{r_2}), (t^c_{r_1}\otimes 1 + 1\otimes t^c_{r_2})\}$$

$$= \mathrm{tr}_{r_1}\,[t^a_{r_1}\{t^b_{r_1}, t^c_{r_1}\}]\ \mathrm{tr}_{r_2}\,[1]$$

$$+ \mathrm{tr}_{r_1}[1]\cdot \mathrm{tr}\,[t^a_{r_2}\,\{t^b_{r_2}, t^c_{r_2}\}]$$

$$+ [\text{terms} \propto \mathrm{tr}(t^a_{r_1})] + [\text{terms} \propto \mathrm{tr}(t^a_{r_2})]$$

$$= \tfrac{1}{2}\,A(r_1)\,d^{abc}\cdot d_{r_2} + \tfrac{1}{2}\,d_{r_1}\,A(r_2)\,d^{abc}$$

$$+ 0 + 0$$

since $\mathrm{tr}(t^a) = 0$

on the other hand, if we write $r_1 \times r_2 = \bigoplus r_j$ where r_j are irreducible,

$$= \sum_j \mathrm{tr}\,[t^a_{r_j}\{t^b_{r_j}, t^c_{r_j}\}] = \tfrac{1}{2}\sum_j A(r_j)\,d^{abc}$$

so

$$d_1\,A(r_2) + d_2\,A(r_1) = \sum_j A(r_j)$$

b f) We can compute the anomaly for any SU(N) rep. by looking in an SU(3) subgroup. Under SU(3)

$$N \to \underset{\sim}{3} + (N-3)\,\underset{\sim}{1}$$

$$(N\times N)_a \to (3\times 3)_a + (N-3)\,\underset{\sim}{3} + \underline{1}\text{'s}$$

$$\underset{\sim}{\bar{3}} + (N-3)\,\underset{\sim}{3} + 1\text{'s}.$$

$$A(a) = A(\bar{3}) + (N-3)\,A(3)$$

$$= -1 + (N-3)\cdot 1 = N-4$$

check: in $\underline{SU(4)}$ $a = 6$ is $\underline{real}$ and so has $A(6) = 0$

then

$$N\times N = a + s$$

$$d(N)\cdot A(N)\cdot 2 = A(a) + A(s)$$

$$2N = (N-4) + A(s)$$

$$A(a) = N-4$$

$$A(s) = N+4$$

In SU(3) $s = 6$. A chiral system with $\underset{\sim}{6} + 7\,\underset{\sim}{\bar{3}}$ is anomaly-free.

alternately

$$(N\times N)_s = (3\times 3)_s + (N-3)\,\utilde{3} + 1^{\prime s}$$

$$= \utilde{6} + (N-3)\,\utilde{3} + 1^{\prime s}$$

In $SU(3)$ $\quad (3\times 3)_a = \bar{3} \qquad A(\bar{3}) = -1$

$$3\cdot \underbrace{A(3)}_{1}\cdot 2 = A(6) + \underbrace{A(\bar{3})}_{-1} \qquad \text{so} \qquad A(6) = 7$$

then

$$(N\times N)_s \Rightarrow \utilde{6} + (N-3)\,\utilde{3} + \cdots$$

has anomaly coefficient $\quad A(6) + (N-3)\,A(3) =$

$$7 + (N-3) = N+4$$

by the same method

$$(N\times N\times N)_a = (3\times 3\times 3)_a + (N-3)\,(3\times 3)_a + \frac{(N-3)(N-4)}{2}\,3 + 1^{\prime s}$$

$$A\left((N\times N\times N)_a\right) = A(1) + (N-3)\,A(\bar{3}) + \frac{(N-3)(N-4)}{2}\,A(3)$$

$$= \frac{(N-3)(N-4)}{2} - (N-3)$$

$$= \frac{(N-3)(N-6)}{2}$$

$$A((N\times N\times N\times N)_a) = A\Big\{(N-3)\,(3\times3\times3)_a + \frac{(N-3)(N-4)}{2}(3\times3)_a + \frac{(N-3)(N-4)(N-5)}{3!}(3)_a + 1\text{'s}.\Big\}$$

$$= \frac{(N-3)(N-4)(N-5)}{3!} - \frac{(N-3)(N-4)}{2}$$

$$= \frac{(N-3)(N-4)(N-8)}{6}$$

similarly

$$A(\underbrace{N\times\cdots\times N}_{a})_{a}) = \frac{(N-3)(N-4)\cdots(N-j-1)}{(j-1)!} - \frac{(N-3)\cdots(N-j)}{(j-2)!}$$

$$= \frac{(N-3)(N-4)\cdots(N-j)\cdot(N-2j)}{(j-1)!}$$

this must vanish for $N=3$, $N=4$, .. $N=j$, since

$$(\overbrace{N\times N\times\cdots\times N}^{j})_a$$

has at most singlets for $N\le j$

it must vanish for $N=2j$ because in $SU(2j)$

$$A^{\overbrace{ab\cdots k}^{j}} = \epsilon_{ab-k\ell-m}\,A_{\ell-m}$$

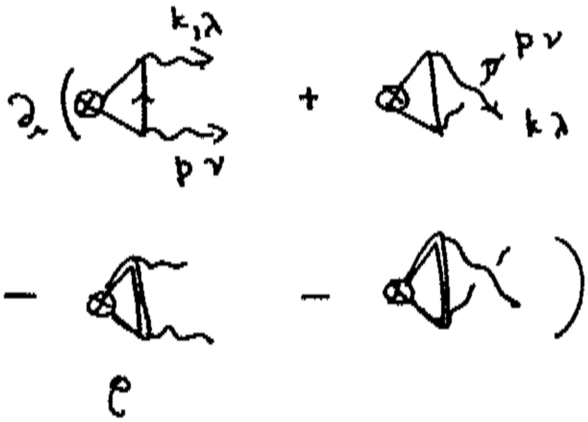
is a <u>real</u> representation.

4.) a.) The Adler-Bell-Jackiw anomaly correspond to

$$\langle p,k|\,\partial_\mu j^{\mu 5}\,|0\rangle = +\frac{e^2}{2\pi^2}\,\varepsilon^{\alpha\nu\beta\lambda}\,p_\alpha\,\varepsilon^*_\nu(p)\,k_\beta\,\varepsilon^*_\lambda(k)$$

If we regularize the current matrix element by Pauli Villars regularization:

$\langle p,k|\,\partial_\mu j^{\mu 5}\,|0\rangle = \partial_\mu\Big($



heavy fermion of mass $M \to \infty$

the divergence gives 0 on the top diagrams, but in the bottom line we should use

$$\partial_\mu j^{\mu 5}_M = \partial_\mu(\bar\Psi\gamma^\mu\gamma^5\Psi) = 2iM\,\bar\Psi\gamma^5\Psi$$

then we should have

$$\lim_{M\to\infty}\langle p,k|\,2iM\,\bar\Psi\gamma^5\Psi(0)\,|0\rangle = -\frac{e^2}{2\pi^2}\,\varepsilon^{\alpha\nu\beta\lambda}\,p_\alpha\,\varepsilon^*_\nu\,k_\beta\,\varepsilon^*_\lambda$$

b.) I did not discuss the trace anomaly in class, but it is discussed in Section 19.5 of Peskin & Schroeder.

In QED,

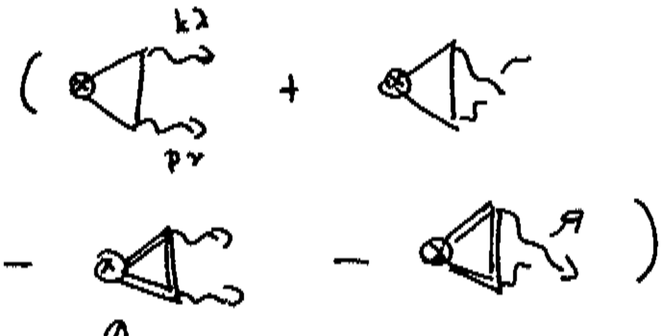
$$\Theta^\mu{}_\mu = \frac{\beta(e)}{2e^3}(eF_{\lambda\sigma})^2 = \frac{e^2}{24\pi^2}(F_{\lambda\sigma})^2$$

this corresponds to the matrix element

$$\langle p,k|\, \Theta^\mu{}_\mu \,|0\rangle = -\frac{e^2}{6\pi^2}\left[p\cdot k \;\epsilon^*_\nu(p)\,\epsilon^{\nu *}(k) - p\cdot\epsilon^*(k)\; k\cdot\epsilon^*(p)\right]$$

If we define $\Theta^\mu{}_\mu$ by Pauli-Villars regularization,

$$\langle p,k|\, \Theta^\mu{}_\mu \,|0\rangle =$$



heavy fermion of mass $M \to \infty$

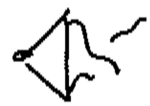
In the bottom line, we can use

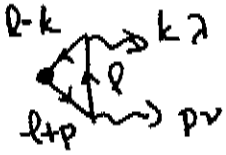
$$(\Theta^\mu{}_\mu)_M = M\,\bar\Psi\Psi$$

so we should have

$$\lim_{M\to\infty} \langle p,k|\, M\bar\Psi\Psi(0)\,|0\rangle = +\frac{e^2}{6\pi^2}\left[p\cdot k \;\epsilon^*(p)\cdot\epsilon^*(k) - p\cdot\epsilon^*(k)\; k\cdot\epsilon^*(p)\right]$$

c.) Now check these predictions explicitly:

$\langle p,k|\, 2iM\, \bar{\Psi}\gamma^5\Psi(0)\,|0\rangle$ =  + 



$$= (-1)(-ie)^2 \int \frac{d^4\ell}{(2\pi)^4}\, \mathrm{tr}\Big\{(2iM\gamma^5)\, \frac{i(\not{\ell}-\not{k}+M)}{(\ell-k)^2-M^2}\,\gamma^\lambda\, \frac{i(\not{\ell}+M)}{\ell^2-M^2}\,\gamma^\nu\, \frac{i(\not{\ell}+\not{p})+M}{(\ell+p)^2-M^2}\Big\}$$

Combine denominators:

$$\text{Den} = x\,((\ell-k)^2-M^2) + y\,((\ell+p)^2-M^2) + z\,(\ell^2-M^2)$$

$$= \ell^2 + 2\ell(-xk+yp) + xk^2 + yp^2 - M^2$$

$$= \mathbb{l}^2 + xk^2 + yp^2 - x^2k^2 - y^2p^2 + 2k\cdot p\, xy - M^2$$

$$= \mathbb{l}^2 + xy\,(k+p)^2 + xz\,k^2 + yz\,p^2 - M^2$$

$$= \mathbb{l}^2 - \Delta$$

where $\Delta = M^2 - xy\,(k+p)^2 - xz\,k^2 - yz\,p^2$

$\mathbb{l} = \ell - xk + yp$

so $\ell = \mathbb{l} + xk - yp$

$\ell - k = \mathbb{l} - (1-x)k - yp$

$\ell + p = \mathbb{l} + xk + (1-y)p$

to get a non zero trace, we need

$$\mathrm{tr}\ \gamma^5 \quad \gamma^\lambda \quad \gamma^\nu \quad \cdot \quad (\, 2\, (\not{k})'_s \; + \; 1\, M\,)$$

if both $\not{k}'$s are $\not{\ell}$, we get 0, so both must be finite momenta. Then the integral is finite and equals

$$= \; -ie^2 \int dx\,dy\,dz\; \delta(x+y+z-1)\cdot 2 \int \frac{d^4 \ell}{(2\pi)^4}\; \frac{1}{[\ell^2-\Delta]^3}\;\cdot (2iM)$$

$$\cdot\ \mathrm{tr}\, \{ \gamma^5\; M\; \gamma^\lambda\, (+x\not{k} - y\not{p})\; \gamma^\nu\, (x\not{k} + (1-y)\not{p})$$

$$+\; \gamma^5 (-(1-x)\not{k} - y\not{p})\; \gamma^\lambda\; M \;\; \gamma^\nu\, (x\not{k} + (1-y)\not{p})$$

$$+\; \gamma^5\, (-(1-x)\not{k} - y\not{p})\; \gamma^\lambda\, (+x\not{k} - y\not{p})\; \gamma^\nu \;\; M \;\; \}$$

$$= \quad + \; 2Me^2 \int dx\,dy\,dz\; \delta(x+y+z-1) \;\cdot\; 2 \int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{[\ell^2-\Delta]^3}$$

$$\cdot\; (-\,4i\,M\,)\, [\;\; \varepsilon^{\lambda\alpha\nu\beta}\; k_\alpha\, p_\beta\, [(+x)(1-y) \;+\; xy\,]$$

$$+\;\; \varepsilon^{\alpha\lambda\nu\beta}\; k_\alpha\, p_\beta\; [-(1-x)(1-y) \;+\; xy]$$

$$+\;\; \varepsilon^{\alpha\lambda\beta\nu}\; k_\alpha\, p_\beta\; [+(1-x)y \;+ xy\,]\;]$$

$$= \; -8iM^2e^2 \int dx\,dy\,dz\; \delta(x+y+z-1) \;\cdot\; 2 \int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{[\ell^2-\Delta]^3}$$

$$(\,\varepsilon^{\alpha\nu\beta\lambda}\, p_\alpha\, k_\beta\,)\; \{ +x(1-y) \;+\; xy \;+\; (1-x)(1-y) \;-\; xy$$

$$+\; (1-x)\,y \;+\; xy\;\}$$

$$\{\} = x - \cancel{xz} + \cancel{xz} + 1-x-y + \cancel{yz} - \cancel{xy}$$
$$+ \; y - \cancel{xy} + \cancel{xy} \qquad = 1$$

so

$$= -8i M^2 e^2 \int dx\,dy\,dz\, \delta(x+y+z-1) \cdot \left[\frac{-i}{(4\pi)^2} \frac{1}{\Delta} \right] \varepsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta$$

$$\lim_{M\to\infty} \frac{M^2}{\Delta} = 1 \quad \text{then} \quad \int dx\,dy\,dz\,\delta(\,) \cdot 1 = \tfrac{1}{2}$$

$$= -\frac{1}{4\pi^2} e^2 \; \varepsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta$$

adding the second diagram with $p \leftrightarrow k$ $\nu \leftrightarrow \lambda$, we have

$$\langle p,k|\; 2iM\, \bar{\Psi}\gamma^5\Psi\, |0\rangle = -\frac{e^2}{2\pi^2} \varepsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta$$

d.) similarly

$\langle p,k|\; M\bar{\Psi}\Psi\; |0\rangle =$  + 

$$= (-1)(-ie)^2 \int \frac{d^4 l}{(2\pi)^4} \operatorname{tr}\Big\{ M \; \frac{i(\not{l}-\not{k}+M)}{(l-k)^2-M^2} \; \gamma^\lambda$$
$$\frac{i\,(\not{l}+M)}{l^2-M^2} \; \gamma^\nu \; \frac{i\,((\not{l}+\not{p})+M)}{l^2-M^2} \Big\}$$

$$= -ie^2 M \int \frac{d^4\ell}{(2\pi)^4} \, \mathrm{tr} \frac{[(\not\ell - \not k) + M]\gamma^\lambda (\not\ell + M)\gamma^\nu (\not\ell + \not p + M)}{((\ell-k)^2 - M^2)(\ell^2 - M^2)((\ell+p)^2 - M^2)}$$

$$= -ie^2 M \int dx\,dy\,dz\, \delta(x+y+z-1) \cdot 2 \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta]^3}$$

$$\mathrm{tr}\{ [(\not\ell - \not k) + M]\gamma^\lambda (\not\ell + M)\gamma^\nu (\not\ell + \not p + M) \}$$

To obtain a term with $p^\lambda k^\nu$, we need to keep 2 powers of finite momenta and one M in the trace. This gives a finite integral. By gauge invariance, the full structure should be

$$t^{\lambda\nu} = (p\cdot k \, g^{\lambda\nu} - p^\lambda k^\nu)$$

since this automatically satisfies $k_\lambda t^{\lambda\nu} = p_\nu t^{\lambda\nu} = 0$

Better, let's evaluate the full integral by dimensional regularization.

$$\mathrm{tr}\{\} = \mathrm{tr}\{ M\gamma^\lambda \not\ell \gamma^\nu \not\ell + \not\ell \gamma^\lambda M \gamma^\nu \not\ell + \not\ell \gamma^\lambda \not\ell \gamma^\nu M$$
$$+ M^3 \gamma^\lambda \gamma^\nu$$
$$+ (-(1-x)\not k - y\not p)\gamma^\lambda (x\not k - y\not p)\gamma^\nu M$$
$$+ (-(1-x)\not k - y\not p)\gamma^\lambda M \gamma^\nu (x\not k + (1-y)\not p)$$
$$+ M\gamma^\lambda (x\not k - y\not p)\gamma^\nu (x\not k + (1-y)\not p) \}$$

$$
\begin{aligned}
= 4M \{ & (l^\lambda l^\nu + l^\nu l^\lambda - g^{\nu\lambda} l^2) + l^2 g^{\nu\lambda} \\
& + (l^\lambda l^\nu + l^\nu l^\lambda - g^{\nu\lambda} l^2) + M^2 g^{\nu\lambda} \\
& + (2k^\lambda k^\nu - k^2 g^{\nu\lambda})(-x(1-x) + x^2) - x(1-x) k^2 g^{\nu\lambda} \\
& + (2p^\lambda p^\nu - p^2 g^{\nu\lambda})(y^2 - y(1-y)) - y(1-y) p^2 g^{\nu\lambda} \\
& + [k^\lambda p^\nu + k^\nu p^\lambda - g^{\nu\lambda} k\cdot p](y(1-x) - xy + x(1-y) - xy) \\
& + (k^\lambda p^\nu + g^{\mu\nu} k\cdot p - k^\nu p^\lambda)[-(1-x)(1-y)] \\
& + (p^\lambda k^\nu + g^{\mu\nu} k\cdot p - p^\nu k^\lambda)(-xy) \}
\end{aligned}
$$

$$
\begin{aligned}
= 4M \{ & (4 l^\lambda l^\nu - g^{\nu\lambda} l^2) + M^2 g^{\nu\lambda} \\
& + k^\lambda k^\nu (-2x + 4x^2) - k^2 g^{\nu\lambda} x^2 \\
& + p^\lambda p^\nu (-2y + 4y^2) - p^2 g^{\nu\lambda} y^2 \\
& + k^\lambda p^\nu (y - 2xy + x - 2xy - 1 + x + y - \cancel{xy} + \cancel{xy}) \\
& + k^\nu p^\lambda (y - 2xy + x - 2xy + 1 - x - y + \cancel{xy} - \cancel{xy}) \\
& + g^{\mu\nu} k\cdot p (-y + 2xy - x + 2xy - 1 + x + y - xy - xy) \}
\end{aligned}
$$

$$
\begin{aligned}
= 4M \{ & (4 l^\lambda l^\nu - g^{\nu\lambda} l^2) + M^2 g^{\nu\lambda} \\
& + g^{\nu\lambda} (-x^2 k^2 - y^2 p^2 - (1 - 2xy) k\cdot p) \\
& + k^\lambda p^\nu (-1 + 2x + 2y - 4xy) + k^\nu p^\lambda (1 - 4xy) \}
\end{aligned}
$$

$$2\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2-\Delta]^3}\ \mathrm{tr}\{\}$$

$$= \frac{-i}{(4\pi)^{d/2}} \frac{1}{\Delta^{3-d/2}} \cdot 4M \Big\{ -\Gamma(2-\tfrac{d}{2})\big(4\ \underbrace{\tfrac{1}{2} g^{\nu\lambda}}_{2-\frac{d}{2}\, g^{\nu\lambda}} - \tfrac{d}{2} g^{\nu\lambda}\big)\Delta$$

$$+ \Gamma(3-\tfrac{d}{2}) M^2 g^{\nu\lambda} + \Gamma(3-\tfrac{d}{2}) \big[g^{\nu\lambda}(-x^2k^2 - y^2p^2 - (1-2xy)\, k\cdot p$$

$$+ k^\lambda p^\nu (-1 + 2x + 2y - 4xy) + k^\nu p^\lambda (1-4xy)\big]\Big\}$$

$$= \frac{-i\, 4M\, \Gamma(3-\frac{d}{2})}{(4\pi)^{d/2}\, \Delta^{3-d/2}} \Big\{ - g^{\nu\lambda}(\cancel{M^2} - (x-x^2)k^2 - (y-y^2)p^2 - 2xy\, k\cdot p)$$

$$+ g^{\nu\lambda}\cancel{M^2}$$

$$+ g^{\nu\lambda}(-x^2k^2 - y^2p^2 - (1-2xy)\, k\cdot p)$$

$$+ k^\lambda p^\nu(-(1-2x)(1-2y)) + k^\nu p^\lambda (1-4xy)\Big\}$$

take $d \to 4$; this is finite. take $M \to \infty$

$$= -4i \frac{1}{(4\pi)^2} \frac{1}{M} \Big\{ g^{\nu\lambda}\big((x-2x^2)k^2 + (y-2y^2)p^2$$

$$+ [-(1-4xy)\, k\cdot p]\big)$$

$$+ k^\lambda p^\nu(-(1-2x)(1-2y)) + k^\nu p^\lambda (1-4xy)\Big\}$$

now

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\cdot 1 = \tfrac{1}{2}$$

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\cdot (x,y,z) = \tfrac{1}{2}\cdot\left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right)$$

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\cdot x^2 = \int_0^1 dx\; x^2(1-x) = \frac{1}{12}$$

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\;xy = \int_0^1 dx\; x\int_0^{1-x} dy\; y$$

$$= \int_0^1 dx\; x\;\frac{(1-x)^2}{2} = \frac{1}{24}$$

so $\lim_{M\to\infty}\lim_{d\to 4}$

$$\int dx\,dy\,dz\;\delta(x+y+z-1)\cdot\; 2\int\frac{d^dl}{(2\pi)^d}\frac{1}{[l^2-\Delta]^3}\,\mathrm{tr}\{\;\}$$

$$= -4i\,\frac{1}{(4\pi)^2}\cdot\frac{1}{M}\cdot\Big\{ g^{\nu\lambda}\Big[\overbrace{\left(\tfrac{1}{6}-\tfrac{2}{12}\right)}^{0}k^2 + \overbrace{\left(\tfrac{1}{6}-\tfrac{2}{12}\right)}^{0}p^2$$

$$- \left(\tfrac{1}{2} - 4\,\tfrac{1}{24}\right)k\cdot p\Big]$$

$$+ k^\lambda p^\nu\left(-\tfrac{1}{2}+2\cdot\tfrac{1}{6}+2\cdot\tfrac{1}{6}-\tfrac{4}{24}\right) \leftarrow 0$$

$$+ k^\nu p^\lambda\left(\tfrac{1}{2}-\tfrac{4}{24}\right)\Big\}$$

$$= -4i\,\frac{1}{(4\pi)^2}\,\frac{1}{M}\left(-\tfrac{1}{3}\right)\left(g^{\nu\lambda}k\cdot p - p^\lambda k^\nu\right)$$

so finally

$$= \quad -ie^2 M \left(-\frac{4i}{(4\pi)^2 M}\right)\left(-\frac{1}{3}\right)\left(g^{\nu\lambda} k p - p^\lambda k^\nu\right)$$

$$= \quad + \frac{e^2}{12\pi^2}\left(g^{\nu\lambda} k\cdot p - p^\lambda k^\nu\right)$$

adding the second diagram with $p \leftrightarrow k$ $\nu \leftrightarrow \lambda$, we have

$$\langle p,k|\ M\bar{\Psi}\Psi|0\rangle = + \frac{e^2}{6\pi^2}\left(p\cdot k\, g^{\nu\lambda} - p^\lambda k^\nu\right)$$

as required!