Physics 332 - Problem Set # 6

Solutions

1.) a.)
$$\frac{dg}{d\log q} = -\frac{b_0\, g^3}{(4\pi)^2} - \frac{b_1\, g^5}{(4\pi)^4} + \cdots$$

the solution to

$$\frac{dg^{(0)}}{d\log q} = -\frac{b_0\, g^{(0)3}}{(4\pi)^2} \quad \text{is}$$

$$\frac{d\alpha_s^{(0)}}{d\log q} = -\frac{b_0}{2\pi}\, \alpha_s^{(0)2}$$

$$\alpha_s^{(0)} = \frac{2\pi}{b_0}\, \frac{1}{\log q/\Lambda}$$

now write

$$\alpha_s = \alpha_s^{(0)} + \alpha_s^{(1)}$$

$$\frac{d\alpha_s}{d\log q} = -\frac{b_0}{2\pi}\alpha_s^2 - \frac{b_1}{8\pi^2}\alpha_s^3$$

$$\frac{d\alpha_s^{(1)}}{d\log q} = -\frac{b_0}{2\pi}\cdot 2\,\alpha_s^{(0)}\alpha_s^{(1)} - \frac{b_1}{8\pi^2}\left(\alpha_s^{(0)}\right)^3$$

$$\frac{d\alpha_s^{(1)}}{d\log q/\Lambda} = -2\,\frac{1}{\log q/\Lambda}\,\alpha_s^{(1)} - \pi b_1\,\frac{1}{b_0^3}\,\frac{1}{(\log q/\Lambda)^3}$$

the solution of the homogeneous equation

$$\frac{d\alpha_s^{(1h)}}{d\log q/\Lambda} = -2\,\frac{1}{\log q/\Lambda}\,\alpha_s^{(1h)}$$

is

$$\alpha_s^{(1h)} = \text{const}\cdot\frac{1}{(\log q/\Lambda)^2}$$

so we cannot solve the full equation with a function of this form.

try

$$\alpha_s^{(1)} = \frac{A\,\log(\log q/\Lambda) + B}{(\log q/\Lambda)^2}$$

$$\frac{d\alpha_s^{(1)}}{d\log q/\Lambda} = -\frac{2}{\log q/\Lambda}\,\alpha_s^{(1)} + A\,\frac{1}{(\log q/\Lambda)^3}$$

this solves the equation for $A = -\frac{\pi b_1}{b_0^3}$

then

$$\alpha_s(q) = \frac{2\pi}{b_0}\,\frac{1}{\log q/\Lambda} - \frac{\pi b_1}{b_0^3}\,\frac{\log\log q/\Lambda + (\text{const})}{(\log q/\Lambda)^2}$$

$$= \frac{4\pi}{b_0}\left\{\frac{1}{\log q^2/\Lambda^2} - \frac{b_1}{b_0^2}\,\frac{\log\log q^2/\Lambda^2 + (\text{const})}{\log q^2/\Lambda^2}\right\} + \cdots$$

b.) Now

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma_0 \cdot 3\sum Q_f^2 \cdot \left(1 + \frac{\alpha_s(q)}{\pi} + a_2\left(\frac{\alpha_s(q)}{\pi}\right)^2 + \cdots\right)$$

the factor in parentheses is

$$1 + \frac{4\pi}{\pi b_0}\left(\frac{1}{\lg q^2/\Lambda^2} - \frac{b_1}{b_0^2}\frac{\lg \lg q^2/\Lambda^2 + C}{(\lg q^2/\Lambda^2)^2} + \cdots\right)$$

$$+ \; a_2\left(\frac{4}{b_0}\right)^2 \frac{1}{\lg^2 q^2/\Lambda^2} + \cdots$$

$$= \; 1 + \frac{4}{b_0}\frac{1}{\lg q^2/\Lambda^2} - \frac{4b_1}{b_0^3}\frac{\lg \lg q^2/\Lambda^2}{(\lg q^2/\Lambda^2)^2}$$

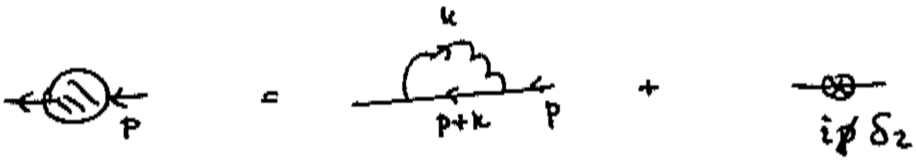
$$+ \; \mathcal{O}\left(\frac{1}{(\lg q^2/\Lambda^2)^2}\right)$$

Since $\sigma(e^+e^- \to \text{hadrons})$ is an observable ~~quantity~~, the factors $4/b_0$ and $-4b_1/b_0^3$ should be independent of the scheme of calculation. Thus, both <u>b_0</u> and <u>b_1</u> should be independent of scheme.

2.) $\beta(g) = M\frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \tfrac{1}{2}\delta_3\right)$

so we need to compute these counterterms. Use Feynman-'t Hooft gauge.

$\underline{\delta_2}$:



$$= (ig)^2\, t^a t^a \int\frac{dk}{(2\pi)}\, \gamma_\mu\, \frac{i(\not p+\not k)}{(p+k)^2}\,\gamma^\mu \left(\frac{-i}{k^2}\right) \qquad \ell = k + xp,\quad k+p = \ell + (1-x)p$$

$$= -g^2 C_2(r)\int_0^1 dx \int\frac{d\ell}{(2\pi)}\,\frac{1}{[\ell^2-\Delta]^2}\,\gamma_\mu\left(\not\ell + (1-x)\not p\right)\gamma^\mu$$

$$= -g^2 C_2(r)\int_0^1 dx \int\frac{d\ell}{2\pi}\,\frac{1}{[\ell^2-\Delta]^2}\cdot(2-d)(1-x)\not p$$

$$= (d-2)\, g^2 C_2(r)\,\frac{i}{(4\pi)^2}\,\Gamma(2-d/2)\,\underbrace{\int_0^1 dx\,(1-x)}_{\frac{1}{2}}\;\not p$$

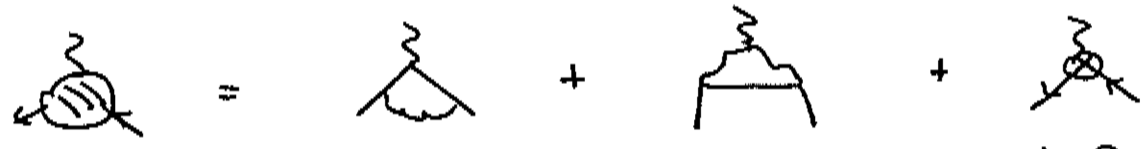
$d\to 4$ $$= \frac{i}{(4\pi)^2}\, g^2 C_2(r)\cdot\left(\frac{2}{\epsilon}\right)$$

so

$$\delta_2 = -\frac{1}{(4\pi)^2}\, g^2 C_2(r)\left(\frac{2}{\epsilon}\right)$$

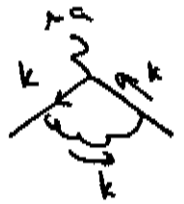
with $\dfrac{2}{\epsilon} \sim \log \Lambda^2/M^2$; $\left(-M\dfrac{\partial}{\partial M}\right)\left(\dfrac{2}{\epsilon}\right) = 2$

$\underline{\delta_1}$:



$ig\,\delta_1$

we can evaluate the diagrams at $\underline{\text{zero}}$ external momentum



$$= (ig)^3 \int \frac{dk}{2\pi}\; \gamma^\lambda t^b \;\frac{i\not k}{k^2}\; \gamma^\mu t^a \;\frac{i\not k}{k^2}\; \gamma_\lambda t^b \;\frac{-i}{k^2}$$

$$= g^3\,(t^b t^a t^b)\;\frac{1}{d}\int \frac{dk}{(2\pi)}\;\frac{1}{(k^2)^2}\;\; \gamma^\lambda \gamma^\sigma \gamma^\mu \gamma_\sigma \gamma_\lambda$$

I have used $k^\alpha k^\beta = \frac{1}{d} k^2 g^{\alpha\beta}$ under the symmetric integral.

$$t^b t^a t^b = t^a t^b t^b - [t^a, t^b]\, t^b$$

$$= t^a\, C_2(r) - i f^{abc} t^c t^b$$

$$= t^a\, C_2(r) - i\left(-\tfrac{1}{2}\right) f^{abc}[t^b, t^c]$$

$$= t^a\, C_2(r) + \tfrac{i}{2}\; i f^{abc} f^{bcd}\, t^d$$

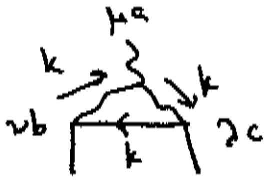
$$= t^a \left[C_2(r) - \tfrac{1}{2} C_2(G)\right]$$

$$\gamma^\lambda \gamma^\sigma \gamma^\mu \gamma_\sigma \gamma_\lambda = (2-d)^2 \gamma^\mu = 4\gamma^\mu \text{ in } \underline{d=4}$$

In all

$$= g^3 \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] \cdot \frac{4}{4} \, \frac{i}{(4\pi)^2} \, \Gamma(2 - d/2) \; t^a \gamma^\mu$$

$$= i \, \frac{g^3}{(4\pi)^2} \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] \cdot \frac{2}{\epsilon} \cdot t^a \gamma^\mu$$



$$= (ig)^2 \cdot g f^{abc} \cdot \int \frac{dk}{2\pi} \; \gamma^\nu t^b \, \frac{i \not{k}}{k^2} \, \gamma^\lambda t^c \left(\frac{-i}{k^2}\right)^2$$

$$\cdot \left[g^{\mu\nu} (-k)^\lambda + g^{\nu\lambda} (2k)^\mu + g^{\lambda\mu} (-k)^\nu \right]$$

$$= i g^3 \left(f^{abc} t^b t^c \right) \int \frac{dk}{2\pi} \; \frac{1}{(k^2)^3}$$

$$\cdot \left\{ -\gamma^\mu \not{k} \not{k} + 2 \gamma^\nu \not{k} \gamma_\nu \, k^\mu - \not{k} \not{k} \gamma^\mu \right\}$$

$$= i g^3 \left(\frac{i}{2} C_2(G) \, t^a \right) \int \frac{dk}{2\pi} \, \frac{1}{(k^2)^3}$$

$$\left\{ -\gamma^\mu k^2 + \frac{2}{d} (2 - d) \, k^2 \gamma^\mu - k^2 \gamma^\mu \right\}$$

$$= - \frac{g^3}{2} \, C_2(G) \, t^a \gamma^\mu \; \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2) \cdot \left(\frac{4}{d} - 4 \right)$$

$$= i \, \frac{3}{2} \, g^3 \, C_2(r) \, \gamma^\mu t^a \, \frac{1}{(4\pi)^2} \, \frac{2}{\epsilon}$$

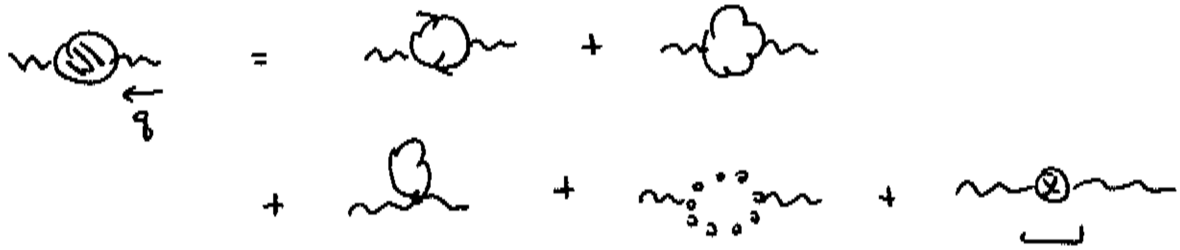
then

$$i\,\frac{g^3}{(4\pi)^2}\left(C_2(r)-\tfrac{1}{2}C_2(G)\right)t^a\gamma^\mu\cdot\frac{2}{\epsilon}$$

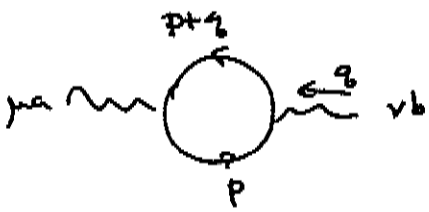
$$+\;i\,\frac{g^3}{(4\pi)^2}\,\tfrac{3}{2}C_2(G)\,t^a\gamma^\mu\cdot\frac{2}{\epsilon}+i\,g\,\delta_1 \;=\; (\text{no divergences})$$

$$\delta_1 \;=\; -\,\frac{g^2}{(4\pi)^2}\left(C_2(r)+C_2(G)\right)\cdot\frac{2}{\epsilon}$$

$\underline{\delta_3}$:



$$-i(q^2g^{\mu\nu}-q^\mu q^\nu)\,\delta^{ab}\,\delta_3$$



$$= \;(ig)^2\underbrace{(-1)}_{\text{fermions!}}\,\mathrm{tr}\,t^at^b\int\frac{d^4p}{(2\pi)^4}$$

$$\cdot\,\mathrm{tr}\left[\gamma^\mu\,\frac{i(\not p+\not q)}{(p+q)^2}\,\gamma^\nu\,\frac{i\not p}{p^2}\right]$$

In all of these graphs, we will combine denominators with

$$\mathbb{P} = p+xq$$

$$p = \mathbb{P}-xq \qquad p+q = \mathbb{P}+(1-x)q$$

$$2p+q = 2\mathbb{P}+(2-x)q$$

$$= \quad -g_s^2 C(r)\, \delta^{ab} \int \frac{d\mathbb{P}}{2\pi} \int dx \; \frac{1}{[\mathbb{P}^2 + x(1-x) q^2]^2}$$

$$\cdot \, \mathrm{tr}\, [\gamma^\mu \, (\not{\mathbb{P}} + (1-x)\not{q}) \, \gamma^\nu \, (\not{\mathbb{P}} - x\not{q})]$$

$$= \quad -g^2 C(r) n_f \delta^{ab} \int_0^1 dx \int \frac{d\mathbb{P}}{2\pi} \; \frac{1}{[\mathbb{P}^2 + x(1-x) q^2]^2}$$

$$\cdot \, 4 \, [2\mathbb{P}^\mu \mathbb{P}^\nu - g^{\mu\nu} \mathbb{P}^2 - 2x(1-x) q^\mu q^\nu + x(1-x) g^{\mu\nu} q^2]$$

$$= \quad -4 g_s^2 n_f C(r)\, \delta^{ab} \int_0^1 dx \quad \frac{i}{(4\pi)^{d_L}} \frac{1}{[-x(1-x)q^2]^{2-d_L}}$$

$$\Big\{ -\Gamma(1-d_L) \cdot [-x(1-x) q^2] \cdot \left(2 \cdot \tfrac{1}{2} g^{\mu\nu} - \tfrac{d}{2} g^{\mu\nu}\right)$$

$$+ \Gamma(2-d_L) \, (-2x(1-x) q^\mu q^\nu + x(1-x) g^{\mu\nu} q^2) \Big\}$$

now $\quad \Gamma(1-d_L) \cdot (1-d_L) = \Gamma(2-d_L)$

$$= \quad -4 g_s^2 n_f C(r)\, \delta^{ab} \int_0^1 dx \; \frac{i}{(4\pi)^{d_L}} \frac{1}{[-x(1-x) q^2]^{2-d_L}} \cdot \Gamma(2-d_L)$$

$$\cdot \, \{ x(1-x) q^2 g^{\mu\nu} \cdot 2 - 2x(1-x) q^\mu q^\nu \}$$

$$= \quad -8i \, g_s^2 n_f C(r) \, \delta^{ab} \, \frac{1}{(4\pi)^{d_L}} \Gamma(2-d_L)$$

$$\cdot \int_0^1 dx \;\; x(1-x) \, \frac{1}{[-x(1-x)q^2]^{2-d_L}}$$

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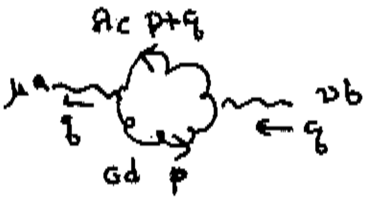
$$= -8i\, g^2 n_f C(r)\, \delta^{ab}\, \frac{1}{(4\pi)^2}\cdot\frac{2}{\epsilon}\cdot\frac{1}{6}$$

$$= i g^2 \frac{C(r) n_f \delta^{ab}}{(4\pi)^2}\cdot\left(-\frac{4}{3}\right)\cdot\frac{2}{\epsilon}$$



$$g f^{acd}\, g f^{bcd}\cdot \overset{\text{sym. factor}}{\frac{1}{2}} \int \frac{d^d p}{(2\pi)^d}\, \frac{-i}{(p+q)^2}\, \frac{-i}{p^2}$$

$$\cdot\left[g^{\mu\lambda}(-q-(p+q))^\sigma + g^{\lambda\sigma}[(p+q)+p]^\mu + g^{\sigma\mu}[-p+q]^\lambda \right]$$

$$\left[g^{\nu\lambda}[q+(p+q)]^\sigma + g^{\lambda\sigma}[-(p+q)-p]^\nu + g^{\sigma\nu}[p-q]^\lambda \right]$$

$$= -\frac{g^2}{2} C_2(G)\, \delta^{ab} \int_0^1 dx \int \frac{d^d \mathbb{P}}{(2\pi)^d}\, \frac{1}{(\mathbb{P}^2 + x(1-x) q^2]^2}$$

$$\{ g^{\mu\nu}[-(2q+p)^2 - (p-q)^2] + (-d)\,(2p+q)^\mu (2p+q)^\nu$$

$$+ (p+2q)^\mu (2p+q)^\nu + (p+2q)^\nu (2p+q)^\mu - (p-q)^\mu (2q+p)^\nu$$

$$- (p-q)^\nu (2q+p)^\mu + (2p+q)^\mu (p-q)^\nu + (2p+q)^\nu (p-q)^\mu \}$$

$$= -\frac{g^2}{2} C_2(G)\, \delta^{ab} \int_0^1 dx \int \frac{d^d \mathbb{P}}{(2\pi)^d}\, \frac{1}{[\mathbb{P}^2 + x(1-x) q^2]^2}$$

$$\{ g^{\mu\nu}[-2p^2 - 3p\cdot q - 5q^2] - d\, (2p+q)^\mu (2p+q)^\nu$$

$$+ p^\mu p^\nu \{2+2-1-1+2+2\} + (p^\mu q^\nu + p^\nu q^\mu)[5-1-1]$$

$$+ q^\mu q^\nu \{2+2+2+2-1-1\} \}$$

$$= -\frac{g^2}{2} C_2(G)\, \delta^{ab} \int_0^1 dx \int \frac{d^d \mathbb{P}}{(2\pi)^d} \frac{1}{[\mathbb{P}^2 + x(1-x) q^2]^2}$$

$$\{ - g^{\mu\nu} [(p+2q)^2 + (p-q)^2] - d\ (2p+q)^\mu (2p+q)^\nu$$

$$+ 6 p^\mu p^\nu + 3(p^\mu q^\nu + p^\nu q^\mu) + 6 q^\mu q^\nu \}$$

substitute $p = \mathbb{P} - xq$ & drop terms linear in $\mathbb{P}$

$$= -\frac{g^2}{2} C_2(G)\, \delta^{ab} \int_0^1 dx \int \frac{d^d \mathbb{P}}{(2\pi)^d} \frac{1}{[\mathbb{P}^2 + x(1-x) q^2]^2}$$

$$\{ - g^{\mu\nu} [\mathbb{P}^2 + (2-x)^2 q^2 + \mathbb{P}^2 + (1+x)^2 q^2]$$

$$- d\ [4 \mathbb{P}^\mu \mathbb{P}^\nu + (1-2x)^2 q^\mu q^\nu]$$

$$+ 6(\mathbb{P}^\mu \mathbb{P}^\nu + x^2 q^\mu q^\nu) + 3(-x)\cdot 2\, q^\mu q^\nu$$

$$+ 6 q^\mu q^\nu \}$$

$$= -\frac{g^2}{2} C_2(G)\, \delta^{ab} \int_0^1 dx \frac{i}{[-x(1-x)q^2]^{2-d/2}} \frac{1}{(4\pi)^{d/2}}$$

$$\{ - \Gamma(1-\tfrac{d}{2}) [-x(1-x)q^2] \left(-g^{\mu\nu} \tfrac{d}{2}\cdot 2 - 4d\cdot \tfrac{1}{2} g^{\mu\nu} + 6 \cdot \tfrac{1}{2} \cdot g^{\mu\nu} \right)$$

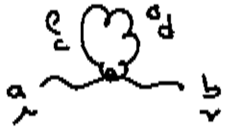
$$+ \Gamma(2-\tfrac{d}{2}) \left(-g^{\mu\nu} [(2-x)^2 q^2 + (1+x)^2 q^2] \right.$$

$$\left. + q^\mu q^\nu \left(-d(1-2x)^2 + 6x^2 - 6x + 6 \right) \right)$$

$$= -i \frac{g^2}{2} \frac{1}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int_0^1 dx \frac{1}{(-x(1-x)q^2)^{2-d/2}}$$

$$\{ \Gamma(1-\tfrac{d}{2})\, x(1-x) q^2 g^{\mu\nu} [-3d + 3]$$

$$+ \Gamma(2-\tfrac{d}{2}) [-g^{\mu\nu} q^2 (5 - 2x + 2x^2) + q^\mu q^\nu (-d(1-2x)^2 + 6(x^2 - x + 1))] \}$$



$$= \quad -ig^2 \cdot \frac{1}{2} \cdot \int \frac{d^dp}{(2\pi)^d} \quad \frac{-i}{p^2}\, g^{\rho\sigma}\, \delta^{cd}$$

$$\Big[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$

$$+ f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho})$$

$$+ f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \Big]$$

$$= \quad -\frac{g^2}{2} \int \frac{d^dp}{(2\pi)^d} \; \frac{1}{p^2} \cdot \frac{(p+q)^2}{(p+q)^2}$$

$$[\, 0 \; + g^{\mu\nu} C_2(G)\, \delta^{ab}\, (d-1) \; \cdot 2 \,]$$

$P + (1-x)q$ $\quad$ $(P - xq)$

$$= \quad -g^2\, C_2(G)\, \delta^{ab}\, (d-1) \int \frac{d^dp}{(2\pi)^d} \quad g^{\mu\nu}_{(d-1)} \frac{[(P+q)^2 + P^2]\cdot\frac{1}{2}}{P^2 (P+q)^2}$$

$$= \quad -g^2\, C_2(G)\, \delta^{ab} (d-1)\, g^{\mu\nu} \; \frac{i\,(d-1)}{(4\pi)^{d/2}} \cdot \int_0^1 dx \; \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

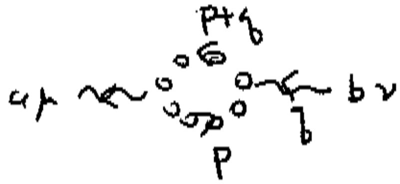
$$\Big\{ -\Gamma(1-d/2)\,(-x(1-x)q^2) \cdot 2\, \frac{d}{2} \cdot \frac{1}{2}$$

$$+ \; \Gamma(2-d/2) \cdot \frac{1}{2} [(1-x)^2 q^2 + x^2 q^2] \Big\}$$

$$= \quad -\frac{ig^2}{2}\, C_2(G)\, \delta^{ab}\, \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \; \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

$$\Big\{ \Gamma(1-d/2) \cdot x(1-x)q^2 \cdot g^{\mu\nu} \cdot d\,(d-1)$$

$$+ \; \Gamma(2-d/2) \;\; g^{\mu\nu} q^2 (d-1)(1 - 2x + 2x^2) \Big\}$$



$$= (-1)(-g)^2 f^{cad} f^{dbc} \int \frac{d^d p}{(2\pi)^d} \frac{i}{(p+q)^2}\frac{i}{p^2} \cdot (p+q)^\nu p^\mu$$

$$= -g^2 C_2(G)\,\delta^{ab} \int_0^1 dx \int \frac{d^d P}{(2\pi)^d} \frac{1}{[P^2+x(1-x)q^2]^2}\left(P^\mu P^\nu - x(1-x) q^\mu q^\nu\right)$$

$$= -g^2 C_2(G)\,\delta^{ab} \int_0^1 dx \; \frac{i}{(4\pi)^{d/2}} \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

$$\left(-\Gamma(1-\tfrac{d}{2})[-x(1-x)q^2]\tfrac{1}{2} g^{\mu\nu} + \Gamma(2-\tfrac{d}{2})(-x(1-x)q^\mu q^\nu)\right)$$

$$= -i\frac{g^2}{2}\frac{1}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab} \int_0^1 dx \; \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

$$\left(\Gamma(1-\tfrac{d}{2})\, x(1-x) q^2 g^{\mu\nu} + \Gamma(2-\tfrac{d}{2})(-2x(1-x) q^\mu q^\nu)\right)$$

$\Rightarrow$ all

 +  + 

$$= -i\frac{g^2}{2}\frac{1}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab} \int_0^1 dx \; \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

$$\Big\{ \Gamma(1-\tfrac{d}{2})\, x(1-x) q^2 g^{\mu\nu} \left(-3d+3+d^2-d+1\right)$$

$$+ \Gamma(2-\tfrac{d}{2})\, q^2 g^{\mu\nu}\left(-5+2x-2x^2+(d-1)(1-2x+2x^2)\right)$$

$$+ \Gamma(2-\tfrac{d}{2})\, q^\mu q^\nu \left(-d(1-2x)^2 + (1-2x)^2 + (5-2x+2x^2) - 2x(1-x)\right)\Big\}$$

now $\Gamma(1-\frac{d}{2})(d^2-4d+4) = \Gamma(1-\frac{d}{2})\cdot 4(1-\frac{d}{2})^2 = \Gamma(2-\frac{d}{2})(4-2d)$

so

$$\{\ \} = \Gamma(2-d/2)\Big([2-2(d-1)]\, x(1-x)\, q^2 g^{\mu\nu}$$
$$+ [(d-1)(1-2x+2x^2) - 5 + 2x - 2x^2]\, q^2 g^{\mu\nu}\Big)$$
$$+ \Gamma(2-d/2)(-q^\mu q^\nu)\big[(d-1)(1-2x)^2 + (-5+2x-2x^2)$$
$$+ 2x(1-x)\big]$$

$$= \Gamma(2-d/2)(q^2 g^{\mu\nu} - q^\mu q^\nu)\{(d-1)(1-2x)^2 - 5 + 4x - 4x^2\}$$

the result is transverse; doing the $\int dx$ we find

$$\int_0^1 dx \ \{\ \} = \frac{2}{\epsilon}\cdot(q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot\Big\{3\cdot\left(1 - \frac{4}{2} + \frac{4}{3}\right)$$
$$- 5 + \frac{4}{2} - \frac{4}{3}\Big\}$$

$$= \frac{2}{\epsilon}(q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot\Big\{1 - 3 - \frac{4}{3}\Big\}$$

$$= \frac{2}{\epsilon}(q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot\Big\{-\frac{10}{3}\Big\}$$

so

$$= \frac{i g^2}{(4\pi)^2} C_2(G)\, \delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot\frac{5}{3}\cdot\frac{2}{\epsilon}$$

adding this to the result at the top of p.9 and

cancelling w. δ_3, we find:

$$\delta_3 = \frac{g^2}{(4\pi)^2}\left[\frac{5}{3} C_2(G) - \frac{4}{3} C(r) n_f\right] \cdot \frac{2}{\epsilon}$$

Then, finally

$$\beta(g) = M\frac{\partial}{\partial M}\left(-g\delta_1 + g\delta_2 + \frac{1}{2} g\delta_3 \right)$$

using $\quad M\frac{\partial}{\partial M}\frac{2}{\epsilon} = -2$

$$\beta = -2\frac{g^3}{(4\pi)^2}\left[\cancel{C_2(r)} + C_2(G)\right] + \cancel{2\frac{g^3}{(4\pi)^2} C_2(r)}$$

$$- \frac{1}{2}\cdot 2 \frac{g^3}{(4\pi)^2}\left[\frac{5}{3} C_2(G) - \frac{4}{3} C(r) n_f\right]$$

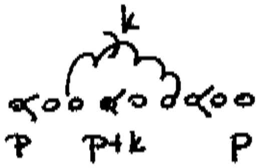
$$\beta = -\left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r)\right]\frac{g^3}{(4\pi)^2}$$

For reference in the next problem:

$$\delta_1 - \delta_2 = -\frac{g^2}{(4\pi)^2} C_2(G) \cdot \frac{2}{\epsilon}$$

3.) a.) The ghost field strength counterterm is given by

$i p^2 \delta_{2g}$



$$= -g f^{acd} f^{dcb} \int \frac{d^dk}{(2\pi)^d}\, p^\mu \frac{i}{(p+k)^2} (p+k)_\mu \frac{-i}{k^2}$$

$$= (-g)^2(-C_2(G))\, \delta^{ab} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 + x(1-x)p^2]^2} \cdot$$

$k = \ell - xp$

$p+k = \ell + (1-x)p$

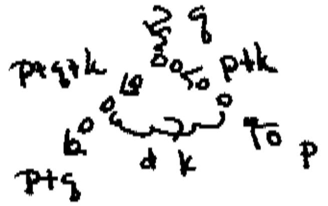
$$\cdot\, p^\mu (\ell + (1-x)p)_\mu$$

$$= \frac{-i g^2}{(4\pi)^2} C_2(G)\, \delta^{ab} \cdot \frac{2}{\epsilon} \cdot \underbrace{\int dx\, (1-x)}_{\frac{1}{2}}\, p^2$$

so $$\delta_{2g} = \frac{1}{2} C_2(G) \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon}$$

The ghost vertex counterterm is given by

$-g f^{abc} (p+q)^\mu \delta_{1g}$



$$= (-g f^{ade})(-g f^{ebf})(-g f^{fdc})$$

$$\cdot \int \frac{d^dk}{(2\pi)^d} \frac{i}{(p+k)^2} \frac{i}{(p+q+k)^2} \frac{-i}{k^2} (p+q)^\lambda (p+q+k)^\mu (p+k)_\lambda$$

$$= -i g^3 (f^{ade} f^{ebf} f^{fdc}) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p+k)^2 (p+q+k)^2 k^2}$$

$$\cdot (p+q)^\lambda (p+q+k)^\mu (p+k)_\lambda$$

To simplify the product of f's use:

$$[f^{ade} f^{ebf} + f^{dbe} f^{eaf} + f^{bae} f^{edf}] f^{fdc} = 0 \quad \text{by the Jacobi id.}$$

$$f^{ade} f^{ebf} f^{fdc} + f^{afe}(-f^{ebd})(-f^{dfc}) + (-f^{abe}) f^{edf}(-f^{dfc}) = 0$$

$$2 f^{ade} f^{ebf} f^{fdc} + f^{abc} C_2(G) = 0$$

so

$$f^{ade} f^{ebf} f^{fdc} = \left(-\tfrac{1}{2} C_2(G)\right) f^{abc}$$

To simplify the numerator, combine denominators with Feynman parameters

$$\text{Denom} = k^2 + x k\cdot p + x p^2 + y\, k\cdot(p+q) + y (p+q)^2$$

$$= \ell^2 - [xp + y(p+q)]^2 + x p^2 + y (p+q)^2$$

$$= \ell^2 + x\underbrace{(1-x)}_{y+z} p^2 + y \underbrace{(1-y)}_{x+z} (p+q)^2 - 2xy\, p\cdot(p+q)$$

Denom $= \ell^2 + xzp^2 + yz(p+q)^2 + xyq^2$

$= \ell^2 - \Delta \qquad -\Delta = xzp^2 + yz(p+q)^2 + xyq^2$

$k = \ell - xp - y(p+q) = \ell - (1-z)p - yq$

$k+p = \ell + zp - yq$

$k+p+q = \ell + zp + (1-y)q$

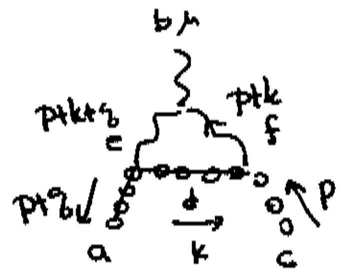
then

$$= -ig^3\left(-\tfrac{1}{2} C_2(G) f^{abc}\right) \int dx\,dy\,dz\,\delta() \cdot 2 \cdot \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^3}$$

$$\cdot \left\{ (p+q)^\lambda \ell^\mu \ell^\lambda + (\text{finite}) \right\}$$

$$= +\frac{ig^3}{2} C_2(G) f^{abc} (p+q)^\lambda \cdot \frac{i}{(4\pi)^{d/2}} \underbrace{\int dx\,dy\,dz\,\delta()}_{1/2} \cdot \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \cdot \frac{1}{2} g^{\lambda\mu}$$

$$= -\frac{g^3}{8}(p+q)^\mu f^{abc} C_2(G) \frac{1}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$



$$= (-g f^{aed})(-g f^{dfc})(g f^{ebf})$$

$$\cdot \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2} \frac{-i}{(p+k)^2} \frac{-i}{(p+k+q)^2} (p+q)^\lambda (-k)^\eta$$

$$\left[g^{\lambda\mu}(-(p+k+q)-q)^\eta + g^{\mu\eta}(q-(p+k))^\lambda + g^{\lambda\eta}(p+k+p+k+q)^\mu \right]$$

$$= \quad i g^3 \left(f^{ade} f^{ebf} f^{fdc}\right) \int dx\,dy\,dz\,\delta(\,) \cdot 2 \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta]^3}$$

$$\cdot \; (p+q)^\lambda \, \ell^\nu \left[g^{\lambda\mu}(-\ell)^\nu + g^{\mu\nu}(-\ell)^\lambda + g^{\lambda\nu}\, 2\ell^\mu \right]$$

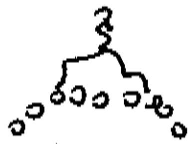
$$+ \text{finite}$$

$$= \quad i g^3 \left(f^{ade} f^{ebf} f^{fdc}\right) \cdot \frac{1}{2} \cdot \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2)$$

$$\left[(p+q)^\mu \left(-\frac{d}{2}\right) + (p+q)^\mu \left(-\frac{1}{2}\right) + (p+q)^\mu \cdot 2 \cdot \frac{1}{2} \right]$$

$$= \quad + \frac{(-g^3)}{(4\pi)^2} \cdot \frac{1}{2} \left(-\frac{1}{2} C_2(G) f^{abc} \right) (p+q)^\mu \, \frac{2}{\epsilon} \cdot \left[-\frac{(d-1)}{2} \right]$$

$\approx$ all


$+$


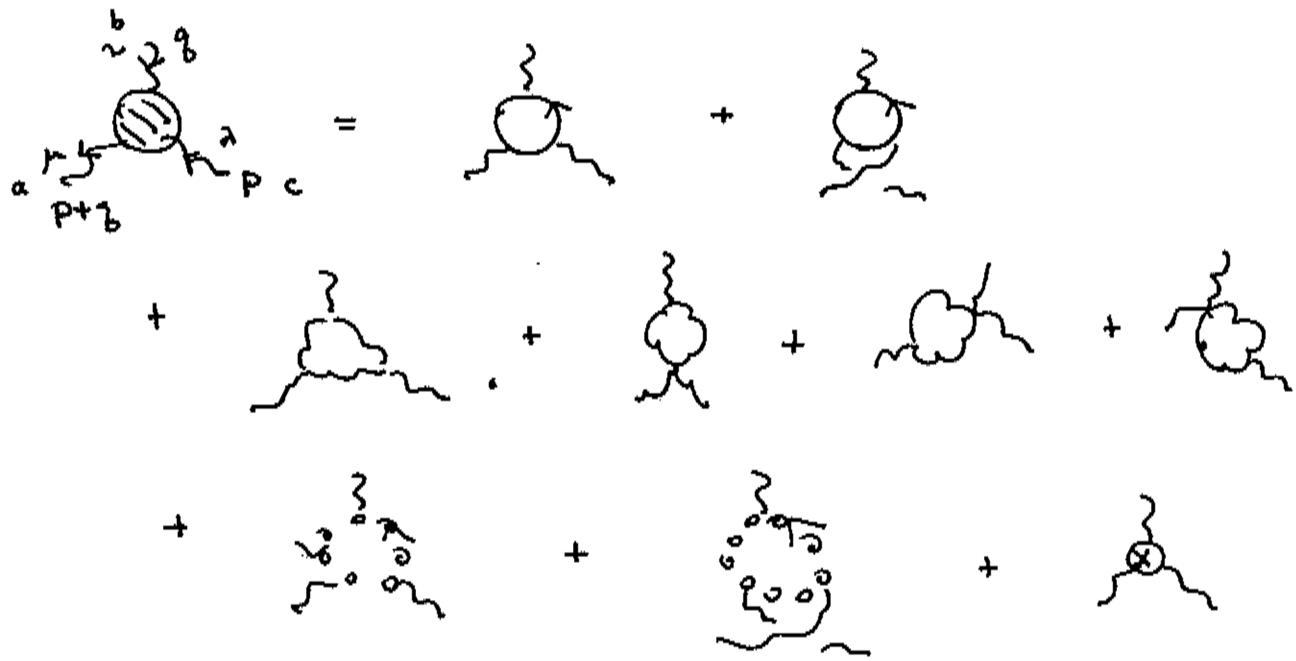
$$= \quad \left[\frac{g^3 C_2(G)}{(4\pi)^2} (p+q)^\mu f^{abc} \cdot \frac{2}{\epsilon} \right] \Big(\underbrace{-\frac{1}{8} - \frac{3}{8}}_{-\frac{1}{2}} \Big)$$

so $$\delta_{1g} = -\frac{1}{2} \frac{g^2}{(4\pi)^2} C_2(G) \cdot \frac{2}{\epsilon}$$

$$\delta_{1g} - \delta_{2g} = - \frac{g^2}{(4\pi)^2} C_2(G) \cdot \frac{2}{\epsilon}$$

as required!

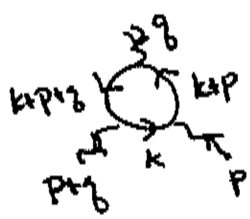
b.) The 3-gluon vertex counterterm is given by



with

$$= g f^{abc}\left[g^{\mu\nu}(-p-2q)^{\lambda} + g^{\nu\lambda}(q-p)^{\mu} + g^{\lambda\mu}(2p+q)^{\nu}\right]$$

$$\times\ \delta_1^{3g}$$



$$= (ig)^3\, \mathrm{tr}[t^a t^b t^c] \int \frac{d^dk}{(2\pi)^d}\ \frac{i}{(k+p)^2}\, \frac{i}{(k+p+q)^2}\, \frac{i}{k^2}$$

$$\cdot \underbrace{(-1)}_{\text{fermion}} \cdot\ \mathrm{tr}\left[\gamma^{\mu}(\not k+\not p+\not q)\,\gamma^{\nu}\,(\not k+\not p)\,\gamma^{\lambda}\,\not k\right]$$

$$= +\, g^3\, \mathrm{tr}[t^a t^b t^c] \int dx\,dy\,dz\ \delta(x+y+z-1)\cdot 2 \int \frac{d^d\ell}{(2\pi)^d}\ \frac{1}{(\ell^2-\Delta)^3}$$

$$\cdot\, \mathrm{tr}\left[\gamma^{\mu}(\not\ell + z\not p + (1-y)\not q)\,\gamma^{\nu}\,(\not\ell + z\not p - y\not q)\,\gamma^{\lambda}\,(\not\ell - (1-z)\not p - y\not q)\right]$$

to get a divergence, we must take two factors of k in the numerator. Then we find (using expressions on p. 17)

$$= g^3 \, tr(t^a t^b t^c) \int dx\,dy\,dz\, \delta(\,) \cdot \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \Gamma(2-\tfrac{d}{2}) \cdot \tfrac{1}{2}$$

$$\{ \; tr[\gamma^\mu [zp + (1-y)q]\, \gamma^\nu \gamma^\alpha \gamma^\lambda \gamma_\alpha] \quad \leftarrow \text{evaluate in } d=4 \quad \gamma^\alpha\gamma^\lambda\gamma_\alpha = -2\gamma^\lambda$$

$$+ tr[\gamma^\mu \gamma^\alpha \gamma^\nu (zp - yq) \gamma^\lambda \gamma_\alpha]$$

$$+ tr[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma_\alpha \gamma^\lambda (-(1-z)p - yq)] \}$$

$$= \frac{ig^3}{2(4\pi)^2} tr(t^a t^b t^c) \int dx\,dy\,dz\, \delta(\,) \cdot \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}$$

$$\{(-8)\} \cdot \{ (zp^\mu + (1-y)q^\mu)\, g^{\nu\lambda} - (zp^\lambda + (1-y)q^\lambda)\, g^{\mu\nu}$$

$$+ (zp^\nu + (1-y)q^\nu)\, g^{\mu\lambda}$$

$$+ (-(zp^\mu - yq^\mu))\, g^{\nu\lambda} + (zp - yq)^\lambda g^{\mu\nu}$$

$$+ (zp - yq)^\nu g^{\mu\lambda}$$

$$+ (-(1-z)p - yq)^\mu g^{\nu\lambda} + (-(1-z)p - yq)^\lambda g^{\mu\nu}$$

$$+ ((1-z)+yq)^\nu g^{\mu\lambda} \}$$

$$\{\} = g^{\nu\lambda}[-(1-z)p + (1-y)q]^\mu + g^{\mu\nu}[-(1-z)p - (1+y)q]^\lambda$$

$$+ g^{\mu\lambda}((1+z)p + (1-y)q)^\nu$$

using

$$\int dx\,dy\,dz\;\delta(1-x-y-z)\cdot 1 = \tfrac{1}{2}$$

$$\int dx\,dy\,dz\;\delta(1-x-y-z)\cdot x = \tfrac{1}{2}\cdot\tfrac{1}{3} = \tfrac{1}{6}$$

$$\int dx\,dy\,dz\;\delta(1-x-y-z)\{\} = \tfrac{1}{3}\left(g^{\mu\nu}[-p-2q]^{\lambda} + g^{\nu\lambda}[q-p]^{\mu} + g^{\lambda\mu}(2p+q)^{\nu}\right)$$

$\therefore$ all

 $$= \frac{g^3}{2}\,\mathrm{tr}\, t^a t^b t^c \cdot \frac{i}{(4\pi)^{d_2}}\,\Gamma(2-d_2)\,(-8)\cdot\left(\tfrac{1}{3}\right)$$

$$\cdot\left[g^{\mu\nu}(-p-2q)^{\lambda} + g^{\nu\lambda}(q-p)^{\mu} + g^{\lambda\mu}(2p+q)^{\nu}\right]$$

similarly, for

 we exchange $a \leftrightarrow c$, $p \leftrightarrow -(p+q)$, $\mu \leftrightarrow \lambda$

$$(-p-2q) \to p-q$$
$$(q-p) \to p+2q$$
$$(2p+q) \to -(2p+q)$$

$$= \frac{g^3}{2}\,\mathrm{tr}\, t^c t^b t^a\;\frac{i}{(4\pi)^{d_2}}\,\Gamma(2-d_2)\,\left(-\tfrac{8}{3}\right)$$

$$\left[g^{\lambda\nu}(p-q)^{\mu} + g^{\nu\mu}(p+2q)^{\lambda} + g^{\mu\lambda}(-(2p+q)^{\nu})\right]$$

so

 +  $$= \frac{-i\,g^3}{(4\pi)^2}\cdot\frac{4}{3}\,\Gamma(2-d_2)$$

$$\left[g^{\mu\nu}(-p-2q)^{\lambda} + g^{\nu\lambda}(q-p)^{\mu} + g^{\lambda\mu}(2p+q)^{\nu}\right]$$

$$\cdot\;\mathrm{tr}\left[t^a t^b t^c - t^c t^b t^a\right]$$

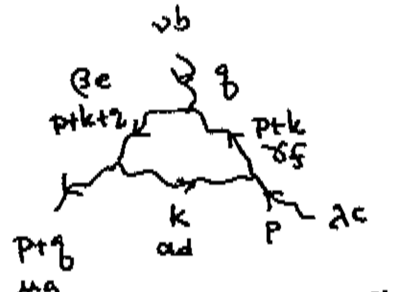
with, finally,

$$\mathrm{tr}\,[t^a t^b t^c - t^c t^b t^a] = \mathrm{tr}\, t^a [t^b, t^c] = \mathrm{tr}\,[t^a \, i f^{bcd} t^d]$$

$$= i f^{abc} \, n_f \, C(r)$$

in all

 +  $= \dfrac{g^3}{(4\pi)^2} f^{abc} \left(\frac{4}{3} n_f C(r)\right) \cdot \dfrac{2}{\epsilon} \cdot [g^{\mu\nu}(-p-2q)^\lambda + \ldots]$



$$= g^3 f^{ade} f^{ebf} f^{fdc} \int \frac{d^dk}{(2\pi)^d} \frac{-i}{(k+p)^2} \frac{-i}{(k+p+q)^2} \frac{-i}{k^2}$$

$$[\, g^{\mu\alpha}[-(p+q)+k]^\beta + g^{\alpha\beta}[-k-(p+k+q)]^\mu + g^{\beta\mu}[p+k+q+p+q]^\alpha \,]$$

$$\cdot [\, g^{\beta\nu}[-(p+k+q)-q]^\gamma + g^{\nu\gamma}[q-(p+k)]^\beta + g^{\gamma\beta}[p+k+p+k+q]^\nu \,]$$

$$\cdot [\, g^{\gamma\alpha}[-(p+k)-k]^\lambda + g^{\alpha\lambda}[k-p]^\gamma + g^{\lambda\gamma}[p+p+k]^\alpha \,]$$

Evaluate this as above: combine denominators, use the formulae on p. 17 to represent momenta in the numerators, extract the terms proportional to $I(2-\frac{d}{2})$. These terms contain one power of finite momenta in the numerator and two powers of ℓ. The finite terms can come from either the first, the second, or the third line above. Using

$$\ell^\alpha \ell^\beta = \frac{1}{d} g^{\alpha\beta} \ell^2$$

we find $\underbrace{\int dx\,dy\,dz\,\delta(\,)}_{\int dF \;\leftarrow\; \text{(Feynman parameters)}} \cdot [\;][\;][\;] = ① + ② + ③$

$$① \quad = \int dF \quad \left[g^{\mu\alpha}(-p - q - (1-z)p - yq)^{\beta} + g^{\alpha\beta}(-p-q+2(1-z)p+2yq)^{\mu} + g^{\beta\mu}(2p+2q-(1-z)p-yq)^{\alpha} \right]$$

$$\cdot \left[g^{\beta\nu}(-k^{\gamma}) + g^{\nu\gamma}(-k)^{\beta} + g^{\gamma\beta}(2k)^{\nu} \right]$$

$$\cdot \left[g^{\gamma\alpha}(-2k)^{\lambda} + g^{\alpha\lambda}(k^{\gamma}) + g^{\lambda\gamma}(k^{\alpha}) \right]$$

$$\hookrightarrow \quad k^{\gamma}k^{\lambda} = \frac{1}{d} k^2 g^{\gamma\lambda} \qquad \int dF \cdot 1 \cdot = \frac{1}{2} \qquad \int dF \,(x,y,z) = \frac{1}{3}\cdot\frac{1}{2}$$

$$= \frac{1}{2}\frac{1}{d} k^2 \left[g^{\mu\alpha}\left(-\frac{5}{3}p - \frac{4}{3}q\right)^{\beta} + g^{\alpha\beta}\left(\frac{1}{3}p - \frac{1}{3}q\right)^{\mu} + g^{\beta\mu}\left(\frac{4}{3}p + \frac{5}{3}q\right)^{\alpha} \right]$$

$$\big[2g^{\beta\nu}g^{\gamma\lambda}g^{\gamma\alpha} - g^{\beta\nu}g^{\alpha\lambda}g^{\gamma\gamma} - g^{\beta\nu}g^{\gamma\alpha}g^{\lambda\gamma} \qquad \leftarrow \text{set } g^{\gamma\gamma} = 4 \ \sim \underline{d=4}$$

$$+ 2g^{\nu\gamma}g^{\beta\lambda}g^{\gamma\alpha} - g^{\nu\gamma}g^{\beta\gamma}g^{\alpha\lambda} - g^{\nu\gamma}g^{\alpha\beta}g^{\lambda\gamma}$$

$$- 4g^{\gamma\beta}g^{\nu\lambda}g^{\gamma\alpha} + 2g^{\gamma\beta}g^{\alpha\lambda}g^{\nu\gamma} + 2g^{\gamma\beta}g^{\nu\alpha}g^{\lambda\gamma}\big]$$

$$= \frac{1}{6}\frac{1}{d} k^2 \Big[g^{\beta\nu}g^{\alpha\lambda}\,[2 - 4 - 1 - 1 + 2]$$

$$g^{\beta\lambda}g^{\alpha\nu}\,[2 + 2]$$

$$g^{\alpha\beta}g^{\lambda\nu}\,[-1 - 4] \Big]$$

$$\cdot \left[g^{\mu\alpha}(-5p-4q)^{\beta} + g^{\alpha\beta}(p-q)^{\mu} + g^{\beta\mu}(4p+5q)^{\alpha} \right]$$

$$= \frac{1}{6}\cdot\frac{1}{d} k^2 \Big[(-2)\cdot\left(g^{\mu\lambda}(-5p-4q)^{\nu} + g^{\nu\lambda}(p-q)^{\mu} + g^{\mu\nu}(4p+5q)^{\lambda} \right)$$

$$+ 4\cdot\left(g^{\mu\nu}(-5p-4q)^{\lambda} + g^{\nu\lambda}(p-q)^{\mu} + g^{\mu\lambda}(4p+5q)^{\nu} \right)$$

$$- 5\left(g^{\lambda\nu}(-5p-4q)^{\mu} + 4g^{\nu\lambda}(p-q)^{\mu} + g^{\lambda\nu}(4p+5q)^{\mu} \right]$$

$$= \frac{1}{6}\,\frac{1}{d}\,k^2 \Big\{ g^{\mu\nu}\left[-8p - 10q - 20p - 16q \right]^\lambda$$

$$+ g^{\nu\lambda}\left[(p-q)(\underbrace{-2+4-20}_{-18}) + 25p + 20q - 20p - 25q \right]^\mu$$

$$+ g^{\lambda\mu}\left[+10p + 8q + 16p + 20q \right]^\nu \Big\}$$

$$① = \frac{1}{6}\,\frac{1}{d}\,k^2 \Big\{ g^{\mu\nu}\left[-28p - 26q \right]^\lambda + g^{\nu\lambda}\left[-13p + 13q \right]^\mu$$

$$+ g^{\lambda\mu}\left[26p + 28q \right]^\nu \Big\}$$

$$② = \int dF \Big[g^{\beta\nu}\left[-p - 2q + (1-z)p + yq \right]^\gamma + g^{\nu\gamma}\left[q - p + (1-z)p + yq \right]^\beta$$

$$+ g^{\gamma\beta}\left[2p + q - 2(1-z)p - 2yq \right]^\nu \Big]$$

$$\left[g^{\mu\alpha} k^\beta - 2 g^{\alpha\beta} k^\mu + g^{\beta\mu} k^\alpha \right]$$

$$\left[-2 g^{\gamma\alpha} k^\lambda + g^{\alpha\lambda} k^\gamma + g^{\lambda\gamma} k^\alpha \right]$$

$$= \frac{1}{2}\cdot\frac{1}{d}\,k^2 \Big[g^{\beta\nu}\left[-\tfrac{1}{3}p - \tfrac{5}{3}q \right]^\gamma + g^{\nu\gamma}\left[-\tfrac{1}{3}p + \tfrac{4}{3}q \right]^\beta$$

$$+ g^{\gamma\beta}\left[\tfrac{2}{3}p + \tfrac{1}{3}q \right]^\nu \Big]$$

$$\Big[g^{\mu\alpha}(-2) g^{\beta\lambda} g^{\gamma\alpha} + g^{\mu\alpha} g^{\beta\gamma} g^{\alpha\lambda} + g^{\mu\alpha} g^{\alpha\beta} g^{\lambda\gamma}$$

$$+ 4 g^{\alpha\beta} g^{\mu\lambda} g^{\gamma\alpha} - 2 g^{\alpha\beta} g^{\mu\gamma} g^{\alpha\lambda} - 2 g^{\alpha\beta} g^{\mu\alpha} g^{\lambda\gamma}$$

$$- 2 g^{\beta\mu} g^{\alpha\lambda} g^{\gamma\alpha} + g^{\beta\mu} g^{\alpha\gamma} g^{\alpha\lambda} + g^{\beta\mu} g^{\alpha\alpha} g^{\lambda\gamma} \Big]$$

$$= \frac{1}{6d} k^2 \; [\, g^{\beta\nu}(-p-5q)^\lambda + g^{\nu\lambda}[-p+4q]^\beta + g^{\lambda\beta}[2p+q]^\nu \,]$$

$$[\, g^{\mu\lambda} g^{\beta\lambda} [-2 -2]$$

$$+ g^{\mu\lambda} g^{\beta\lambda} [\,1 + 4\,]$$

$$+ g^{\mu\beta} g^{\lambda\lambda} [\,1 - 2 - 2 + 1 + 4\,] \,]$$

$$= \frac{1}{6d} k^2 \; [\, g^{\lambda\nu} (-p-5q)^\mu \cdot (-4) + g^{\mu\nu}(-p+4q)^\lambda (-4)$$

$$+ g^{\mu\lambda} (2p+q)^\nu (-4)$$

$$+ 5 \cdot (g^{\mu\lambda}) \, (-p-5q - p + 4q + 4\cdot(2p+q))^\nu$$

$$+ 2 \cdot (g^{\mu\nu}(-p-5q)^\lambda + g^{\nu\lambda}(-p+4q)^\mu + g^{\mu\lambda}(2p+q)^\nu \,]$$

$$= \frac{1}{6d} k^2 \; [\, g^{\mu\nu} (+4p - 16q - 2p - 10q)^\lambda$$

$$+ g^{\nu\lambda} (4p + 20q - 2p + 8q)^\mu$$

$$+ g^{\lambda\mu} (-8p - 4q + 5(6p + 3q) + 4p + 2q)$$

$$(2) = \frac{1}{6} \frac{1}{d} k^2 \; [\, g^{\mu\nu} (2p - 26q)^\lambda + g^{\nu\lambda} (2p + 28q)^\mu$$

$$+ g^{\lambda\mu} (26p + 13q)^\nu \,]$$

$$③ = \int dF \; \left[g^{\gamma\alpha}(-p+2(1-z)p+2yq)^{\lambda} + g^{\alpha\lambda}(-(1-z)p-yq-p)^{\gamma} + g^{\lambda\gamma}(2p-(1-z)p-yq)^{\alpha} \right]$$

$$\cdot \left[g^{\mu\alpha} k^{\beta} - 2g^{\alpha\beta}k^{\mu} + g^{\mu\beta}k^{\alpha} \right]$$

$$\cdot \left[g^{\beta\nu}(-k^{\gamma}) - g^{\nu\gamma}k^{\beta} + 2g^{\gamma\beta}k^{\nu} \right]$$

$$= \tfrac{1}{2}\tfrac{1}{d}k^2 \left[g^{\gamma\alpha}(\tfrac{1}{3}p+\tfrac{2}{3}q)^{\lambda} + g^{\alpha\lambda}(-\tfrac{5}{3}p-\tfrac{1}{3}q)^{\gamma} + g^{\lambda\gamma}(\tfrac{4}{3}p-\tfrac{1}{3}q)^{\alpha} \right]$$

$$\left[-g^{\mu\alpha}g^{\gamma\beta}g^{\nu\beta} - g^{\mu\alpha}g^{\nu\gamma}g^{\beta\beta} + 2g^{\mu\alpha}g^{\gamma\beta}g^{\beta\nu} \right.$$
$$+ 2g^{\alpha\beta}g^{\mu\gamma}g^{\beta\nu} + 2g^{\alpha\beta}g^{\mu\beta}g^{\nu\gamma} - 4g^{\alpha\beta}g^{\gamma\beta}g^{\mu\nu}$$
$$\left. - g^{\mu\beta}g^{\alpha\gamma}g^{\beta\nu} - g^{\mu\beta}g^{\alpha\beta}g^{\nu\gamma} + 2g^{\mu\beta}g^{\alpha\nu}g^{\gamma\beta} \right]$$

$$= \tfrac{1}{6}\tfrac{1}{d}k^2 \left[g^{\gamma\alpha}(p+2q)^{\lambda} + g^{\alpha\lambda}(-5p-q)^{\gamma} + g^{\lambda\gamma}(4p-q)^{\alpha} \right]$$

$$\left[g^{\mu\alpha}g^{\nu\gamma}[-1-4+2+2-1] \right.$$
$$+ g^{\alpha\nu}g^{\mu\gamma}[2+2]$$
$$\left. + g^{\alpha\gamma}g^{\mu\nu}[-4-1] \right]$$

$$= \tfrac{1}{6}\tfrac{1}{d}k^2 \left[(-2)\cdot\left[g^{\mu\nu}(p+2q)^{\lambda} + g^{\mu\lambda}(-5p-q)^{\nu} + g^{\lambda\nu}(4p-q)^{\mu} \right] \right.$$
$$+ 4\left[g^{\mu\nu}(p+2q)^{\lambda} + g^{\nu\lambda}(-5p-q)^{\mu} + g^{\mu\lambda}(4p-q)^{\nu} \right]$$
$$\left. - 5\left[g^{\mu\nu} \right]\left[4(p+2q)^{\lambda} - 5p^{\lambda} - q^{\lambda} + 4p^{\lambda} - q^{\lambda} \right] \right]$$

$$= \tfrac{1}{6}\tfrac{1}{d}k^2 \left[g^{\mu\nu}[-2p+4p-15p-4q+8q-30q]^{\lambda} \right.$$
$$\left. + g^{\mu\lambda}[+10p+2q+16p-4q]^{\nu} + g^{\lambda\nu}[-8p+2q-20p-4q]^{\mu} \right]$$

$$③ = \frac{1}{6}\frac{1}{d}k^2\left[g^{\mu\nu}(-13p-26q)^\lambda + g^{\nu\lambda}(-28p-2q)^\mu + g^{\lambda\mu}(26p-2q)^\nu\right]$$

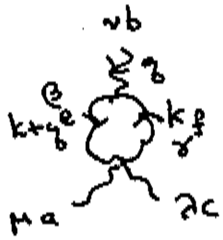
$$①+②+③ = \frac{1}{6}\frac{1}{d}k^2\left[g^{\mu\nu}(-39p-78q)^\lambda + g^{\nu\lambda}(-39p+39q)^\mu + g^{\lambda\mu}(78p+39q)^\nu\right]$$

$$= \frac{13}{2}\frac{1}{d}k^2\left[g^{\mu\nu}(-p-2q)^\lambda + g^{\nu\lambda}(-p+q)^\mu + g^{\lambda\mu}(2p+q)^\nu\right]$$

exact the $\approx$ p.22

$$= ig^3\left(-\tfrac{1}{2}C_2(G)f^{abc}\right)\cdot\frac{i}{(4\pi)^{d_L}}\Gamma\left(\frac{2-d_L}{2}\right)\cdot\frac{d}{2}\cdot 2 \cdot\frac{13}{2}\cdot\frac{1}{d}\left[g^{\mu\nu}(-p-2q)^\lambda+\dots\right]$$

$$= +\frac{13}{8}g^3 C_2(G) f^{abc}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\left[g^{\mu\nu}(-p-2q)^\lambda + \dots\right]$$



$$= g f^{ebf}\cdot\underbrace{\frac{1}{2}}_{\text{symt factor}}\cdot(-ig^2)\Big[f^{aed}f^{efd}(g^{\mu\beta}g^{\lambda\gamma}-g^{\mu\gamma}g^{\lambda\beta}) + f^{aed}f^{cfd}(g^{\mu\lambda}g^{\beta\gamma}-g^{\mu\gamma}g^{\lambda\beta}) + f^{afd}f^{ced}(g^{\mu\lambda}g^{\beta\gamma}-g^{\mu\beta}g^{\lambda\gamma})\Big]$$

$$\cdot\int\frac{d^dk}{(2\pi)^d}\frac{-i}{k^2}\frac{-i}{(k+q)^2}$$

$$\cdot\left[g^{\beta\nu}(-(k+q)-q)^\gamma + g^{\nu\gamma}(q-k)^\beta + g^{\gamma\beta}(k+k+q)^\nu\right]$$

combine denominators w. Feynman parameters. $\mathbb{k} = k + xq$

$$k = \mathbb{k} - xq \qquad (k+q) = \mathbb{k} + (1-x) q$$

To obtain a divergence, keep the finite part of [ ].

$$= \frac{ig^3}{2} \int_0^1 dx \; \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2)$$

$$\cdot \{ f^{ebf} f^{acd} f^{efd} (g^{\mu\beta} g^{\lambda\gamma} - g^{\mu\gamma} g^{\lambda\beta})$$

$$+ f^{ebf} f^{aed} f^{cfd} (g^{\mu\lambda} g^{\beta\gamma} - g^{\mu\gamma} g^{\lambda\beta})$$

$$+ f^{ebf} f^{afd} f^{ced} (g^{\mu\lambda} g^{\beta\gamma} - g^{\mu\beta} g^{\lambda\gamma}) \}$$

$$\cdot [g^{\beta\nu} (-2q + xq)^\gamma + g^{\nu\gamma} (q + xq)^\beta + g^{\gamma\beta} (-2xq + q)^\nu]$$

$$f^{ebf} f^{acd} f^{efd} = - f^{aeb} C_2(G) = + C_2(G) f^{abc}$$

$$f^{ebf} f^{aed} f^{efd} = (-f^{ade}) f^{ebf} f^{fdc} = + \tfrac{1}{2} C_2(G) f^{abc}$$

$$f^{ebf} f^{afd} f^{ced} = (-f^{adf})(-f^{fbe})(f^{edc}) = - \tfrac{1}{2} C_2(G) f^{abc}$$

$$= \frac{ig^3}{2} \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2) \; C_2(G) f^{abc}$$

$$\{ (g^{\mu\beta} g^{\lambda\gamma} - g^{\mu\gamma} g^{\lambda\beta}) + \tfrac{1}{2} (\cancel{g^{\mu\lambda} g^{\beta\gamma}} - g^{\mu\gamma} g^{\lambda\beta}) - \tfrac{1}{2} (\cancel{g^{\mu\lambda} g^{\beta\gamma}} - g^{\mu\beta} g^{\lambda\gamma}) \}$$

$$\cdot [g^{\beta\nu} (-\tfrac{3}{2}) q^\gamma + g^{\nu\gamma} \tfrac{3}{2} q^\beta + 0]$$

$$= - \frac{g^3}{2} \frac{1}{(4\pi)^{d/2}} \Gamma(2 - d/2) \; C_2(G) f^{abc} \; \tfrac{3}{2} (g^{\mu\beta} g^{\lambda\gamma} - g^{\mu\gamma} g^{\lambda\beta}) (-\tfrac{3}{2})$$

$$\cdot (g^{\beta\nu} q^\gamma - g^{\nu\gamma} q^\beta)$$

$$= + \frac{9}{8} g^3 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} C_2(G) f^{abc}$$

$$\cdot \left\{ g^{\mu\nu} q^\lambda - g^{\nu\lambda} q^\mu - g^{\nu\lambda} q^\mu + g^{\mu\nu} q^\lambda \right\}$$

$$= \frac{9}{4} g^3 C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \left\{ g^{\mu\nu} q^\lambda - g^{\nu\lambda} q^\mu \right\}$$

now cross

$$\begin{pmatrix} a \\ \mu \\ p+q \end{pmatrix} \leftrightarrow \begin{pmatrix} b \\ \nu \\ -q \end{pmatrix}$$

$$= \frac{9}{4} g^3 C_2(G)(-f^{abc}) \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \left\{ g^{\mu\nu}(-p-q)^\lambda + g^{\mu\lambda}(p+q)^\nu \right\}$$

$$\begin{pmatrix} c \\ \lambda \\ -p \end{pmatrix} \leftrightarrow \begin{pmatrix} b \\ \nu \\ -q \end{pmatrix}$$

$$= \frac{9}{4} g^3 C_2(G) (-f^{abc}) \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \left\{ g^{\mu\lambda} p^\nu - g^{\nu\lambda} p^\mu \right\}$$

so

 +  + 

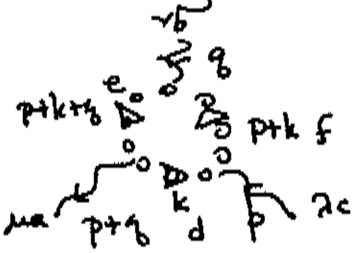
$$= \frac{9}{4} g^3 C_2(G) f^{abc} \frac{1}{(4\pi)^2} \cdot \frac{2}{\epsilon} \cdot \Big[g^{\mu\nu} (q^\lambda + (p+q)^\lambda)$$

$$+ g^{\nu\lambda}(-q^\mu + p^\mu) + g^{\lambda\mu}(-p-q-p)^\nu \Big]$$

$$= -\frac{9}{4} g^3 C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \Big[g^{\mu\nu}(-p-2q)^\nu + g^{\nu\lambda}(q-p)^\mu$$

$$+ g^{\lambda\mu}(2p+q)^\nu \Big]$$

finally, the ghost loops:



$$= (-g)^3 f^{dae} f^{ebf} f^{fcd} \underbrace{(-1)}_{\text{fermions}}$$

$$\cdot \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2}\frac{i}{(p+k)^2}\frac{i}{(p+k+q)^2}\, k^\mu (p+k+q)^\nu (p+k)^\lambda$$

$$= (-i)\, g^3 \left(f^{ade} f^{ebf} f^{fdc}\right) \cdot \int dx\,dy\,dz\,\delta(1-x-y-z)\cdot 2 \cdot \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{[\ell^2-\Delta]^3}$$

(p. 17) $\rightarrow \quad \cdot\left(\ell-(1-z)p-yq\right)^\mu\left(\ell+zp+(1-y)q\right)^\nu\left(\ell+zp-yq\right)^\lambda$

For the divergent term, pick up 2 ℓ's and one finite term.

$$= (-i)\, g^3 \left(-\tfrac12 C_2(G) f^{abc}\right) \frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\tfrac d2\right)\cdot\frac12$$

$$\cdot\int dF \left\{ g^{\mu\nu}(zp-yq)^\lambda + g^{\mu\lambda}(zp+(1-y)q)^\nu + g^{\nu\lambda}(-(1-z)p-yq)^\mu\right\}$$

$$= -\frac{g^3}{2\cdot2}C_2(G) f^{abc}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\cdot\frac16$$

$$\cdot\left\{g^{\mu\nu}(p-q)^\lambda + g^{\nu\lambda}(-2p-q)^\mu + g^{\lambda\mu}(p+2q)^\nu\right\}$$

$$= -\frac{g^3}{24}C_2(G) f^{abc}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}$$

$$\cdot\left\{g^{\mu\nu}(p-q)^\lambda + g^{\nu\lambda}(-2p-q)^\mu + g^{\lambda\mu}(p+2q)^\nu\right\}$$

for  exch. $\mu \leftrightarrow \lambda$

$a \leftrightarrow c$

$p+q \leftrightarrow -p$

 $= -\frac{g^3}{24} C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$

$$\cdot (-1) \cdot \{ g^{\lambda\nu}(-p-2q)^\mu + g^{\mu\nu}(2p+q)^\lambda + g^{\mu\lambda}(q-p)^\nu \}$$

 $+$  $= -\frac{g^3}{24} C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$

$$\cdot \{ g^{\mu\nu}(p-q-2p-q)^\lambda + g^{\nu\lambda}(-2p-q+p+2q)^\mu + g^{\lambda\mu}(p+2q-q+p)^\nu \}$$

$$= -\frac{g^3}{24} C_2(G) f^{abc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \{ g^{\mu\nu}(-p-2q)^\lambda + g^{\nu\lambda}(q-p)^\mu + g^{\lambda\mu}(p+2q)^\nu \}$$

so!

 $+$  $= \left(\frac{4}{3} n_f C(r)\right) \cdot \frac{g^3 f^{abc}}{(4\pi)^2} \frac{2}{\epsilon} \left[g^{\mu\nu}(-p-2q)^\lambda + \dots \right]$

 $= \frac{13}{8} C_2(G) \cdot \frac{g^3 f^{abc}}{(4\pi)^2} \frac{2}{\epsilon} \left[g^{\mu\nu}(-p-2q)^\lambda + \dots \right]$

 $+$  $+$  $= -\frac{9}{4} C_2(G) \cdot \frac{g^3 f^{abc}}{(4\pi)^2} \frac{2}{\epsilon} \left[g^{\mu\nu}(-p-2q)^\lambda + \dots \right]$

 $+$  $= -\frac{1}{24} C_2(G) \frac{g^3 f^{abc}}{(4\pi)^2} \frac{2}{\epsilon} \left[g^{\mu\nu}(-p-2q)^\lambda + \dots \right]$

to cancel these :

$$\delta_1^{3g} = \left(-\tfrac{4}{3} n_f C(r) + \underbrace{\left(-\tfrac{13}{8} + \tfrac{9}{4} + \tfrac{1}{24} \right)}_{\frac{5}{8} + \frac{1}{24} = \frac{2}{3}} C_2(G) \right) \frac{g^2}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$

$$\delta_1^{3g} = \left(\tfrac{2}{3} C_2(G) - \tfrac{4}{3} n_f C(r) \right) \frac{g^2}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$

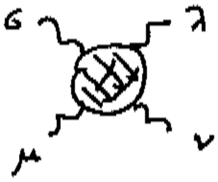
from p. 14

$$\delta_3 = \left(\tfrac{5}{3} C_2(G) - \tfrac{4}{3} n_f C(r) \right) \frac{g^2}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$

so $$\delta_1^{3g} - \delta_3 = - \frac{g^2}{(4\pi)^2} \cdot \frac{2}{\epsilon} \cdot C_2(G)$$

as required.

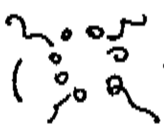
c.) The 4-gluon vertex is given by :



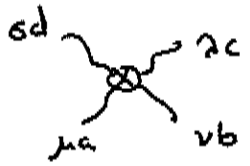
= ( + ...) (3! diagrams)

+ ( + ...) (3!/2 diagrams)

+ ( + ...) ($\frac{4 \cdot 3}{2}$ diagrams)

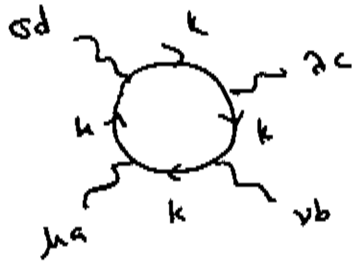
+ ( + ...)(3 diagrams) + ( + ...) (3! diagrams) +

the counterterm is



$$= -ig^2 \delta_1^{4g} \left[f^{abe} f^{cde} (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\sigma} g^{\nu\lambda}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma}) \right]$$

This has no powers of external momenta, so we can set the external momenta in the diagrams $= 0$. Also we can set $d=4$ in the divergent terms, e.g. $\gamma^\alpha \gamma^\mu \gamma_\alpha = (2-d)\gamma^\mu = -2\gamma^\mu$.



$$= (ig)^4 \underset{\text{fermion}}{(-1)} \int \frac{d^dk}{(2\pi)^d}\, tr\left[\gamma^\mu \frac{i\not k}{k^2} \gamma^\nu \frac{i\not k}{k^2} \gamma^\lambda \frac{i\not k}{k^2} \gamma^\sigma \frac{i\not k}{k^2}\right] \times tr[t^a t^b t^c t^d]$$

we need

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\alpha k^\beta k^\gamma k^\delta}{[k^2]^4} = \frac{i}{(4\pi)^{d/2}} \Gamma(2-d/2) \frac{1}{3!} \cdot \frac{1}{4} \left[g^{\alpha\beta} g^{\gamma\delta} + g^{\alpha\delta} g^{\gamma\beta} + g^{\alpha\gamma} g^{\beta\delta}\right]$$

$$= -\frac{g^4}{24} \cdot \frac{i}{(4\pi)^2} \cdot \frac{2}{\epsilon} \cdot tr[t^a t^b t^c t^d] \cdot tr\left[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma_\alpha \gamma^\lambda \gamma^\gamma \gamma^\sigma \gamma_\gamma + \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\lambda \gamma_\beta \gamma^\sigma \gamma_\alpha + \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\lambda \gamma_\alpha \gamma^\sigma \gamma_\beta\right]$$

$$= -i\,\frac{g^4}{24}\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}\;\mathrm{tr}(t^a t^b t^c t^d)$$

$$\cdot\,\mathrm{tr}\left[(-2)^2(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)\cdot 2 \;+\; (-2)\,\gamma^\mu\gamma^\lambda\gamma^\beta\gamma^\nu\gamma^\sigma\gamma_\beta\right]$$

$$= -i\,\frac{g^4}{24}\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}\;\mathrm{tr}(t^a t^b t^c t^d)$$

$$\cdot\,\mathrm{tr}\left[8\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma \;-\; 2\cdot 4\,\gamma^\mu\gamma^\lambda g^{\nu\sigma}\right]$$

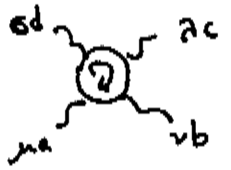
$$= -i\,\frac{g^4}{24}\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}\;\mathrm{tr}(t^a t^b t^c t^d)\cdot 32$$

$$\cdot\,(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\lambda}g^{\nu\sigma})$$

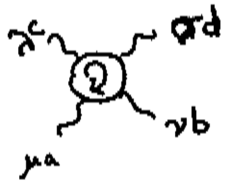
$$= -i\,\frac{4}{3}\,\frac{g^4}{(4\pi)^2}\,\frac{2}{\epsilon}\;\mathrm{tr}(t^a t^b t^c t^d)$$

$$(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} - 2g^{\mu\lambda}g^{\nu\sigma})$$

now look at the coefficient of $g^{\mu\lambda}g^{\nu\sigma}$ in all 6 diagrams.

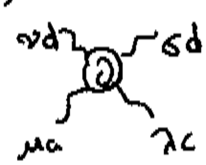


$$= +i\,\frac{4}{3}\,\frac{g^4}{(4\pi)^2}\,\frac{2}{\epsilon}\;\left[2\,\mathrm{tr}(t^a t^b t^c t^d)\right]\,g^{\mu\lambda}g^{\nu\sigma}$$



$$= +i\,\frac{4}{3}\,\frac{g^4}{(4\pi)^2}\,\frac{2}{\epsilon}\;\left[-\,\mathrm{tr}(t^a t^b t^d t^c)\right]\,g^{\mu\lambda}g^{\nu\sigma}$$

$$= \quad '' \quad \left[-\,\mathrm{tr}(t^a t^c t^b t^d)\right] \quad ''$$



$$= \quad '' \quad \left[-\,\mathrm{tr}(t^a t^c t^d t^b)\right] \quad ''$$

$$= \quad " \quad \left[-\,\mathrm{tr}\left[t^a t^d t^b t^c \right] \right] \quad "$$

$$= \quad " \quad \left[\, 2\,\mathrm{tr}\left[t^a t^d t^c t^b \right] \right] \quad "$$

in all

$$= \; + \frac{4}{3} i \frac{g^4}{(4\pi)^2} \frac{2}{\epsilon} \cdot g^{\mu\lambda} g^{\nu\sigma}$$

$$\cdot\, \mathrm{tr}\Big[\; t^a t^b [t^c, t^d] \; + \; t^a [t^b, t^c]\, t^d \; + \; t^a t^d [t^c, t^b] \; + \; t^a [t^d, t^c]\, t^b \; \Big]$$

$$+ \; (\text{2 more tensor structures})$$

the trace is:

$$\mathrm{tr}\Big\{ t^a t^b \, i f^{cde}\, t^e \; + \; t^a \, i f^{bce} t^e t^d \; + \; t^a t^d (-i f^{bce}\, t^e) \; - \; i \, t^a f^{cde} t^e t^b \Big\}$$

$$= \mathrm{tr}\left(t^a \; i f^{cde} [t^b, t^e] \; - \; i\, f^{bce}\, t^a [t^d, t^e] \right)$$

$$= \mathrm{tr}\; t^a \left(i f^{cde}\, i f^{bef}\, t^f \; - \; i\, f^{bce}\; i f^{def} t^f \right)$$

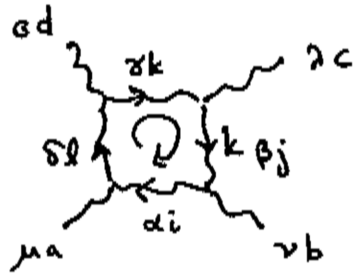
$$= -n_f\, C(r) \left[[f^{abe} f^{cde}] - [f^{ade} f^{bce}] \right]$$

so!

$$= \; - \frac{4}{3} i \frac{g^4}{(4\pi)^2} \cdot \frac{2}{\epsilon} \; g^{\mu\lambda} g^{\nu\sigma} \; n_f\, C(r) \left(f^{abe} f^{cde} - f^{ade} f^{bce} \right)$$

$$+ \; \text{2 more tensor structures.}$$

cycling. The other two tensor structures give the remaining terms in the expression at the top of p. 33.



$$= \quad g^4 f^{lai} f^{ibj} f^{jck} f^{kdl}$$

$$\cdot \int \frac{d^dk}{(2\pi)^d} \left(\frac{-i}{k^2}\right)^4 \cdot \left[g^{\delta\mu}(-k)^\alpha + g^{\mu\alpha}(-k)^\delta + g^{\alpha\delta}(k+k)^\mu \right]$$

$$\cdot \left[g^{\alpha\nu}(-k)^\beta + g^{\nu\beta}(-k)^\alpha + g^{\beta\alpha}(2k)^\nu \right]$$

$$\cdot \left[g^{\beta\lambda}(-k)^\gamma + g^{\lambda\gamma}(-k)^\beta + g^{\gamma\beta}(2k)^\lambda \right]$$

$$\cdot \left[g^{\gamma\sigma}(-k)^\delta + g^{\sigma\delta}(-k)^\gamma + g^{\delta\gamma}(2k)^\sigma \right]$$

Let $\quad (T^a)_{ij} = i f^{iaj} \quad$ then $\quad f^{lai} f^{ibj} f^{jck} f^{kdl} = \mathrm{tr}[T^a T^b T^c T^d]$

$$= \mathrm{tr}[T^d T^c T^b T^a]$$

$$= \quad g^4 \, \mathrm{tr}[T^a T^b T^c T^d] \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^4}$$

$$\cdot \{ g^{\mu\delta} k^\nu k^\beta + g^{\mu\delta} k^2 g^{\nu\beta} + (-2) g^{\mu\delta} k^\beta k^\nu$$

$$+ g^{\mu\nu} k^\delta k^\beta + g^{\nu\beta} k^\delta k^\mu - 2 g^{\mu\beta} k^\delta k^\nu$$

$$-2 g^{\nu\delta} k^\mu k^\beta - 2 g^{\nu\beta} k^\delta k^\mu + 4 g^{\delta\beta} k^\mu k^\nu \}$$

$$\{ g^{\beta\lambda} k^\delta k^\sigma + g^{\beta\lambda} k^2 g^{\sigma\delta} - 2 g^{\beta\lambda} k^\delta k^\sigma$$

$$+ g^{\lambda\sigma} k^\beta k^\delta + g^{\sigma\delta} k^\lambda k^\beta - 2 g^{\lambda\delta} k^\beta k^\sigma$$

$$-2 g^{\beta\sigma} k^\lambda k^\delta - 2 g^{\sigma\delta} k^\lambda k^\beta + 4 g^{\beta\delta} k^\lambda k^\sigma \}$$

$$= g^4 \, tr[T^a T^b T^c T^d] \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2]^4}$$

$$\{ g^{\mu\delta} g^{\nu\beta} k^2 - g^{\mu\delta} k^\beta k^\nu - g^{\nu\beta} k^\mu k^\delta$$
$$- 2 g^{\mu\beta} k^\nu k^\delta - 2 g^{\nu\delta} k^\mu k^\beta + g^{\mu\nu} k^\beta k^\delta + 4 g^{\beta\delta} k^\mu k^\nu \}$$

$$\cdot \{ g^{\beta\lambda} g^{\sigma\delta} k^2 - g^{\beta\lambda} k^\sigma k^\delta - g^{\delta\sigma} k^\beta k^\lambda$$
$$- 2 g^{\lambda\delta} k^\beta k^\sigma - 2 g^{\beta\sigma} k^\lambda k^\delta + g^{\lambda\sigma} k^\beta k^\delta + 4 g^{\beta\delta} k^\lambda k^\sigma \}$$

$$= g^4 \, tr[T^a T^b T^c T^d] \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2]^4}$$

$$\{ \underline{g^{\mu\sigma} g^{\nu\lambda} (k^2)^2} - g^{\nu\lambda} k^\mu k^\sigma (k^2) - g^{\mu\sigma} k^2 k^\nu k^\lambda$$
$$- 2 g^{\mu\lambda} k^\nu k^\sigma k^2 - 2 g^{\nu\sigma} k^\lambda k^\mu k^2$$
$$+ g^{\lambda\sigma} k^\mu k^\nu k^2 + 4 g^{\mu\nu} k^\lambda k^\sigma k^2$$

$$- g^{\mu\sigma} k^\lambda k^\nu (k^2) + k^\mu k^\nu k^\lambda k^\sigma + g^{\mu\sigma} k^2 k^\nu k^\lambda$$
$$+ 2 g^{\mu\lambda} k^2 k^\nu k^\sigma + 2 k^\mu k^\nu k^\lambda k^\sigma - g^{\lambda\sigma} k^\mu k^\nu k^2 - 4 k^\mu k^\nu k^\lambda k^\sigma$$

$$- g^{\nu\lambda} k^\mu k^\sigma k^2 + g^{\lambda\nu} k^\mu k^\sigma k^2 + k^\mu k^\nu k^\lambda k^\sigma$$
$$+ 2 k^\mu k^\nu k^\lambda k^\sigma + 2 g^{\nu\sigma} k^\mu k^\lambda k^2 - g^{\lambda\sigma} k^\mu k^\nu k^2 - 4 k^\mu k^\nu k^\lambda k^\sigma$$

$$- 2 g^{\mu\lambda} k^\nu k^\sigma k^2 + 2 g^{\mu\lambda} k^\nu k^\sigma k^2 + 2 k^\mu k^\nu k^\lambda k^\sigma$$
$$+ 4 k^\mu k^\nu k^\lambda k^\sigma + 4 g^{\mu\sigma} k^\nu k^\lambda k^2 - 2 g^{\lambda\sigma} k^\mu k^\nu k^2 - 8 k^\mu k^\nu k^\lambda k^\sigma$$

$$- 2 g^{\nu\sigma} k^\mu k^\lambda k^2 + 2 k^\mu k^\nu k^\lambda k^\sigma + 2 g^{\nu\sigma} k^2 k^\mu k^\lambda$$
$$+ 4 g^{\nu\lambda} k^2 k^\mu k^\sigma + 4 k^\mu k^\nu k^\lambda k^\sigma - 2 g^{\lambda\sigma} k^\mu k^\nu k^2 - 8 k^\mu k^\nu k^\lambda k^\sigma$$

$+$ (more) $\longrightarrow$

$$+ g^{\mu\nu}k^2k^\lambda k^\sigma - g^{\mu\nu}k^2k^\lambda k^\sigma - g^{\mu\nu}k^2k^\lambda k^\sigma$$

$$- 2g^{\mu\nu}k^2k^\lambda k^\sigma - 2g^{\mu\nu}k^2k^\lambda k^\sigma + g^{\mu\nu}\underline{g^{\lambda\sigma}}(k^2)^2 + 4g^{\mu\nu}k^\lambda k^\sigma k^2$$

$$+ 4g^{\lambda\sigma}k^2k^\mu k^\nu - 4k^\mu k^\nu k^\lambda k^\sigma - 4k^\mu k^\nu k^\lambda k^\sigma$$

$$-8k^\mu k^\nu k^\lambda k^\sigma - 8k^\mu k^\nu k^\lambda k^\sigma + 4k^\mu k^\nu g^{\lambda\sigma}k^2 + 16 \cdot 4k^\mu k^\nu k^\lambda k^\sigma \Big\}$$

$$= g^4 \, tr(T^aT^bT^cT^d) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^4}$$

$$\Big\{ g^{\mu\sigma}g^{\lambda\nu}(k^2)^2 \cdot 1 + g^{\mu\nu}g^{\lambda\sigma}(k^2)^2 \cdot 1$$

$$+ g^{\mu\sigma}k^2k^\nu k^\lambda[-1-1+1+4]$$

$$+ g^{\nu\lambda}k^2k^\mu k^\sigma[-1-1+1+4]$$

$$+ g^{\mu\nu}k^2k^\lambda k^\sigma[+4+1-1-1-2-2+4]$$

$$+ g^{\lambda\sigma}k^2k^\mu k^\nu[+1-1-1-2-2+4+4]$$

$$+ g^{\mu\lambda}k^2k^\nu k^\sigma[-2+2-2+2]$$

$$+ g^{\nu\sigma}k^2k^\mu k^\lambda[-2+2-2+2]$$

$$+ k^\mu k^\nu k^\lambda k^\sigma[1+2-4+1+2-4+2+4-8$$

$$+2+4-8-4-4-8-8+64]\Big\}$$

$$= g^4 \, tr(T^aT^bT^cT^d) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^4}$$

$$\cdot\Big\{ g^{\mu\sigma}g^{\lambda\nu}(k^2)^2 + g^{\mu\nu}g^{\lambda\sigma}(k^2)^2 + (g^{\mu\sigma}k^\nu k^\lambda + g^{\nu\lambda}k^\mu k^\sigma)k^2\, 3$$

$$+ (g^{\mu\nu}k^\lambda k^\sigma + g^{\lambda\sigma}k^\mu k^\nu) \cdot 3$$

$$+ k^\mu k^\nu k^\lambda k^\sigma \cdot 34 \Big\}$$

now
$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^2} = \frac{i}{(4\pi)^{d/2}} \Gamma(2-d/2)$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^3} k^\mu k^\nu = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{2} \cdot \frac{1}{2} g^{\mu\nu} = \frac{i}{(4\pi)^{d/2}} \Gamma(2-d/2) \cdot \frac{1}{4} g^{\mu\nu}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^4} k^\mu k^\nu k^\lambda k^\sigma = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{6} \cdot \frac{1}{4}(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma} + g^{\nu\lambda}g^{\mu\sigma})$$

$$= \frac{i}{(4\pi)^{d/2}} \Gamma(2-d/2) \frac{1}{24} (g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma} + g^{\nu\lambda}g^{\mu\sigma})$$

so

$$= g^4 \,\mathrm{tr}\, T^aT^bT^cT^d \frac{i}{(4\pi)^{d/2}} \Gamma(2-d/2)$$

$$\cdot \left\{ (g^{\mu\sigma}g^{\lambda\nu} + g^{\mu\nu}g^{\lambda\sigma}) + (g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\nu}g^{\lambda\sigma}) \cdot 2 \cdot 3 \cdot \frac{1}{4} \right.$$

$$\left. + \; 34 \; \frac{1}{24} (g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} + g^{\nu\sigma}g^{\mu\lambda}) \right\}$$

$$= \frac{ig^4}{12} \,\mathrm{tr}\, T^aT^bT^cT^d \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$$

$$\cdot \left\{ (g^{\mu\sigma}g^{\lambda\nu} + g^{\mu\nu}g^{\lambda\sigma}) \cdot [12 + 18 + 17] \right.$$

$$\left. + \; g^{\mu\lambda}g^{\nu\sigma} \cdot 17 \right\}$$

$$= \frac{ig^4}{12} \,\mathrm{tr}\, T^aT^bT^cT^d \frac{1}{(4\pi)^2} \frac{2}{\epsilon}$$

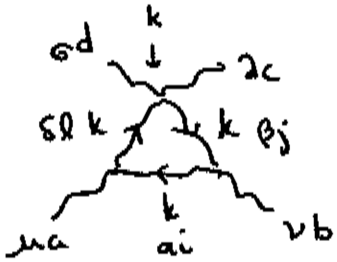
$$\cdot \left\{ (g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\lambda\nu}) \, 47 + g^{\mu\lambda}g^{\nu\sigma} \cdot 17 \right\}$$

then, summing over 3 diagrams:

$$= \frac{ig^4}{12}\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}\cdot g^{\mu\lambda}g^{\nu\sigma}$$

$$\cdot\left\{ 47\,\mathrm{tr}\,T^aT^cT^bT^d + 47\,\mathrm{tr}\,T^aT^bT^dT^c + 17\,\mathrm{tr}\,T^aT^bT^cT^d \right\}$$

$$+\ (\text{cyclic})$$



$$= -ig^2\Big[f^{ldk}f^{jck}\left(g^{\delta\beta}g^{\sigma\lambda} - g^{\delta\lambda}g^{\sigma\beta}\right) + f^{ljk}f^{dck}\left(g^{\delta\sigma}g^{\beta\lambda} - g^{\delta\lambda}g^{\beta\sigma}\right) + f^{lck}f^{djk}\left(g^{\delta\sigma}g^{\beta\lambda} - g^{\delta\beta}g^{\sigma\lambda}\right)\Big]$$

$$\cdot \int \frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)^2 \cdot g^2 f^{lai}f^{ibj}$$

$$\cdot\left[-g^{\delta\mu}k^\alpha - g^{\mu\alpha}k^\delta + 2k^\mu g^{\alpha\delta}\right]\left[-g^{\alpha\nu}k^\beta - g^{\nu\beta}k^\alpha + 2k^\nu g^{\alpha\beta}\right]$$

Let's write the f's as traces of T's.

$$f^{ldk}f^{jck}f^{lai}f^{ibj} = f^{lai}f^{ibj}f^{jck}(-f^{kdl}) = -\mathrm{tr}\,T^aT^bT^cT^d$$

$$f^{lai}f^{ibj}f^{ljk}f^{dck} = f^{lai}f^{ibj}f^{jkl}(-f^{cdk}) = -if^{cdk}\,\mathrm{tr}\,T^aT^bT^k$$

$$= -\mathrm{tr}\,T^aT^b[T^c,T^d]$$

$$f^{lai}f^{ibj}f^{lck}f^{djk} = f^{lai}f^{ibj}(-f^{jdk})(-f^{kcl}) = +\mathrm{tr}\,T^aT^bT^dT^c$$

so

$$= g^4 \{ tr(T^a T^b T^c T^d) [- g^{\delta\beta} g^{\sigma\lambda} + g^{\delta\lambda} g^{\sigma\beta} - g^{\delta\sigma} g^{\beta\lambda} + g^{\delta\lambda} g^{\beta\sigma}]$$

$$+ tr(T^a T^b T^d T^c) [g^{\delta\sigma} g^{\beta\lambda} - g^{\delta\lambda} g^{\beta\sigma} + g^{\delta\sigma} g^{\beta\lambda} - g^{\delta\beta} g^{\lambda\sigma}] \}$$

$$\int \frac{d^4 k}{(2\pi)^d} \left(\frac{1}{k^2}\right)^3 \left[\frac{k^2}{4}\right] \{ g^{\mu\delta} g^{\alpha\beta} g^{\alpha\nu} + g^{\mu\delta} g^{\alpha\alpha} g^{\nu\beta} - 2 g^{\mu\delta} g^{\alpha\nu} g^{\alpha\beta}$$

$$+ g^{\mu\alpha} g^{\delta\beta} g^{\alpha\nu} + g^{\mu\alpha} g^{\delta\alpha} g^{\nu\beta} - 2 g^{\mu\alpha} g^{\delta\nu} g^{\alpha\beta}$$

$$- 2 g^{\alpha\delta} g^{\mu\beta} g^{\alpha\nu} - 2 g^{\alpha\delta} g^{\mu\alpha} g^{\nu\beta} + 4 g^{\alpha\delta} g^{\mu\nu} g^{\alpha\beta} \}$$

$$= g^4 \{ tr(T^a T^b T^c T^d) [2 g^{\delta\lambda} g^{\sigma\beta} - g^{\delta\beta} g^{\sigma\lambda} - g^{\delta\sigma} g^{\beta\lambda}]$$

$$+ tr(T^a T^b T^d T^c) [2 g^{\delta\sigma} g^{\beta\lambda} - g^{\delta\lambda} g^{\beta\sigma} - g^{\delta\beta} g^{\sigma\lambda}]$$

$$\cdot \{ g^{\mu\delta} g^{\nu\beta} (\underbrace{1 + 4 - 2 + 1 - 2}_{2})$$

$$+ g^{\mu\nu} g^{\beta\delta} (\underbrace{1 + 4}_{5})$$

$$+ g^{\mu\beta} g^{\nu\delta} (\underbrace{-2 - 2}_{-4}) \} \cdot \left(\frac{i}{(4\pi)^2} \Gamma(2 - \tfrac{d}{2}) \cdot \frac{1}{4} \right)$$

$$= \frac{i g^4}{4} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \{ tr(T^a T^b T^c T^d) [4 g^{\mu\lambda} g^{\nu\sigma} + 10 g^{\mu\nu} g^{\lambda\sigma} - 8 g^{\mu\sigma} g^{\nu\lambda}$$

$$- 2 g^{\mu\nu} g^{\lambda\sigma} - 20 g^{\mu\nu} g^{\lambda\sigma} + 4 g^{\mu\nu} g^{\lambda\sigma}$$

$$- 2 g^{\mu\sigma} g^{\lambda\nu} - 5 g^{\mu\nu} g^{\lambda\sigma} + 4 g^{\mu\lambda} g^{\nu\sigma}]$$

$$+ tr(T^a T^b T^d T^c) [4 g^{\mu\sigma} g^{\nu\lambda} + 10 g^{\mu\nu} g^{\lambda\sigma} - 8 g^{\mu\lambda} g^{\nu\sigma}$$

$$- 2 g^{\mu\lambda} g^{\nu\sigma} - 5 g^{\mu\nu} g^{\lambda\sigma} + 4 g^{\mu\sigma} g^{\nu\lambda}$$

$$- 2 g^{\mu\nu} g^{\lambda\sigma} - 20 g^{\mu\nu} g^{\lambda\sigma} + 4 g^{\mu\nu} g^{\lambda\sigma}] \}$$

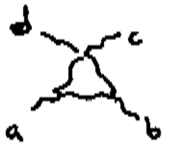
$$= \frac{ig^4}{4}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}$$

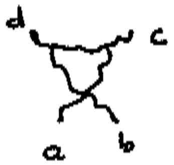
$$\Big\{ \mathrm{tr}(T^aT^bT^cT^d)\cdot\left(-13\, g^{\mu\nu}g^{\lambda\sigma} + 8\, g^{\mu\lambda}g^{\nu\sigma} - 10\, g^{\mu\sigma}g^{\nu\lambda}\right)$$

$$+ \mathrm{tr}(T^aT^bT^dT^c)\left(-13\, g^{\mu\nu}g^{\lambda\sigma} - 10\, g^{\mu\lambda}g^{\nu\sigma} + 8\, g^{\mu\sigma}g^{\nu\lambda}\right)\Big\}$$

as before, focus on the structure $g^{\mu\lambda}g^{\nu\sigma}$ and sum over the 6 diagrams.

 $= \frac{ig^4}{4}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\; g^{\mu\lambda}g^{\nu\sigma}$

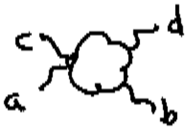
 $\Big\{ \mathrm{tr}(T^aT^bT^cT^d)\cdot 8 + \mathrm{tr}\, T^aT^bT^dT^c\cdot(-10)$

 $+ \mathrm{tr}(T^cT^dT^aT^b)\cdot 8 + \mathrm{tr}\, T^cT^dT^bT^a\,(-10)$

$+ \mathrm{tr}(T^aT^bT^cT^d)\cdot 8 + \mathrm{tr}\, T^aT^dT^bT^c\,(-10)$

 $+ \mathrm{tr}(T^aT^bT^cT^d)\cdot 8 + \mathrm{tr}\, T^aT^cT^bT^d\,(-10)$

 $+ \mathrm{tr}[T^aT^bT^dT^c](-13) + \mathrm{tr}\, T^aT^dT^bT^c\,(-13)$

  $+ \mathrm{tr}(T^aT^bT^dT^c)(-13) + \mathrm{tr}(T^aT^cT^bT^d)(-13)\Big\}$

now, since $\mathrm{tr}\, T^aT^bT^cT^d = \mathrm{tr}\, T^dT^cT^bT^a$ and we have cyclic invariance, there are only 3 structures, and we may write the above as:

$$= \frac{ig^4}{4} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \; g^{\mu\lambda} g^{\nu\sigma}$$

$$\cdot \Big\{ tr\left(T^a T^b T^c T^d\right) \left[8 + 8 + 8 + 8 \right]$$

$$+ \; tr\left(T^a T^b T^d T^c\right) \left[-10 - 10 - 13 - 13 \right]$$

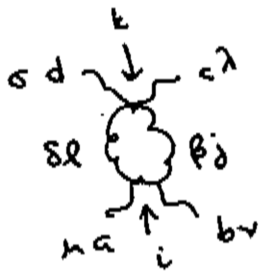
$$+ \; tr\left(T^a T^c T^b T^d\right) \left[-10 - 10 - 13 - 13 \right] \Big\}$$

$$= \frac{ig^3}{4} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \; g^{\mu\lambda} g^{\nu\sigma}$$

$$\Big\{ tr\left(T^a T^b T^c T^d\right) \cdot 32$$

$$+ \; tr\left(T^a T^b T^d T^c + T^a T^c T^b T^d\right) (-46) \Big\}$$

$$+ \text{ cyclic.}$$



$$= (-ig^2)^2 \Big[f^{ldk} f^{jck} \left(g^{\delta\beta} g^{\lambda\sigma} - g^{\delta\lambda} g^{\beta\sigma} \right)$$

$$+ \; f^{ljk} f^{dck} \left(g^{\delta\sigma} g^{\beta\lambda} - g^{\delta\lambda} g^{\beta\sigma} \right)$$

$$+ \; f^{lck} f^{djk} \left(g^{\delta\sigma} g^{\beta\lambda} - g^{\delta\beta} g^{\lambda\sigma} \right) \Big]$$

$$\Big[f^{ali} f^{bji} \left(g^{\mu\nu} g^{\beta\delta} - g^{\mu\beta} g^{\nu\delta} \right)$$

$$+ \; f^{abi} f^{lji} \left(g^{\mu\delta} g^{\nu\beta} - g^{\mu\beta} g^{\nu\delta} \right)$$

$$+ \; f^{aji} f^{lbi} \left(g^{\mu\delta} g^{\nu\beta} - g^{\mu\nu} g^{\beta\delta} \right) \Big]$$

$$\cdot \; \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \left(\frac{-i}{k^2} \right)^2$$

$$
\begin{aligned}
= \left(\frac{ig^4}{2}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\right)\Big\{ & (f^{ali} f^{bji} f^{ldk} f^{jck}) \\
& \cdot \left[4g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} \right] \\
& + (-f^{ali} f^{bji} f^{ljk} f^{dck}) \left[\cancel{g^{\mu\nu}g^{\lambda\sigma}} - \cancel{g^{\mu\nu}g^{\lambda\sigma}} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} \right] \\
& + (f^{ali} f^{bji} f^{lck} f^{djk}) \left[g^{\mu\nu}g^{\lambda\sigma} - 4g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\nu}g^{\lambda\sigma} \right] \\
& + (f^{abi} f^{lji} f^{ldk} f^{jck}) \left[\cancel{g^{\mu\nu}g^{\lambda\sigma}} - \cancel{g^{\mu\nu}g^{\lambda\sigma}} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} \right] \\
& + (f^{abi} f^{lji} f^{ljk} f^{dck}) \left[g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} \right] \\
& + (f^{abi} f^{lji} f^{lck} f^{djk}) \left[g^{\mu\sigma}g^{\nu\lambda} - \cancel{g^{\mu\lambda}g^{\lambda\sigma}} - \underset{\text{ok}}{g^{\mu\lambda}g^{\nu\sigma}} + \cancel{g^{\mu\nu}g^{\lambda\sigma}} \right] \\
& + (f^{aji} f^{lbi} f^{ldk} f^{jck}) \left[g^{\mu\nu}g^{\lambda\sigma} - 4g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\nu}g^{\lambda\sigma} \right] \\
& + (f^{aji} f^{lbi} f^{ljk} f^{dck}) \left[g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} - \cancel{g^{\mu\nu}g^{\lambda\sigma}} + \cancel{g^{\mu\nu}g^{\lambda\sigma}} \right] \\
& + (f^{aji} f^{lbi} f^{lck} f^{djk}) \left[g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\nu}g^{\lambda\sigma} + 4g^{\mu\nu}g^{\lambda\sigma} \right] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{ig^4}{2}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\Big\{ & (-f^{lai}) f^{ibj} f^{jck} (-f^{kdl}) \left(2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} \right) \\
& + (-f^{lai})(f^{ibj})(f^{jkl})(-f^{cdk}) \left(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} \right) \\
& + (-f^{lai})(f^{ibj})(-f^{jdk})(-f^{kcl}) \left(-2g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} \right) \\
& + (f^{abi})(-f^{lij})(f^{jck})(-f^{kdl}) \left(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} \right) \\
& + f^{abi}(-f^{lij})(+f^{jkl})(-f^{cdk}) \left(2g^{\mu\sigma}g^{\nu\lambda} - 2g^{\mu\lambda}g^{\nu\sigma} \right) \\
& + f^{abi}(-f^{lij})(-f^{jdk})(-f^{kcl}) \left(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} \right)
\end{aligned}
$$

$+$ (more $\rightarrow$)

$$+ (-f^{jai})(-f^{ibl})(f^{ldk})(-f^{kcj})(-2g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ (-f^{jai})(-f^{ibl})(-f^{lkj})(-f^{cdk})(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ (-f^{jai})(-f^{ibl})(f^{lck})(f^{kdj})(2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda})$$

$$= \frac{ig^4}{2}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\Big\{ tr(T^aT^bT^cT^d)(2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda})$$

$$+ tr(T^aT^b[T^c,T^d])(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$- tr(T^aT^bT^dT^c)(-2g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ tr[T^a,T^b]T^cT^d\,(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ tr[T^a,T^b][T^c,T^d]\,(2g^{\mu\sigma}g^{\nu\lambda} - 2g^{\mu\lambda}g^{\nu\sigma})$$

$$- tr[T^a,T^b]T^dT^c\,(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$- tr\,T^aT^bT^dT^c\,(-2g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ tr\,T^aT^b[T^c,T^d](g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$+ tr\,T^aT^bT^cT^d\,(2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda})\Big\}$$

$$\left.\begin{array}{l} tr\,T^aT^bT^cT^d \\ - tr\,T^aT^bT^dT^c \\ - tr\,T^bT^aT^cT^d \\ + tr\,T^bT^aT^dT^c \end{array}\right) = \Big) \rightarrow$$

(equal in pairs).

$$= \frac{ig^4}{2}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\,\cdot$$

$$\Big\{ tr(T^aT^bT^cT^d)\big[2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma}$$

$$+ g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\lambda} - 4g^{\mu\lambda}g^{\nu\sigma}$$

$$+ g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma}$$

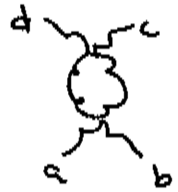
$$+ 2g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda}\big]$$

+ more $\longrightarrow$

$$+ \operatorname{tr}(T^a T^b T^d T^c) \, (-g^{\mu\sigma} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\sigma} + 2 g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\lambda} g^{\nu\sigma}$$
$$- g^{\mu\sigma} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\sigma} - 4 g^{\mu\sigma} g^{\nu\lambda} + 4 g^{\mu\lambda} g^{\nu\sigma}$$
$$+ (-g^{\mu\sigma} g^{\nu\lambda}) + g^{\mu\lambda} g^{\nu\sigma} + 2 g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\lambda} g^{\nu\sigma}$$
$$- g^{\mu\sigma} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\sigma}) \}$$

$$= \frac{ig^4}{2} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \Big\{ \operatorname{tr}(T^a T^b T^c T^d)$$
$$\cdot (4 g^{\mu\nu} g^{\lambda\sigma} + 10 g^{\mu\sigma} g^{\nu\lambda} - 8 g^{\mu\lambda} g^{\nu\sigma})$$
$$+ \operatorname{tr}(T^a T^b T^d T^c)$$
$$(4 g^{\mu\nu} g^{\lambda\sigma} + 10 g^{\mu\lambda} g^{\nu\sigma} - 8 g^{\mu\sigma} g^{\nu\lambda}) \Big\}$$

again, track the coefficient of $g^{\mu\lambda} g^{\nu\sigma}$ thru the 3 diagrams:



$$= \frac{ig^4}{2} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \; g^{\mu\lambda} g^{\nu\sigma} \left[-8 \operatorname{tr} T^a T^b T^c T^d + 10 \operatorname{tr} T^a T^b T^d T^c \right]$$

$$= \quad " \quad \left[-8 \operatorname{tr} T^a T^b T^c T^d + 10 \operatorname{tr} [T^a T^c T^b T^d] \right]$$

$$= \quad " \quad \left[4 \operatorname{tr} T^a T^b T^d T^c + 4 \operatorname{tr} [T^a T^c T^b T^d] \right]$$

~ all

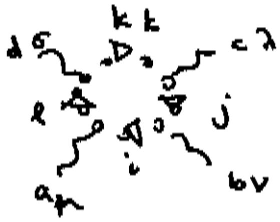
 $= \dfrac{ig^4}{2}\dfrac{1}{(4\pi)^2}\dfrac{2}{\epsilon}\; g^{\mu\lambda}g^{\nu\sigma}$

$$\cdot\Big[\mathrm{tr}\,(T^aT^bT^cT^d)\,(-16) + \mathrm{tr}\,(T^aT^bT^dT^c)\cdot 14$$

$$+\ \mathrm{tr}\,(T^aT^cT^bT^d)\cdot 14\ \Big]$$

$+$ cyclic.

finally

 $= (-g)^4 f^{lai}f^{ibj}f^{jck}f^{kdl}\,(-1)$ fermion.

$$\int\frac{d^dk}{(2\pi)^d}\left(\frac{i}{k^2}\right)^4 k^\mu k^\nu k^\lambda k^\sigma$$

$$= -g^4\,\frac{1}{24}\left(\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\right)\left(g^{\mu\nu}g^{\lambda\sigma}+g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}\right)\mathrm{tr}\,T^aT^bT^cT^d$$

there are 6 diagrams. tracking the coefficient of $g^{\mu\lambda}g^{\nu\sigma}$, we find

 $= -\dfrac{g^4}{24}\dfrac{i}{(4\pi)^2}\dfrac{2}{\epsilon}\; g^{\mu\lambda}g^{\nu\sigma}\Big\{\mathrm{tr}\,T^aT^bT^cT^d$

$\quad + \mathrm{tr}\,T^aT^bT^dT^c$

$\quad + \mathrm{tr}\,T^aT^cT^bT^d\Big\}\cdot 2$

now add all of the contributions:

$$= i\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}\left\{(f^{abe}f^{cde}-f^{ade}f^{bce})\left[-\frac{4}{3}n_f C(r)\right]\right\}$$

$$\begin{aligned} = i\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}\Big\{ & \mathrm{tr}\, T^aT^cT^bT^d \ \frac{47}{12} \\ & + \mathrm{tr}\, T^aT^bT^dT^c \cdot \frac{47}{12} \\ & + \mathrm{tr}\, T^aT^bT^cT^d \cdot \frac{17}{12} \Big\} \end{aligned}$$

$$\begin{aligned} = i\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}\Big\{ & \mathrm{tr}\, T^aT^bT^cT^d \cdot 8 \\ & + \mathrm{tr}(T^aT^bT^dT^c + T^aT^cT^bT^d)\left(-\frac{23}{2}\right)\Big\} \end{aligned}$$

$$\begin{aligned} = i\frac{g^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda}g^{\nu\sigma}\Big\{ & \mathrm{tr}\, T^aT^bT^cT^d \cdot (-8) \\ & + \mathrm{tr}(T^aT^bT^dT^c + T^aT^cT^bT^d) \cdot 7 \Big\} \end{aligned}$$

$$\begin{aligned} = \frac{ig^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\nu}g^{\lambda\sigma}\Big\{ & \mathrm{tr}\, T^aT^bT^cT^d \left(-\frac{1}{12}\right) \\ & + \mathrm{tr}(T^aT^bT^dT^c + T^aT^cT^bT^d)\cdot\left(-\frac{1}{12}\right)\Big\} \end{aligned}$$

in all

 $= \dfrac{ig^4}{(4\pi)^2}\,\dfrac{2}{\epsilon}\; g^{\mu\nu} g^{\rho\sigma}$

$$\cdot \Big\{ \left(f^{abe} f^{cde} - f^{ade} f^{bce}\right)\left(-\tfrac{4}{3}\, n_f\, C(r)\right)$$

$$+ \quad \mathrm{tr}\; T^a T^b T^c T^d \left(\tfrac{17}{12} - \tfrac{1}{12} + 8 - 8 \right) \leftarrow \;=\; \tfrac{4}{3}$$

$$+ \quad \mathrm{tr}\left(T^a T^b T^d T^c + T^a T^c T^b T^d\right)\left(\tfrac{47}{12} - \tfrac{1}{12} - \tfrac{23}{2} + 7 \right) \Big\}$$

$$\uparrow$$

$$\tfrac{23}{6} - \tfrac{23}{2} + \tfrac{21}{3} = -\tfrac{2}{3} \;!$$

now

$$\mathrm{tr}\left(2 T^a T^b T^c T^d - T^a T^b T^d T^c - T^a T^c T^b T^d\right)$$

$$= \mathrm{tr}\Big\{ [T^a T^b\; i f^{cde}\, T^e] + [T^a (i f^{bce})\, T^e\; T^d] \Big\}$$

since $\quad \mathrm{tr}\; T^a T^b T^e = -\mathrm{tr}\; T^e T^b T^a = -\,\mathrm{tr}\; T^a T^e T^b$

$$= \tfrac{1}{2}\; \mathrm{tr}\; T^a [T^b, T^e]$$

$$= \tfrac{i}{2} f^{fbe}\; \mathrm{tr}\; T^a T^f$$

$$= \tfrac{i}{2} f^{fbe} \left(i f^{iaj}\, i f^{jfi} \right)$$

$$= \tfrac{i}{2} f^{fbe}\; C_2(G)\, \delta^{af} \;=\; \tfrac{i}{2} f^{abe}$$

so

$$= \frac{ig^4}{(4\pi)^2}\frac{2}{\epsilon}\, g^{\mu\lambda} g^{\nu\sigma}$$

$$\cdot\left(f^{abe}f^{cde} - f^{ade}f^{bce}\right)$$

$$\left(-\frac{4}{3}n_f C(r) - \frac{1}{3}C_2(G)\right)$$

$$+ (\text{cyclic})$$

$$\delta_1^{4g} = \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\left(-\frac{4}{3}n_f C(r) - \frac{1}{3}C_2(G)\right)$$

$$\delta_3 = \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\left(-\frac{4}{3}n_f C(r) + \frac{5}{3}C_2(G)\right)$$

$$\frac{1}{2}\left(\delta_1^{4g} - \delta_3\right) = \frac{1}{2}\cdot\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\cdot -2C_2(G)$$

$$= -\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\,C_2(G)$$

as required !!!