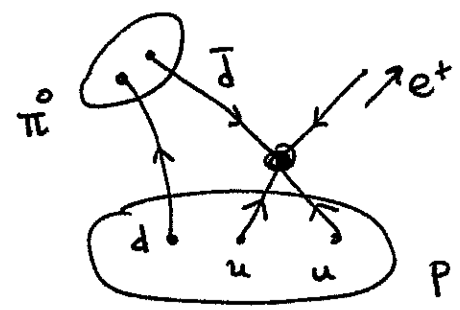
Physics 332 – Problem Set #7

Solutions

1.) a.) consider for definiteness

$p \to \pi^0 e^+$



for a $1 \to 2$ decay, $i\mathcal{M}$ has dimensions of $(\text{mass})^1$

so

$$i\mathcal{M} \sim \frac{2}{m_X^2} \cdot m_p^3$$

we might want to add a factor $g_U^2 \sim 4\pi\left(\frac{g^2}{4\pi}\right) \sim \frac{1}{2}$

$\alpha \sim \frac{1}{24}$

It makes little difference.

$$\Gamma \sim \frac{1}{2m_p}\,\frac{1}{8\pi} \cdot \left|\frac{2m_p^3}{m_X^2}\right|^2$$

$$\sim \frac{1}{4\pi}\,\frac{m_p^5}{m_X^4} \sim \frac{1}{4\pi}\,\frac{(1\ \text{GeV})^5}{(10^{16}\ \text{GeV})^4}$$

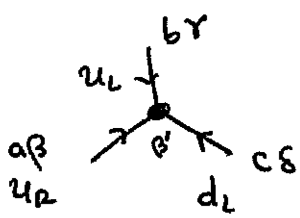
so

$$\Gamma \sim 10^{-65}\,\text{GeV}^{+1} \cdot (6.58 \times 10^{-25}\,\text{GeV-sec})^{-1}$$

$$\sim (10^{41}\,\text{sec})^{-1} \sim (10^{34}\,\text{yr})^{-1}$$

b) for $[\mathcal{O}_\beta]_M = \epsilon_{abc}\,\epsilon^{\gamma\delta}\,u_{Ra\beta}\,u_{Lb\gamma}\,d_{Lc\delta}$

a renormalization condition is



$= \epsilon_{abc}\,\delta_{\beta\beta'}\,\varepsilon_{\gamma\delta}$ at $Q \sim M$

this vertex is corrected by QCD diagrams:

to compute these diagrams, we need the Fierz identities for

$\sigma^\mu = (1, \vec{\sigma})^\mu \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma})^\mu \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$

$$(\sigma^\mu)_{\alpha\beta}\,(\sigma_\mu)_{\gamma\delta} = -2\,\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}$$

check: $(\sigma^\mu \bar{\sigma}^\nu \sigma_\mu)_{\alpha\delta} = -2(\sigma^\nu)_{\alpha\delta} \overset{?}{=} -2\,(i\sigma^2)_{\alpha\gamma}\,(\bar{\sigma}^\nu)^T_{\gamma\beta}\,(i\sigma^2)_{\beta\delta}$

$= +2\,[\sigma^2\,(\bar{\sigma}^\nu)^T\,\sigma^2]_{\alpha\delta}$

$= 2\,(\sigma^\nu)_{\alpha\delta}$

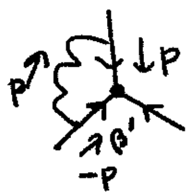
✓

all

$$(\bar{\sigma}^\mu)_{\alpha\beta} \quad (\sigma_\mu)_{\gamma\delta} \quad = \quad 2\,\delta_{\alpha\delta}\,\delta_{\gamma\beta}$$

check: $$(\bar{\sigma}^\mu \cdot 1 \cdot \sigma_\mu)_{\alpha\delta} \overset{?}{=} 2\,\delta_{\alpha\delta}\,\delta_{\gamma\gamma}$$
$$= 4\,\delta_{\alpha\delta} \overset{\checkmark}{=} 4\,\delta_{\alpha\delta}$$

now



$$= (ig)^2 \int \frac{d^dp}{(2\pi)^d} \; \frac{-i}{p^2} \; \varepsilon_{a'b'c}\,(t^A)_{a'a}(t^A)_{b'b}$$
$$\varepsilon_{\gamma'\delta'}\;\delta_{\delta'\delta} \cdot \left(\left(-i\frac{p\cdot\bar{\sigma}}{p^2}\right)\sigma^\mu\right)_{\beta'\beta}$$
$$\cdot \left(\frac{i p\cdot\sigma}{p^2}\;\bar{\sigma}_\mu\right)_{\gamma'\gamma}$$

$$= \; i g^2 \;\; \varepsilon_{\gamma'\delta} \;\; \varepsilon_{a'b'c}\,(t^A)_{a'a}(t^A)_{b'b}$$
$$\cdot \int \frac{d^dp}{(2\pi)^d}\,\frac{1}{(p^2)^3}\,(p\cdot\bar{\sigma})_{\beta'\theta}\;(p\cdot\sigma)_{\gamma'\eta} \cdot 2\,\delta_{\eta\beta}\,\delta_{\theta\gamma}$$

now $$\epsilon_{a'b'c}\,(t^A)_{a'a}\,(t^A)_{b'b} \quad = \quad t^A t^A \quad \text{for} \quad 3\times 3 \to \bar{3}$$
$$= \frac{1}{2}\left[C_2(\bar{3}) - C_2(3) - C(3)\right]$$
$$= \frac{1}{2}\left[\frac{4}{3} - \frac{4}{3} - \frac{4}{3}\right]$$
$$= -\frac{2}{3}$$

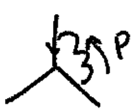
$$= -\frac{2}{3} i g^2 \; \varepsilon_{\gamma'\delta} \, \varepsilon_{abc} \cdot \frac{i}{(4\pi)^2} \frac{\Gamma(2-d/2)}{2} \cdot \frac{1}{2} g^{\alpha\beta}$$

$$\cdot\, 2 \, (\bar{\sigma}_\alpha)_{\beta'\gamma} \, (\sigma_\beta)_{\gamma'\beta}$$

$$= -\frac{2}{3} i g^2 \cdot \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \cdot \frac{1}{4} \cdot 2 \cdot 2 \quad \varepsilon_{abc} \, \varepsilon_{\gamma\delta} \, \delta_{\beta\beta'}$$

$$= \frac{2}{3} g^2 \; \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \quad \delta_{\beta\beta'} \, \epsilon_{\gamma\delta} \, \epsilon_{abc}$$

 gives the same result.

 p

$$= \delta_{\beta\beta'} \; (ig)^2 \int \frac{d^d p}{(2\pi)^d} \; \frac{-i}{p^2} \; \varepsilon_{ab'c'} \, (t^A)_{b'b} (t^A)_{ck}$$

$$\varepsilon_{\gamma'\delta'} \left(\frac{-i\, p\cdot\sigma}{p^2} \, \bar{\sigma}^\mu \right) \left(\frac{+i\, p\cdot\sigma}{p^2} \, \bar{\sigma}_\mu \right)_{\delta'\delta}$$

$$= \left(-\frac{2}{3}\right) \; i g^2 \delta_{\beta'\beta} \int \frac{d^d p}{(2\pi)^d} \; \frac{1}{(p^2)^3} \; \varepsilon_{abc} \, \varepsilon_{\gamma'\delta'} \, (p\cdot\sigma)_{\gamma'\eta} \, (p\cdot\sigma)_{\delta'\zeta} \, (-2 \varepsilon_{\eta\zeta} \, \varepsilon_{\gamma\delta})$$

$$= -i \frac{2}{3} g^2 \delta_{\beta'\beta} \, \varepsilon_{abc} \, \varepsilon_{\gamma'\delta'} \; \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{2} \; \frac{1}{2} g^{\alpha\beta} \; \underbrace{(\sigma^\alpha)_{\gamma'\eta} \, (\sigma_\alpha)_{\delta'\zeta} \, \varepsilon_{\eta\zeta}}_{(-2)} \, \varepsilon_{\gamma\delta}$$

$$= \frac{2}{3} g^2 \, \delta_{\beta\beta'} \, \varepsilon_{abc} \, \varepsilon_{\gamma'\delta'} \, \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \; \frac{1}{4} \left(-2 \, \varepsilon_{\gamma'\delta'} \, \varepsilon_{\eta\delta}\right) \varepsilon_{\eta\delta} \, \varepsilon_{\gamma\delta} (-2)$$

$$= \frac{2}{3} g^2 \, \delta_{\beta\beta'} \, \varepsilon_{abc} \, \varepsilon_{\gamma\delta} \; \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \; \frac{1}{4} \cdot 4 \cdot 4$$

so

 =  $= \frac{2}{3}\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\,\delta_{\beta\beta'}\delta_{\gamma\delta}\epsilon_{abc}$

 $= \frac{8}{3}\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\,\delta_{\beta\beta'}\delta_{\gamma\delta}\epsilon_{abc}$

$$\delta_{O} = -\frac{12}{3}\frac{g^2}{(4\pi)^2}\cdot\frac{2}{\epsilon} = -4\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}$$

recall $\delta_2 = -\frac{g^2}{(4\pi)^2}\cdot\frac{4}{3}\cdot\frac{2}{\epsilon}$

with $2/\epsilon \to \log \Lambda^2/M^2$ $\qquad M\frac{\partial}{\partial M}\left(2/\epsilon\right) = -2$

$$\gamma_O = M\frac{\partial}{\partial M}\left(-\delta_O + \tfrac{3}{2}\delta_2\right)$$

$$= \left(-8 + \tfrac{3}{2}\cdot\tfrac{4}{3}\cdot 2\right)\frac{g^2}{(4\pi)^2}$$

$$\gamma_O = -4\frac{g^2}{(4\pi)^2}$$

writing $\gamma_O = \frac{a_0}{(4\pi)^2}g^2$, this operator is enhanced at large distances by

$$\left[\frac{\log(M_Z^2/\Lambda^2)}{\log(M_p^2/\Lambda^2)}\right]^{a_0/2b_0}$$

ing $\qquad b_0 = 11 - \frac{2}{3} n_f = 7 \quad$ for $n_f = 6$

$m_p = 1\ \text{GeV} \qquad \Lambda = 200\ \text{MeV}$

$M_X = 10^{16}\ \text{GeV}$

the enhancement factor is:

$$\left(\frac{\lg M_X^2/\Lambda^2}{\lg M_p^2/\Lambda^2}\right)^{2/7} = 2.5$$

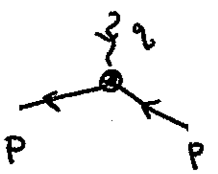
The proton decay rate is increased by the square ~~of~~ this factor

$$\underline{\times 6.}$$

2.) Consider the operator

$$O_f^{\mu_1 \ldots \mu_n}\Big|_M = [\bar{\Psi} \gamma^{\{\mu_1} (iD)^{\mu_2} \cdots (iD)^{\mu_n\}} \Psi]_M$$

at the scale $Q \sim M$. It has matrix element



$$= \gamma^{\{\mu_1} p^{\mu_2} \cdots p^{\mu_n\}}$$

for $q \to 0$ or $|q| \ll |p|$

the vertex obtains QCD correction from

$$= (ig)^2 \int \frac{d^dk}{(2\pi)^d} \frac{-i}{(k-p)^2} \gamma^\lambda t^a \frac{i\not{k}}{k^2} (\gamma^{\mu_1} k^{\mu_2} \cdots k^{\mu_n}) \frac{i\not{k}}{k^2} \gamma_\lambda t^a$$

combine denominators using

$$\left(\frac{1}{k^2}\right)^2 \frac{1}{(k-p)^2} = \int_0^1 dx \, \frac{2(1-x)}{\left[\mathbb{k}^2 + x(1-x)p^2\right]^3}$$

with

$$\mathbb{k} = k - xp \qquad k = \mathbb{k} + xp$$

We need to find a divergent term with the structure $\gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}$.
To do this, set

$$\not{k} \to \not{\mathbb{k}}$$
$$k^{\mu_i} \to x p^{\mu_i}$$

then

$$= -i \, g^2 \underbrace{(t^a t^a)}_{4/3} \int_0^1 dx \, 2(1-x) \int \frac{d^d \mathbb{k}}{(2\pi)^d} \frac{1}{\left[\mathbb{k}^2 + x(1-x)p^2\right]^3}$$
$$\cdot \gamma^\lambda \not{\mathbb{k}} \gamma^{\mu_1} \not{\mathbb{k}} \gamma_\lambda \cdot x^{n-1} p^{\mu_2} \cdots p^{\mu_n}$$

$$= -i \, g^2 \cdot \frac{4}{3} \int_0^1 dx \, 2(1-x) x^{n-1} \frac{i}{(4\pi)^{d/2}} \Gamma\left(\frac{2-d/2}{2}\right) \cdot \frac{1}{2} \gamma^\lambda \gamma^\alpha \gamma^{\mu_1} \gamma_\alpha \gamma_\lambda$$
$$\cdot p^{\mu_2} \cdots p^{\mu_n}$$

Evaluate the residue at $\epsilon = 0$ in $d=4$

$$\gamma^\lambda \gamma^\alpha \gamma^{\mu_1} \gamma_\alpha \gamma_\lambda = (-2)^2 \gamma^{\mu_1}$$

$$= \frac{4}{3} \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} \gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_n} \cdot \underbrace{\int_0^1 dx \; 2(1-x)\, x^{n-1}}_{\downarrow}$$

$$2 \cdot \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{2}{n(n+1)}$$

$$= \frac{8}{3} \frac{1}{n(n+1)} \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} \{\gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}\}$$

!

$$iD^\mu = i\partial^\mu + g A^{\mu a} t^a$$

For the other diagrams, we

$$= \sum_{j=2}^{n} (ig) \cdot g \int \frac{d^d k}{(2\pi)^d} \gamma^\lambda t^a \frac{i \not{k}}{k^2} \gamma^{\mu_1}$$

$$\cdot\; k^{\mu_2} \cdots k^{\mu_{j-1}} t^a g_\lambda^{\ \mu_j} p^{\mu_{j+1}} \cdots p^{\mu_n}$$

$$\cdot\; \frac{-i}{(k-p)^2}$$

$$= \sum_{j=2}^{n} i g^2 \cdot (t^a t^a) \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \gamma^{\mu_j} x \not{p} \gamma^{\mu_1} \frac{1}{(\ell^2 - \Delta)^2}$$

$$\cdot\; \underbrace{(xp)^{\mu_2} \cdots (xp)^{\mu_{j-1}}}_{(j-2\ x\text{'s})} p^{\mu_{j+1}} \cdots p^{\mu_n}$$

now

$$\gamma^{\{\mu_j} \not{p} \gamma^{\mu_1\}} = \frac{1}{2}\left[\gamma^{\mu_j}\not{p}\gamma^{\mu_1} + (\mu_1 \leftrightarrow \mu_j)\right]$$

$$= \frac{1}{2}\left[2p^{\mu_j}\gamma^{\mu_1} - \gamma^{\mu_j}\gamma^{\mu_1} + \mu_1 \leftrightarrow \mu_j\right]$$

$$= 2\gamma^{\{\mu_1} p^{\mu_j\}} - (\text{term} \propto g^{\mu_1\mu_j})$$

$$= \sum_{j=2}^{n} ig^2 \cdot \frac{4}{3} \int_0^1 dx \; \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) \quad x^{j-1} \; 2\gamma^{\mu_1} p^{\mu_2} \text{—} p^{\mu_n}$$

$$= -\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon} \cdot \sum_{j=2}^{n} \frac{2}{j} \cdot \gamma^{\{\mu_1} p^{\mu_2} \text{—} p^{\mu_n\}}$$

$$= \sum_{j=2}^{n} (ig)\cdot g \cdot \int \frac{d^d k}{(2\pi)^d} \; \gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_{j-1}} g^{\mu_j \lambda} t^a$$

$$\cdot \, k^{\mu_{j+1}} \text{—} k^{\mu_n} \; \frac{i\not{k}}{k^2} \gamma_\lambda t^a \quad \frac{-i}{(k-p)^2}$$

$$= \sum_{j=2}^{n} ig^2 \cdot \frac{4}{3} \int_0^1 dx \quad \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2-\Delta]^2}$$

$$\cdot \, \gamma^{\mu_1} p^{\mu_2} \text{—} p^{\mu_{j-1}} \; p^{\mu_{j+1}} \text{—} p^{\mu_n} \quad \not{p}\gamma^{\mu_j} \quad x^{n-j+1}$$

$l = n-j+2$

$$= \sum_{l=n}^{2} ig^2 \cdot \frac{4}{3}\left(\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\right) \int_0^1 dx \; x^{l-1}$$

$$\cdot \; 2\gamma^{\mu_1} p^{\mu_2} \text{—} p^{\mu_n}$$

= (same result as above)

so

$$= \frac{g^2}{(4\pi)^2}\,\frac{2}{\epsilon}\cdot\left(-\frac{4}{3}\right)\cdot\left[\sum_{j=2}^{n}\frac{4}{j} - \frac{2}{n(n+1)}\right]$$

$$\delta_{\mathcal{O}f}^{n} = \frac{g^2}{(4\pi)^2}\,\frac{2}{\epsilon}\,\frac{4}{3}\,\sum_{j=2}^{n}\frac{4}{j} - \frac{2}{n(n+1)}$$

$$\delta_2 = -\frac{4}{3}\,\frac{g^2}{(4\pi)^2}\,\frac{2}{\epsilon}$$

$$\gamma_f^{m} = M\frac{\partial}{\partial M}\left(-\delta_{\mathcal{O}f}^{n} + \delta_2\right)$$

$$= M\frac{\partial}{\partial M}\left(-\frac{g^2}{(4\pi)^2}\cdot\frac{2}{\epsilon}\cdot\frac{4}{3}\left[\left(\sum_{j=2}^{n}\frac{4}{j} - \frac{2}{n(n+1)}\right) + 1\right)\right.$$

$$M\frac{\partial}{\partial M}\left(\frac{2}{\epsilon}\right) = -2$$

$$\gamma_f^{n} = \frac{8}{3}\,\frac{g^2}{(4\pi)^2}\left[1 + \sum_{j=2}^{n}\frac{4}{j} - \frac{2}{n(n+1)}\right]$$

3.) In the problem set, I asked you to work out the Altarelli - Parisi evolution results from

$$\frac{d}{d\lg Q} f(x,Q) = \frac{\alpha_s(Q)}{\pi} \int_x^1 \frac{dz}{z} \, P_{q\leftarrow q}(z) \, f\left(\frac{x}{z}\right)$$

This is the evolution of the non-singlet part of the parton distribution function. For example $(f_u - f_d)$ or $(f_u - f_{\bar{u}})$ obeys this equation.

Integrate $\int_0^1 dx \; x^{n-1}$. Then

$$\frac{d}{d\lg Q} M_n = \frac{\alpha_s(Q)}{\pi} \int_0^1 dx \; x^{n-1} \int_x^1 \frac{dz}{z} \, P_{q\leftarrow q}(z) \, f\left(\frac{x}{z}\right)$$

Let $y = \frac{x}{z}$ $\qquad \frac{dx}{z} = dy$ $\qquad x = yz$

$$= \frac{\alpha_s(Q)}{\pi} \iint dy \, dz \; (yz)^{n-1} \, P_{q\leftarrow q}(z) \, f(y)$$

The region of integration is

$0 < x < 1$

$x < z < 1$ or $y < 1$

so the integral covers $\quad 0 < y < 1$

$\qquad\qquad\qquad\qquad\;\; 0 < z < 1$

then

$$\frac{d}{d\lg Q} M_n = \frac{\alpha_s(Q)}{\pi} \left[\int_0^1 dz\ z^{n-1} P_{q\leftarrow q}(z)\right] \left[\int_0^1 dy\, y^{n-1} f(y)\right]$$

$$= \frac{\alpha_s(Q)}{\pi} \cdot \mathcal{P}_n \cdot M_n$$

where

$$\mathcal{P}_n = \int_0^1 dz\ z^{n-1} P_{q\leftarrow q}(z)$$

$$= \int_0^1 dz\ z^{n-1}\ \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(z-1)\right]$$

$$= \frac{4}{3} \cdot \left\{ \frac{3}{2} + \int_0^1 dz\ \frac{z^{n-1}(1+z^2) - 2}{(1-z)_+} \right\}$$

$$= \frac{4}{3} \cdot \left\{ \frac{3}{2} + \int_0^1 dz\ \frac{(z^{n-1}-1) + (z^{n+1}-1)}{(1-z)} \right\}$$

$$= \frac{4}{3} \left\{ \frac{3}{2} + \int_0^1 dz\ (-1)\left[z^{n-2} + z^{n-3} + \cdots + 1 + z^n + z^{n-1} + z^{n-2} + \cdots + 1\right]\right.$$

$$= \frac{4}{3}\left\{ \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n} - 2\left(\frac{1}{n-1} + \cdots + 1\right)\right\}$$

$$= \frac{4}{3}\left\{ -\frac{1}{2} - 2\sum_2^n \frac{1}{j} + \frac{1}{n} - \frac{1}{n+1}\right\}$$

$$= -\frac{2}{3}\left\{ 1 + 4\sum_2^n \frac{1}{j} - \frac{2}{n(n+1)}\right\}$$

so

$$\frac{d}{d\lg Q} M_n = -\frac{\alpha_s(Q)}{\pi} \cdot \frac{2}{3} \cdot \left\{ 1 + 4\sum_2^n \frac{1}{j} - \frac{2}{n(n+1)} \right\} M_n$$

$$= -\frac{8}{3} \frac{g^2}{(4\pi)^2} \left[1 + 4\sum_2^n \frac{1}{j} - \frac{2}{n(n+1)} \right] M_n$$

or

$$\frac{d}{d\lg Q} M_n = -\gamma_f^n M_n$$